



Connections for Women:
Introduction to the Spring 2008 Programs

January 16-18, 2008

Participant Research Summaries

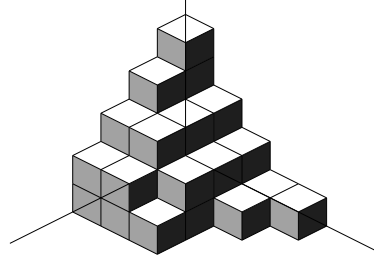
LARGE RANDOM STRICT PLANE PARTITIONS

MIRJANA VULETIĆ*

1. A MEASURE ON STRICT PLANE PARTITIONS

Consider the following two figures:

5	3	2	1	1
4	3	2	1	
3	3	2		
2	2	1		



Both these figures represent a *plane partition* – an infinite matrix with nonnegative integer entries that form nonincreasing rows and columns with only finitely many nonzero entries. The second figure shows the plane partition as a 3-dimensional object where the height of the block positioned at (i, j) is given with the (i, j) th entry of our matrix.

Each diagonal of a plane partition is an (ordinary) partition – a nonincreasing sequence of nonnegative integers with only finitely many nonzero elements. A *strict* partition is an ordinary partition with distinct positive elements. The example given above has all diagonals strict and so it is a *strict plane partition* – a plane partition whose diagonals are strict partitions.

For a plane partition π one defines the volume $|\pi|$ to be the sum of all entries of the corresponding matrix, and $k(\pi)$ to be the number of “white terraces” – connected components of white rhombi of the 3-dimensional representation of π .¹

For a real number q , $0 < q < 1$, we define a probability measure \mathfrak{M}_q on the set of strict plane partitions as follows. If π is a strict plane partition we set

$$\mathfrak{M}_q(\pi) \propto 2^{k(\pi)} q^{|\pi|}.$$

The normalization constant is given by the shifted MacMahon’s formula:

$$\sum_{\substack{\pi \text{ is a strict} \\ \text{plane partition}}} 2^{k(\pi)} q^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^n. \quad (1.1)$$

This formula has appeared very recently in [FW] and [V1]. This is a shifted version of the famous MacMahon’s formula:

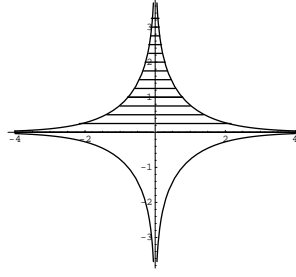
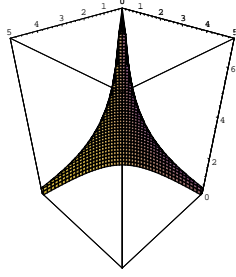
$$\sum_{\substack{\pi \text{ is a plane} \\ \text{partition}}} q^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n} \right)^n. \quad (1.2)$$

The limit shape of large strict plane partitions distributed according to \mathfrak{M}_q is shown in the figure on the left. It is parameterized on the domain representing a half of the amoeba² of the polynomial $P(z, w) = -1 + z + w + zw$ (the figure on the right).

*Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125. E-mail: vuletic@caltech.edu.

¹For the given example $|\pi| = 35$ and $k(\pi) = 7$.

²The amoeba of a polynomial $P(z, w)$ is $\{(\xi, \omega) = (\log |z|, \log |w|) \in \mathbb{R}^2 \mid (z, w) \in (\mathbb{C} \setminus \{0\})^2, P(z, w) = 0\}$.



2. THE SHIFTED SCHUR PROCESS

The measure described above is a special case of the *shifted Schur process*. This is a measure on sequences of strict (ordinary) partitions. We define it as an analogy with the *Schur process* introduced in [OR] that is a measure on sequences of (ordinary) partitions and as a generalization of the *shifted Schur measure* introduced by [TW] and [Mat].

The Schur process has been extensively studied in recent years and has various applications.

The shifted Schur process is defined using *symmetric Schur P* and *Q functions* that appear in the theory of projective representations of the symmetric groups.

Let Λ denote the algebra of symmetric functions. A *specialization* of Λ is an algebra homomorphism $\Lambda \rightarrow \mathbb{C}$. If ρ is a specialization and $f \in \Lambda$ then we use $f(\rho)$ to denote the image of f under ρ .

