Rationally connected varieties over function fields

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A classical result:

Tsen-Lang Theorem

$K = \mathbb{C}(t)$

$f \in K[x_0, \ldots, x_n]$ homogeneous of degree $d \leq n$

Then $f=0$ has a nontrivial solution over K

Geometrically:

If $X := \{f = 0\} \subset \mathbb{P}^n$

then $X(K) \neq \emptyset$

i.e. hypersurfaces of degree $d \leq n$ in $\mathbb{P}^n$ have K-rational points

Remark: True over $K = \mathbb{C}(B)$

B arbitrary curve

Idea of proof

Clearing denominators we get

$$f \in \mathbb{C}[t][x_0 \dots x_n]$$

$$f = \sum_{\substack{i_0 + \dots + i_n \\ = d}} f_{i_0 \dots i_n}(t)\, x_0^{i_0} \cdots x_n^{i_n}$$

$$M = \max(\deg f_{i_0 \dots i_n})$$

Try $\quad x_j = c_{j0} + c_{j1} t + \dots + c_{je} t^e$

$$= c_j(t)$$

$f(c_0(t), \dots, c_n(t)) = 0 \quad$ solvable

$$\Updownarrow$$

$d \cdot e + M + 1$ homogeneous equations

in $\underbrace{c_{00}, \dots, c_{ne}}_{(e+1)(n+1) \text{ variables}}$ have solution over $\mathbb{C}$

But $\quad de + M + 1 < (e+1)(n+1)$

for $e \gg 0$

<u>Question</u> What geometric properties distinguish hypersurfaces

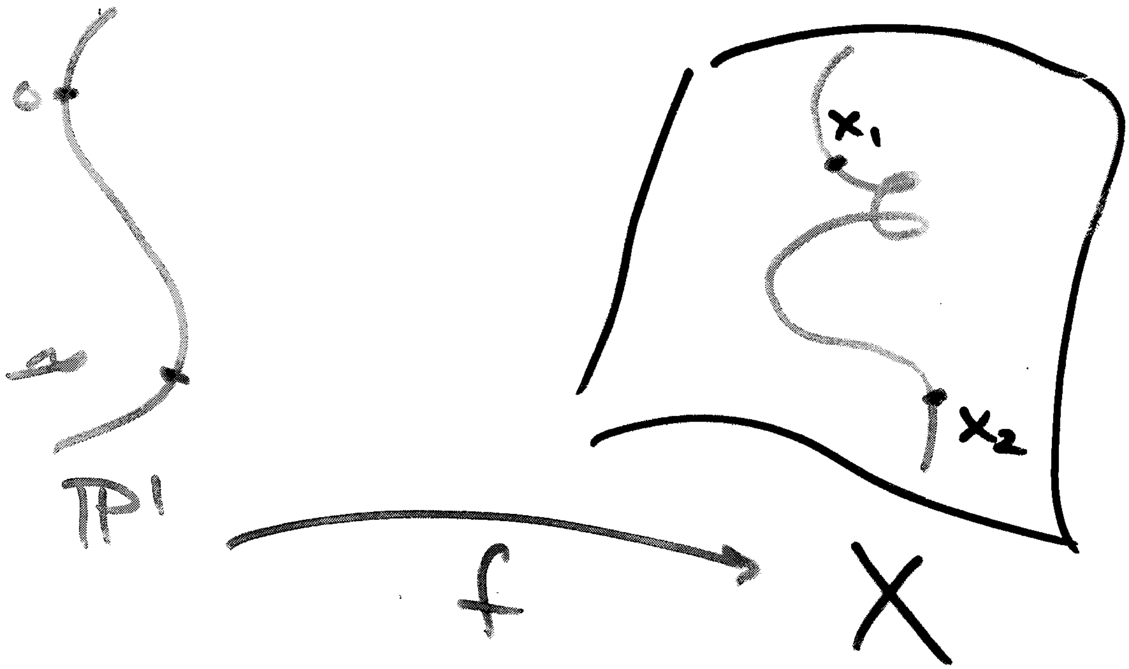
$$X \subset \mathbb{P}^n \quad \deg X \leq n \;?$$