Let $\rho = (\rho_0^+, \rho_1^-, \rho_1^+, \dots, \rho_T^-)$ be a finite sequence of specializations. The shifted Schur process is a measure that to a sequence of strict partitions $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^T)$ assigns

$$\text{Prob}(\lambda) \propto \sum_{\mu} Q_{\lambda^1}(\rho_0^+) P_{\lambda^1/\mu^1}(\rho_1^-) Q_{\lambda^2/\mu^1}(\rho_1^+) \dots Q_{\lambda^T/\mu^{T-1}}(\rho_{T-1}^+) P_{\lambda^T}(\rho_T^-),$$

where the sum goes over all sequences of strict partitions $\mu = (\mu^1, \mu^2, \dots, \mu^{T-1})$.

Let $X = \{(x_i, t_i) : i = 1, \dots, n\} \subset \mathbb{N} \times [1, 2, \dots, T]$ and let $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^T)$ be a sequence of strict partitions. We say that $X \subset \lambda$ if x_i is a part (nonzero element) of the partition λ^{t_i} for every $i = 1, \dots, n$. The *correlation function* is defined by

$$\rho(X) = \text{Prob}(X \subset \lambda).$$

The shifted Schur process is a *Pfaffian process*, i.e. its correlation functions can be expressed as Pfaffians of a certain kernel:

Theorem 2.1. *Let $X \subset \mathbb{N} \times [1, 2, \dots, T]$ with $|X| = n$. The correlation function has the form*

$$\rho(X) = \text{Pf}(M_X)$$

where M_X is a skew-symmetric $2n \times 2n$ matrix

$$M_X(i, j) = \begin{cases} K_{x_i, x_j}(t_i, t_j) & 1 \leq i < j \leq n, \\ (-1)^{x_{j'}} K_{x_i, -x_{j'}}(t_i, t_{j'}) & 1 \leq i \leq n < j \leq 2n, \\ (-1)^{x_{i'} + x_{j'}} K_{-x_{i'}, -x_{j'}}(t_{i'}, t_{j'}) & n < i < j \leq 2n, \end{cases}$$

where $i' = 2n - i + 1$ and $K_{x,y}(t_i, t_j)$ is the coefficient of $z^x w^y$ in the formal power series expansion of

$$\frac{z-w}{2(z+w)} J(z, t_i) J(w, t_j)$$

in the region $|z| > |w|$ if $t_i \geq t_j$ and $|z| < |w|$ if $t_i < t_j$.

Here $J(z, t)$ is given with

$$J(t, z) = \prod_{t \leq m} F(\rho_m^-; z) \prod_{m \leq t-1} F(\rho_m^+; z^{-1}),$$

where $F(x; z) = \prod_i (1 + x_i z) / (1 - x_i z)$.

The proof relies on two tools. One is the Fock space associated to strict plane partitions and the other one is a Wick type formula that yields a Pfaffian.

We use the theorem above to obtain the correlation functions for \mathfrak{M}_q and to study the bulk scaling limit of the correlation functions when $q \rightarrow 1$. This allows us to obtain the limit shape.

These results can be found in [V1].

3. GENERALIZED MACMAHON'S FORMULA

In [V2] we generalize both formulas (1.1) and (1.2). Namely, we define a polynomial $A_\pi(t)$ that gives a generating formula for plane partitions of the form

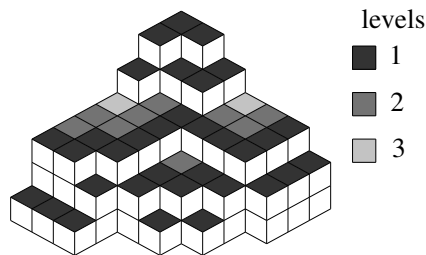
$$\sum_{\pi \in \mathcal{P}(r,c)} A_\pi(t) q^{|\pi|} = \prod_{i=1}^r \prod_{j=1}^c \frac{1 - tq^{i+j-1}}{1 - q^{i+j-1}}$$

with the property that $A_\pi(0) = 1$ and

$$A_\pi(-1) = \begin{cases} 2^{k(\pi)}, & \pi \text{ is a strict plane partition,} \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{P}(r, c)$ are plane partitions with at most r rows and c columns. We describe $A_\pi(t)$ below.

If a box (i, j) belongs to a connected component C then we define its *level* $h(i, j)$ as the smallest positive integer such that $(i + h, j + h)$ does not belong to C . A *border component* is a connected subset of a connected component where all boxes have the same level. Border components and their levels are shown in the figure below:



Let (n_1, n_2, \dots) be a sequence of nonnegative integers where n_i is the number of i -level border components of π . We set

$$A_\pi(t) = \prod_{i \geq 1} (1 - t^i)^{n_i}.$$

For the example above $A_\pi(t) = (1 - t)^{10}(1 - t^2)^3(1 - t^3)^2$.

REFERENCES

- [FW] O. Foda and M. Wheeler, *BKP Plane Partitions*; J. High Energy Phys. JHEP01(2007)075; arXiv: math-ph/0612018
- [Mat] S. Matsumoto, *Correlation functions of the shifted Schur measure*; J. Math. Soc. Japan 57 (2005), no. 3, 619–637.
- [OR] A. Okounkov and N. Reshetikhin, *Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram*; J. Amer. Math. Soc. 16 (2003), no. 3, 581–603.
- [TW] C. A. Tracy and H. Widom, *A limit theorem for shifted Schur measures*; Duke Math. J. 123 (2004), no. 1, 171–208
- [V1] M. Vuletić, *The Shifted Schur Process and Asymptotics of Large Random Strict Plane Partitions*; Int. Math. Res. Notices (2007), Vol 2007, article ID rnm043, 53 pages, arXiv: math-ph/0702068
- [V2] M. Vuletić, *A generalization of MacMahon's formula*; to appear in Transactions of Amer. Math. Soc., arXiv: math.CO/0707.0532

RESEARCH ABSTRACT—MONICA VAZIRANI

My research is in the fields of algebra and combinatorics, and my area of expertise is the representation theory of Hecke algebras. Hecke algebras arise naturally in many areas of mathematics and physics, such as quantum groups, quantum field theory, statistical mechanics, and knot theory. I study their irreducible representations (a.k.a. simple modules), which are the most basic objects whose symmetries are encoded in this algebra.

My research program is driven by extracting combinatorial structure from non-semisimple representation-theoretic phenomena.

I am interested in understanding the representation theory of affine, cyclotomic, and double affine Hecke algebras. These algebras can be thought of as deformations of the group algebra of the wreath product of a Coxeter group (e.g. the symmetric group) with a lattice. One of the primary methods of attack to understand the simple modules of Hecke algebras is to “glue” these basic objects together in such a way that their global controlling structure is apparent. For the affine Hecke algebra, the controlling structure is called a crystal graph, an object which comes from an entirely different area of representation theory, that of Lie theory, which is very central to modern mathematics. Such a rigid structure may exist for the double affine Hecke algebra, but has yet to be discovered, and this is the focus of my current and future research.

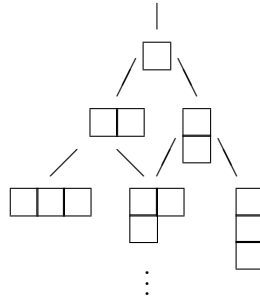
A “multiplicity-one” result (joint with Grojnowski) laid the foundation for later work of Grojnowski who realized the structure of an integrable highest weight representation of an affine Lie algebra on the representation category of the cyclotomic Hecke algebras and showed their crystal graphs controlled the behavior of the simple modules. It also generalized the multiplicity-free results of Kleshchev and Brundan-Kleshchev for the symmetric group in arbitrary characteristic. This work can be seen as giving a categorification of integrable highest weight modules. Related ideas have been further developed in the literature. One such example is the work of Chuang-Rouquier, who gave a class of examples where Broue’s conjecture holds.

This powerful viewpoint has led to a deeper understanding of the representation theory.

It is both surprising and beautiful that crystals appear in this Hecke-theoretic context naturally. A crystal graph is a combinatorial object that was created by Kashiwara to encode algebraic information of modules over quantized universal enveloping algebras. Their definition was motivated by mathematical physics in the context of the quantum spin chain in statistical mechanics. Representations of quantum algebras describe integrable systems (in particular the 2-dimensional lattice statistical system), and their crystal graphs reflect the simpler behavior of the model as temperature goes to absolute zero. The crystal graph of a representation has proved to be an amazing tool for studying it.