- $$K_X = [\omega_X] = [\Omega_X^{n-1}] = [\mathcal{O}_X(d-n-1)] < 0$$

  i.e. X is Fano

- X is consequently rationally connected

Defn A complex variety X is rationally connected if any two general points $x_1, x_2 \in X$ can be joined by a rational curve



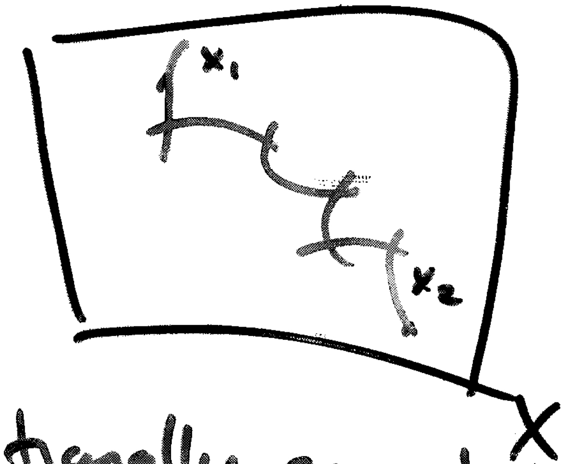
Notes 1) This curve is proper with $f(\mathbb{P}^1) \subset X$

2) Notion was originally developed by Campana and Kollár / Miyaoka / Mori

Variants

X is rationally chain connected

if any two general points can be joined by a chain of rational curves



X is strongly rationally connected

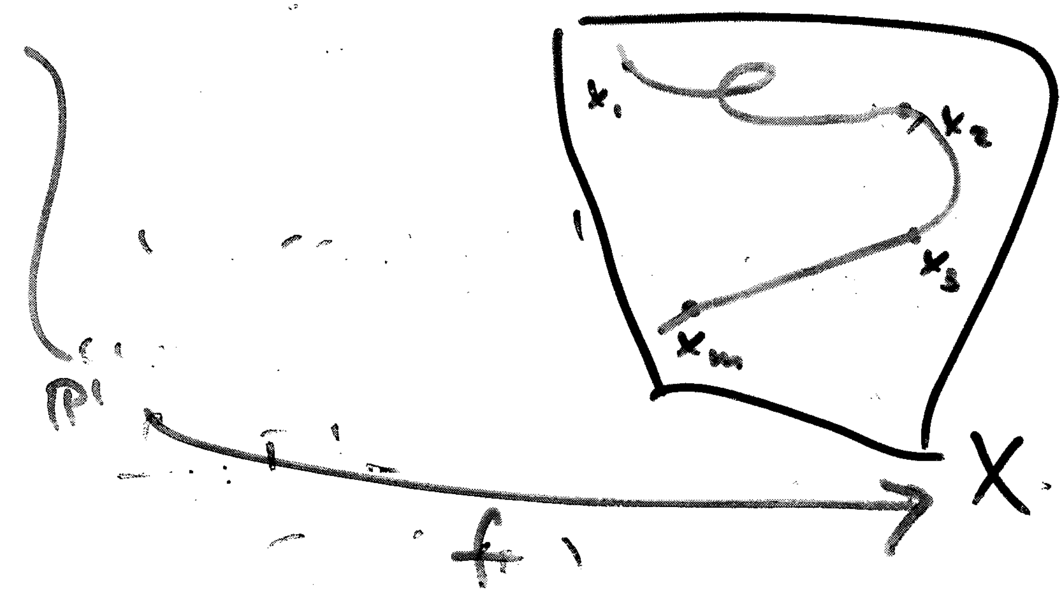
if any point $x \in X$ can be joined to a general point by a rational curve