A crystal graph is a picture (a sort of skeleton) of a single irreducible representation of a quantum algebra or Lie algebra. The work of Grojnowski says it is also a picture of *every* irreducible representation of a whole family of Hecke algebras glued together. For the affine Hecke algebra H_n , that glue comes from the fact that the algebras are nested $H_n \subseteq H_{n+1}$, and so the functors of induction and restriction let us pass back and forth between their simple modules. This is also the case for the cyclotomic Hecke algebras. The double affine Hecke algebra (discussed below) does *not* have this property. One must search for a different structure.

The simplest degeneration of a cyclotomic Hecke algebra is just the group algebra of the symmetric group S_n . (S_n is the set of permutations of n objects.) The following picture encodes all of the simple modules of S_n for all n (only $0 \leq n \leq 3$ are drawn).



Each node of this graph stands for a simple module, and each edge gives information about the glue. Two things stand out. First, this graph is very easy to construct. Second, an expert will recognize this graph (with minor modifications) as a crystal graph—a picture of a single infinite-dimensional representation of an affine Lie algebra. While the second observation sounds more complicated, it is really this that makes the graph easy to draw. It turns out that for a general cyclotomic Hecke algebra, this graph would be incredibly hard to construct. But because the result is also a crystal graph, Lie theory gives us instructions of how to draw it.

Actually, for the symmetric group S_n over a field \mathbb{F} of characteristic 0, people have known how to draw the above graph for over 100 years. That is because $\mathbb{F}S_n$ is semisimple. In general, Hecke algebras are not semisimple, and it is only within the past 10 years we have recognized their corresponding graphs and begun to understand the information they contain. This has been the focus of several of my publications that exploit the connection between crystal graphs and the affine and cyclotomic Hecke algebras to better analyze the structure of the simple modules. This connection can also be used in the other direction. A certain representation-theoretic behavior of the affine Hecke algebra of type A inspired the observation that a corresponding behavior is mirrored in all highest weight crystals of classical type.

Despite the knowledge of this remarkable structure, there still remain many open questions about the simple modules of Hecke algebras, for instance, what are their dimensions? Because the algebras in question are *not* semisimple, this is an extremely difficult question and of key importance to representation theorists.

An algebra being semisimple is analogous to a given matrix being diagonalizable. The information we are trying to extract is akin to asking the Jordan normal form of a matrix knowing only its eigenvalues—an easy task for a diagonalizable matrix, but a trickier task otherwise. This is where the difficulty comes from, and where my results come in. Amazingly, for the Hecke algebra, despite nonsemisimplicity, we are able to extract quite a bit of information, and the combinatorics of how their simple modules fit together obeys the same rules as it does for S_n .

Such a rigid structure may exist for the double affine Hecke algebra, but has yet to be discovered, and this is the focus of my current and future research. In recent groundbreaking work, Cherednik introduced a structure called the double affine Hecke algebra (DAHA) and used its representation theory to study a class of orthogonal symmetric polynomials called Macdonald polynomials. The DAHA relates to the areas of special functions, conformal field theory, harmonic analysis of symmetric spaces, and hypergeometric functions.

In joint work with Suzuki, we parameterize and combinatorially construct a very large class of the DAHA's simple modules (avoiding the use of algebraic geometry and K-theory). My future plans include extending this study to a wider category of simple modules and, in particular, parameterizing them. While the notion of "gluing" does not generalize directly (this family of algebras is not nested), our results suggest there is a controlling structure; and many of the tools I developed to understand affine Hecke algebras do apply here, as affine Hecke algebras sit inside the DAHA. My research also uses the DAHA's connection to Macdonald polynomials to study them, as begun in joint work with Rains. We use the connection that Cherednik developed between the representation theory of the DAHA and Macdonald polynomials to prove certain vanishing conditions their integrals satisfy.