Note If X smooth, proper then all these notions are equivalent

Key property

X smooth, strongly rationally connected

Then for any $x_1, \dots, x_m \in X$ there exists a very free curve passing through those points



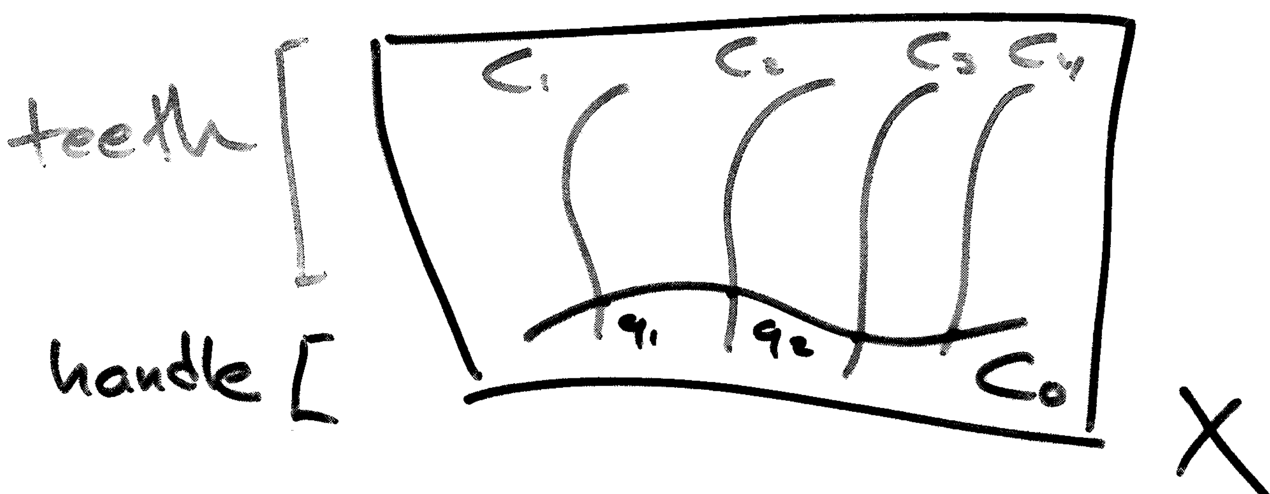
i.e. $f^* T_X$ ample

$$\cong \bigoplus_{i=1}^{\dim X} \mathcal{O}_{\mathbb{P}^1}(a_i) \qquad a_i > 0$$

What good are very free curves?

Comb constructions!

Suppose we are given a rational (or higher genus) curve $C_0 \subset X$



Then we can produce very free curves

$$f_i : \mathbb{P}^1 \longrightarrow X \quad i=1,\dots,s$$
$$C_i = f_i(\mathbb{P}^1)$$

meeting C_0 at points $q_1 \dots q_s$ so that the comb $C_0 \cup C_1 \cup \dots \cup C_s$ admits a deformation smoothing $q_1 \dots q_s$

Graber Harris Starr Theorem

(21st century Tsen-Lang)

B smooth algebraic curve

$K = \mathbb{C}(B)$

X rationally connected variety $/K$

Then $X(K) \neq \emptyset$

Apply valuative criterion

Restrict to generic point

Fibration formulation

$\pi : \mathcal{X} \longrightarrow B$ proper morphism from a smooth variety to a curve, with rationally connected generic fiber