MSRI RESEARCH ABSTRACT

Stephanie van Willigenburg

Associate Professor, University of British Columbia, Vancouver, Canada

My area of expertise is algebraic combinatorics. It is a relatively new area of mathematics, having begun just over a century ago with the work of Schur and Frobenius on the representation theory of the symmetric group. Since then its scope and impact have broadened exponentially due to the following two aspects that essentially define the area. Firstly, it has been able to provide combinatorial interpretations to other areas of mathematics such as algebraic geometry (e.g. Schubert calculus), topology (e.g. Hochschild homology), algebra (e.g. cluster algebras), representation theory (e.g. classical Lie algebras) and physics (e.g. Fock spaces). These interpretations have often afforded a new perspective and understanding of the related problems and the solutions to them. Secondly, these connections to other areas admit proof techniques that have been utilised to solve classical combinatorial problems, for example geometry was used to prove Horn's conjecture, and algebraic geometry was used to prove the $n!$ conjecture.

Consequently, my research focuses on finding combinatorial results that impact on other areas and are derived using both combinatorial and algebraic tools, as my five most prominent papers will illustrate. Initially, my research concerned the family of algebras known as “descent algebras”, which arise naturally in a variety of contexts including the study of Hochschild homology by Bergeron (UQAM), planar binary trees by Loday (IRMA-Strasbourg), and sorting procedures in the guise of card shuffling by Diaconis (Stanford). One focus was on combinatorial “matrix interpretations” to describe the multiplicative structure of the descent algebra of Coxeter groups of type D . Such interpretations had been found for the Coxeter groups of type of A and B by, for example, Garsia (UCSD) and Bergeron-Bergeron (UQAM and CRC York). In **A multiplication rule for the descent algebra of type D** (with Nantel Bergeron) we exploited a framework I developed in my paper *A proof of Solomon's rule* and found such an interpretation for the descent algebra of the Coxeter groups of type D , thus concluding a search that had begun in the late 1980's.

From here my studies naturally led me to investigate the dual Hopf algebra of the descent algebra of the Coxeter groups of type A , known as the Hopf algebra of quasisymmetric functions. Quasisymmetric functions arise as weight enumerators of partitions of a partially ordered set (“poset”), via the work of Gessel (Brandeis) and Stembridge (Michigan, Ann-Arbor); in the Stanley symmetric functions of any type, developed by Stanley (MIT), Billey (U Washington) and Haiman (Berkeley); and as the flag f -vector of a graded poset, via the work of Ehrenborg (Kentucky). In **Non-commutative Pieri operators on posets** (with Nantel Bergeron, Stefan Mykytiuk and Frank Sottile) we naturally generalised Pieri's formula, which describes the action of complete symmetric functions on Schur functions, to an action of a Pieri operator on a given poset. Moreover, we found that depending on the choice of Pieri operator and poset, the result of the action often corresponded to specific well-known quasisymmetric functions including all those mentioned above.

In light of this discovery, a worthwhile avenue to pursue was to search for Pieri operators and posets whose related quasisymmetric functions form a subalgebra. We could then manipulate the

Hopf duality to forge further connections between areas, like the connection between the descent algebra of Coxeter groups of type A and quasisymmetric functions. One instance of success was the family of “rank selection” Pieri operators operating on Eulerian posets. In this case the related quasisymmetric functions lie in the peak algebra of Stembridge, and our theory identified the Hopf dual to be the algebra of chain operators restricted to Eulerian posets described by Billera (Cornell). In this way the study of enumerative properties of Eulerian posets, including associated geometric objects such as polytopes and hyperplane arrangements, became closely linked to questions on peaks and shuffles in permutations. For example, in **Peak quasisymmetric functions and Eulerian enumeration** (with Louis Billera and Samuel Hsiao) we discovered that the Hopf dual to the natural basis of the peak algebra is given by the **cd-index** of a poset, often studied by geometers. We also discovered that a specific map defined and studied by Stembridge could be viewed as giving a random walk on the peak sets of the symmetric group, whose stationary distribution is given by the distribution of peak sets.

Continuing to pursue the more geometric aspects of this discovery led to the study of a cone of fundamental quasisymmetric functions. We discovered that the extreme rays of this cone contained the symmetric functions known as Schur functions, and we conjectured that the facets of the cone were described by the closely related ribbon Schur functions. To make this conjecture more accessible, we found that a combinatorial condition for when two ribbon Schur functions are distinct, or equivalently when they are equal, was required. In **Decomposable compositions, symmetric quasisymmetric functions and equality of ribbon Schur functions** (with Louis Billera and Hugh Thomas) we discovered such a necessary and sufficient condition. Due to the ubiquity of ribbon Schur functions our result had the immediate impact of determining (1) when the coefficients of two fundamental quasisymmetric functions in a symmetric function are equal; (2) when the multisets of partitions determined by the refinement of two compositions are equal; (3) when cardinalities of sets of permutations whose descent sets satisfy certain natural criteria are equal.