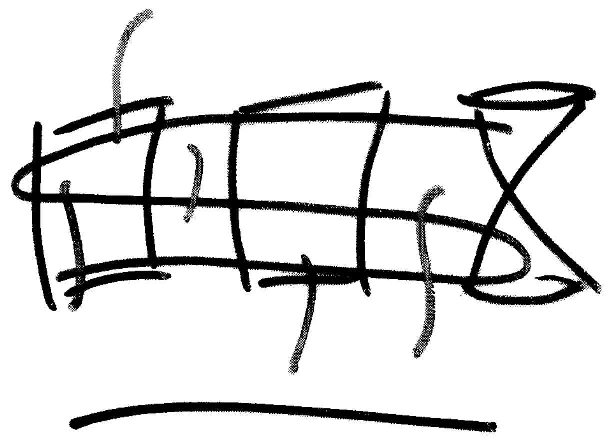
Then π admits a section

$s : B \longrightarrow \mathcal{X}$

Some geometric constructions used in the proof

Start with a multisection in $\mathcal{X}$ mapping finitely to B

$$
\begin{tikzcd}
\mathcal{X} \arrow[d, "\pi"'] & M \arrow[l, hook'] \arrow[dl] \\
B &
\end{tikzcd}
$$



Step 1 Build a comb using very free curves in fibers as teeth and M as the handle

Get a new multisection M′ which is simply ramified (over B) in the open subset where π is smooth

Furthermore the corresponding branch points deform freely over B as M′ deforms in X

Step 2 Add very free curves meeting M′ twice in a fiber to increase the number of simple branch points (and genus)

And the brilliant trick!

$$\begin{tikzcd} \mathcal{X} \arrow[d] & \\ B & M'' \arrow[ul, hook] \arrow[l] \end{tikzcd}$$

Ramification of M'' arising from multiple component

multiple component fiber

Use the resulting multitude of ramification points to "cancel out" local monodromy of

$$M'' \longrightarrow B$$

arising at points where M'' intersects multiple components of fibers of π

$\rightsquigarrow$ Multisection M''' ramified (simply) only where π is smooth, with branch points deforming freely

This necessarily breaks into a union of sections!!

Producing sections with prescribed properties

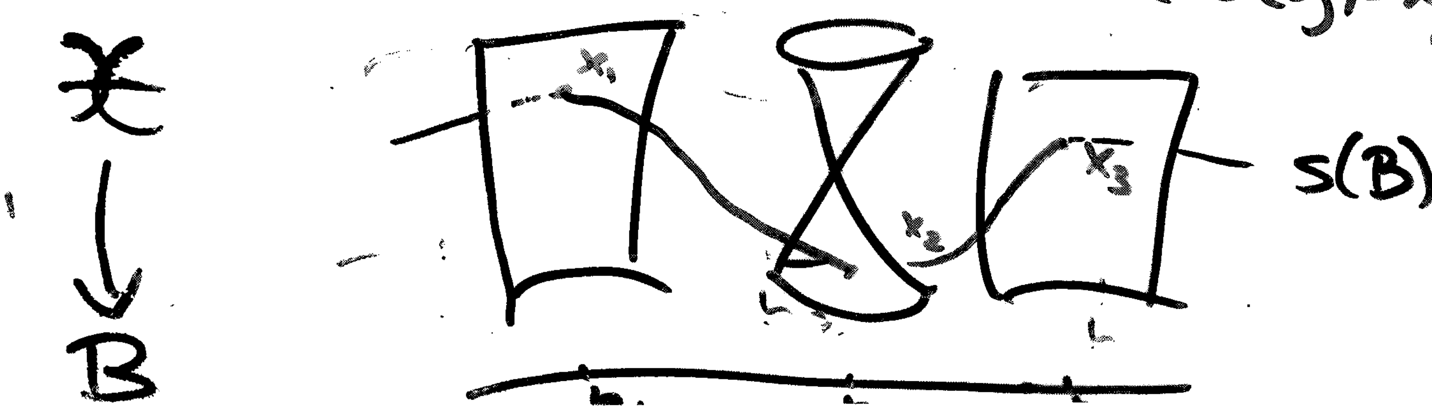
$\pi : \mathcal{X} \longrightarrow B$ proper morphism from a smooth variety with rationally connected generic fiber

GHS $\Longrightarrow$ $\exists$ section $s : B \longrightarrow \mathcal{X}$

In fact, we have $s(B) \subset \mathcal{X}^{sm}$

$$\mathcal{X}^{sm} = \{ x \in \mathcal{X} : \pi \text{ is smooth at } x \}$$

<u>Problem</u> Given distinct $b_1, \ldots, b_r \in B$ and $x_j \in \mathcal{X}^{sm}_{b_j}$, can we find a section $s : B \longrightarrow \mathcal{X}$ with $s(b_j) = x_j$?



Problem′ (Weak approximation)

Given N-jets of sections

$$j_i : \underbrace{\mathrm{Spec}\left(\mathcal{O}_{B,b_i}/\mathfrak{m}_{b_i}^{N+1}\right)}_{\substack{\shortparallel \\ B_{b_i,N}}} \longrightarrow \mathcal{X}^{sm} \times_B B_{b_i,N}$$

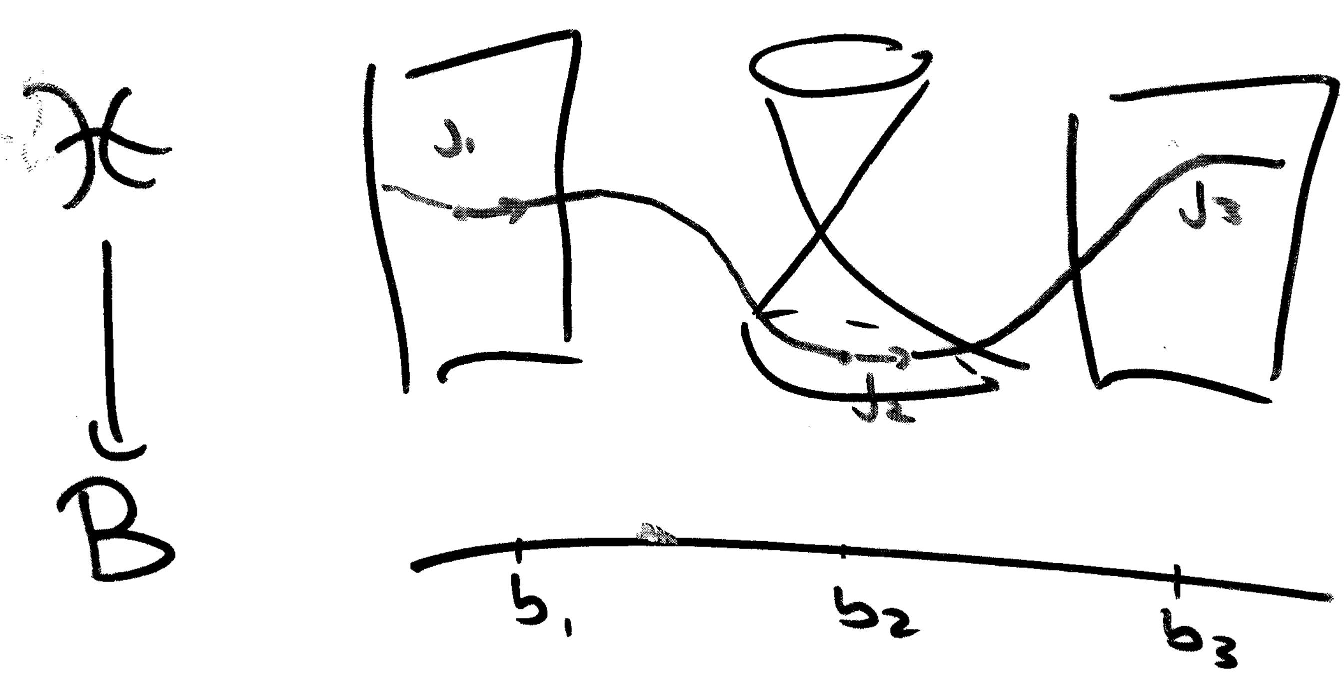
$i=1 \ldots r$

can we find a section $s : B \to \mathcal{X}$

with $s \equiv j_i \ (\mathrm{mod}\ \mathfrak{m}_{b_i}^{N+1})$

$i = 1 \ldots r$?