However, it was the impact of ribbon Schur functions on the relationships between Littlewood-Richardson coefficients that was the most valuable. More precisely, Littlewood-Richardson coefficients arise in variety of areas of mathematics. The three most prominent occurrences are as the structure constants in the algebra of symmetric functions; as intersection numbers of Schubert cells in the cohomology of the Grassmanian; and in the representation theory of the symmetric group and $GL(n)$. Consequently knowledge of them and relations between them is of importance. Unfortunately, Narayanan (Chicago) recently showed that computing them is $\#P$ -complete, and finding relations between them has had limited success. However, via ribbon Schur function equality, infinitely many classes of equal Littlewood-Richardson coefficients were easily identified, and hope was given that a solution to the more general classical problem of skew Schur function equality might be found. In **Towards a combinatorial classification of skew Schur functions** (with Peter MacNamara) we did indeed generalise the sufficient condition of my previous paper and conjecture a closely related necessary combinatorial condition for when two skew Schur functions are equal. A natural avenue to pursue from here is to prove this combinatorial condition is both necessary and sufficient, thus solving a problem that has been considered intractable since the 19th century.

Jie Sun (University of Alberta)

Abstract: My current research interest is infinite dimensional Lie theory and algebraic geometry. My Ph.D. project is to construct central extensions of infinite dimensional Lie algebras by using the descent theory from modern algebraic geometry. So far I have given a natural construction of central extensions of twisted forms of split simple Lie algebras over rings. These types of algebras arise naturally in the construction of extended affine Lie algebras which are generalizations of the well known Kac-Moody algebras. The construction also gives information about the structure of the automorphism groups of such algebras. Most of this work has been published in the paper *Descent constructions for central extensions of infinite dimensional Lie algebras*, A. Pianzola, D. Prelat and J. Sun, *Manuscripta Math.* **122** (2) (2007), 137-148. The paper is also posted on arXiv ([arXiv:0711.3799](https://arxiv.org/abs/0711.3799)). In the next step, my research is to focus on determining when the above construction is universal. Continuing this work I will consider the representation theory of these algebras.

During my master study I have worked on representation theory of algebras. In my M.Sc. thesis (I did my M.Sc. at Beijing Normal University under the supervision of Dr. X. X. Ci), I introduced cyclotomic Temperley-Lieb algebras of type D . Such algebras have cyclotomic Temperley-Lieb algebras as a class of subalgebras. We prove that cyclotomic Temperley-Lieb algebras of type D are cellular by means of diagrammatic way. A cellular basis is given explicitly. After determining all the irreducible representations of these algebras, we give a necessary and sufficient condition for a cyclotomic Temperley-Lieb algebras of type D to be quasi hereditary. My poster presentation in this workshop is about this work.

Global to local bijections in blocks of finite groups of Lie type
Bhama Srinivasan

Let \mathbf{G} be a connected, reductive algebraic group over \mathbf{F}_q , $F : \mathbf{G} \rightarrow \mathbf{G}$ a Frobenius morphism and $G = \mathbf{G}^F$ the finite reductive group of F -fixed points of \mathbf{G} . Let ℓ be a prime not dividing q . We study the ℓ -blocks of G , in particular the unipotent blocks. If the defect group of a unipotent block B is abelian, Broué, Malle and Michel have described a bijection between the characters in B and the characters of $N_G(L)/L$ where L is a Levi subgroup of G . In joint work with Fong and Broué we consider a unipotent block B of G with a possibly non-abelian defect group. We partition B into Lusztig families and conjecture that the characters in each family are in bijection with a set of characters of a “local” subgroup of the form $N_G(L)$, where L is a Levi subgroup, possibly different for different families. The conjecture has been verified, with some restrictions on ℓ , for general linear groups and classical groups. The bijections that are constructed have some interesting arithmetic properties.