In other words, are there sections with prescribed Taylor series?



The two problems are equivalent - even though the second looks stranger

$J = \{ j_1 \dots j_r \}$ N-jet data for $\mathcal{X} \longrightarrow B$

Sections $s : B \longrightarrow \mathcal{X}$ with those jets

$$\Updownarrow$$

Sections $s : B \longrightarrow \mathcal{X}(J)$ with $s(b_i) = x_i$ for suitable $x_i \in \mathcal{X}(J)^{sm}_{b_i}$

where $\mathcal{X}(J) \longrightarrow \mathcal{X}$ is a suitable iterated blow up

[Examples to follow soon!]

General results on weak approximation (CT/Gille)

X/K generic fiber $\quad K = \mathbb{C}(B)$

(1) $X = \mathbb{P}^n$ satisfies weak approximation

(2) $X_1 \overset{\sim}{\dashrightarrow} X_2$ birational$/K$

Then X_1 satisfies WA $\iff$ X_2 satisfies WA

($\Rightarrow$ $\underset{\text{quadric}}{X_2 \subset \mathbb{P}^n}$ $n \geq 2$ satisfies WA)

(3) $X \longrightarrow Y$ fibration so that Y satisfies WA and fibers do as well Then X satisfies WA

($\Rightarrow$ $\underset{\text{cubic}}{X_3 \subset \mathbb{P}^n}$ $n \geq 6$ satisfies WA

$X_{2,2} \subset \mathbb{P}^4$ satisfies WA)

WA at places of good reduction

Defn $K = \mathbb{C}(B)$

X smooth, proper $/K$

A place $b \in B$ is of good reduction if there exists a smooth proper morphism

$$\hat{\mathcal{X}} \longrightarrow \hat{B}_b = \mathrm{Spec}(\hat{\mathcal{O}}_{B,b})$$

$$= \mathrm{Spec}(\varprojlim \mathcal{O}_{B,b}/m_b^n)$$

with generic fiber isomorphic to

$$X \times \underbrace{\hat{K}_b}_{\text{completion of } K \text{ at } b}$$

Proposition There exists a proper algebraic space

$$\mathcal{X} \xrightarrow{\pi} B \quad \text{"good model"}$$

with generic fiber X and smooth over all places of good reduction

<u>Theorem</u> (KMM $N=0$ case; H — Tschinkel generally)

$K = \mathbb{C}(B)$

X smooth, proper, rationally connected $/K$

Then X satisfies WA at places of good reduction

In particular

Given a model $\mathcal{X} \longrightarrow B$ (with $\mathcal{X}$ smooth), points $b_1 \dots b_r \in B$ such that $\mathcal{X}_{b_1} \dots \mathcal{X}_{b_r}$ are smooth, and N-jets $j_1 \dots j_r$ at those points

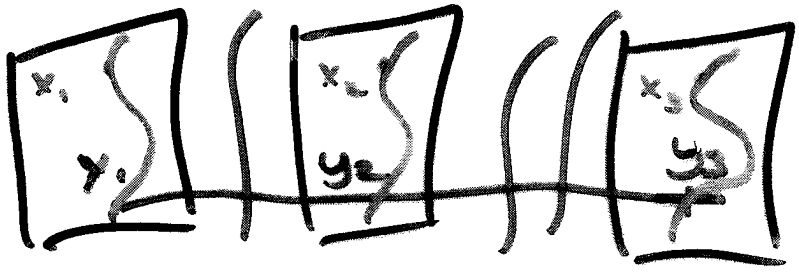
There exists a section $s : B \longrightarrow \mathcal{X}$ with: $s(b_i) = j_i \pmod{m_{b_i}^{N+1}}$

$i = 1 \dots r$

(Kollár Miyaoka Mori did point conditions)

Proof sketch (Induction on N)

($N=0$)



$$\begin{array}{c}\mathcal{X}\\ \downarrow\\ B\end{array}$$

Graber/Harris/Starr $\rightsquigarrow$ $s': B \longrightarrow \mathcal{X}$
initial section
$s'(b_i) = y_i$

$x_1 \ldots x_r$ desired points in $\mathcal{X}_{b_1} \ldots \mathcal{X}_{b_r}$

Join x_i to y_i using a free curve in $\mathcal{X}_{b_i}$

$$f_i : \mathbb{P}^1 \longrightarrow \mathcal{X}_{b_i}$$

Construct comb with handle $C_0 = s'(B)$
and teeth $C_i = f_i(\mathbb{P}^1)$ $i = 1 \ldots r$
and also some additional very free teeth in other fibers

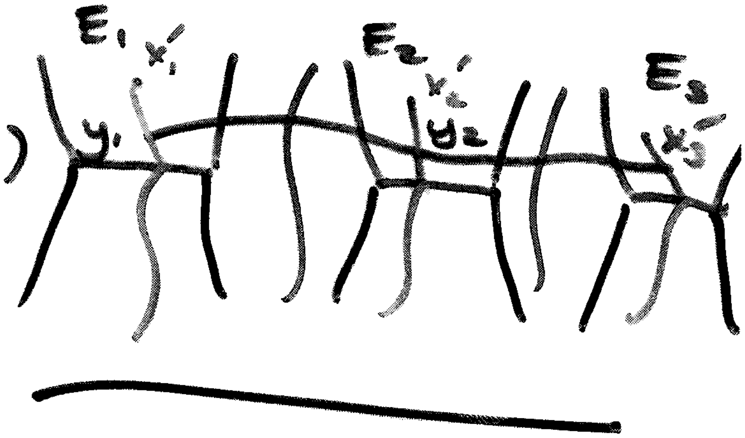
Using deformation theory, show that the comb deforms to a smooth curve in $\mathcal{X}$ containing $x_1 \ldots x_r$

(Inductive step)

$N=1$ for simplicity

Jet data = $\begin{cases} \text{Points } x_i \text{ in smooth fibers } \mathcal{X}_{b_i} \\ \text{Tangent directions } v_i \in T_{x_i}\mathcal{X} - T_{x_i}\mathcal{X}_{b_i} \end{cases}$

$$\begin{array}{c} \mathcal{X}(J) = \mathrm{Bl}_{x_1 \dots x_r}\mathcal{X} \\ \downarrow \\ B \end{array}$$



$E_i \cong \mathbb{P}^{\dim(\mathcal{X})}$ exceptional

$\ni$

x_i' corresponding to v_i

$s' : B \longrightarrow \mathcal{X}(J)$ proper transform of section of $\mathcal{X} \to B$ with $s'(b_i) = x_i$

$y_i = s'(b_i)$

Join x_i' to y_i with line $T_{i,1} \subset E_i$

Then produce very free curve in $\mathrm{Bl}_{x_i}\mathcal{X}_{b_i}$ meeting E_i only at $T_{i,1} \cap \mathrm{Bl}_{x_i}\mathcal{X}_{b_i}$

Adding some additional teeth gives comb smoothing to section $\ni x_1' \dots x_r'$

What about places of bad reduction? (20)

Weak approximation is still open for cubic surfaces!!

<u>Theorem</u> (H-Tschinkel)

$\mathcal{X} \to B$ fibration in cubic surfaces with at worst rational double points

Assume either

1) $\mathcal{X}$ is smooth, or

2) the fibers have just ordinary double points (A_1 singularities)

Then weak approximation holds

<u>Note</u> (1) covers the case of cubic surfaces w/ square free discriminant