Britta Späth: **The McKay conjecture for finite simple groups of Lie type**

The McKay conjecture is an old open conjecture about global-local relations in the representation theory of finite groups. It states that for every finite group G the number of irreducible characters of ℓ' degree of G and $N_G(P)$ coincides, where P is a Sylow ℓ -subgroup of G and $N_G(P)$ the normalizer of P in G .

As the conjecture is well-known for ℓ -solvable groups, it seems natural to inspect the case where G is a finite quasi-simple group of Lie type. According to Cabanes and Malle the normalizer of a Sylow d -torus \mathbf{T} of G includes the normalizer of a Sylow ℓ -subgroup, whenever $\ell > 5$ and G is a reductive group of characteristic $\neq p$.

For the groups $L := C_G(\mathbf{T})$ and $N := N_G(\mathbf{T})$ the Clifford theory seems to be very special: Every irreducible character of L extends to its inertia group in N .

A Sylow 1-torus is a maximal split torus and in this case the proof is closely related to properties of the root system of \mathbf{G} . In other cases the structure of N , which depends on d , has to be analyzed in more detail. This property implies that $\text{Irr}_{\ell'}(N)$ can be parametrized by the same set, which labels according to Malle the set $\text{Irr}_{\ell'}(G)$.

Isaacs, Malle and Navarro have proven that the McKay conjecture is fulfilled for a finite group H and a prime ℓ if every simple non-abelian section of H with ℓ' order is good for ℓ . A simple non-abelian group X with $\ell \nmid |X|$ is called good if for a certain maximal extension G of X a group $M \neq G$ with $N_G(P) \leq M < G$ and a bijection between $\text{Irr}_{\ell'}(G)$ and $\text{Irr}_{\ell'}(M)$ fulfilling some additional equivariance properties exist

These additional properties are still open and are the subject of further research.

Anne Schilling

University of California, Davis

My main research interest lies in the interplay between algebraic combinatorics, representation theory and mathematical physics. In particular, I am currently involved in several projects:

- * The study of finite-dimensional affine crystals also known as Kirillov-Reshetikhin crystals.
- * Affine Schubert calculus, in particular the analogue of k -Schur functions for type C .
- * connection between the Hecke group algebra and the affine Hecke algebra.

My research is divided essentially into 3 areas.

I. PhD Work.

R. Brauer partitioned, the set of irreducible characters, $\text{Irr}(G)$, of the group G into ‘blocks’ associated to the p' -subgroups of G . I tried to do the same with some other kind of subgroups of G (nilpotent subgroups, subgroups with a central Sylow p -subgroup).

Papers:

- [1] G. Navarro, L. Sanus, *Partial characters with respect to a normal subgroup*, J. Austral. Math. Soc. Ser. A **66** (1999), no. 1, 104-124.
- [2] G. Navarro, L. Sanus, *A partition of characters associated to nilpotent subgroups*, Israel J. Math. **114** (1999), 359-380.
- [3] L. Sanus, *Inducing characters and nilpotent injectors*, Bull. Austral. Math. Soc. **62** (2000), no. 1, 155-158.
- [4] L. Sanus, *Blocks relative to a normal subgroup in π -separable groups*, J. Algebra **232** (2000), no. 1, 343-359.
- [5] G. Navarro, L. Sanus, *Characters induced from fully ramified subgroups*, Comm. Algebra **29** (2001), no. 4, 1829-1840.

II. Character Correspondences.

For a finite group G , the set of irreducible characters of G which have degree not divisible by p is denoted by $\text{Irr}_{p'}(G)$. When G is a solvable group of odd order, I. M. Isaacs ([Is]) constructed a natural one-to-one correspondence $*$: $\text{Irr}_{p'}(G) \rightarrow \text{Irr}_{p'}(N_G(P))$ which depends only on G and P , where P is a Sylow p -subgroup of G . Let $\xi^* \in \text{Irr}_{p'}(N_J(P))$ and $\chi^* \in \text{Irr}_{p'}(N_G(P))$ be the Isaacs correspondents of ξ and χ respectively. We proved in [6] that if $\xi^G = \chi$, then $(\xi^*)^{N_G(P)} = \chi^*$.

Suppose that p and q are distinct prime numbers. Let B be a p -block of G such that every irreducible character of B has q' -degree. Let D be a defect group of B . In [7] we proved that there exists a unique p -block B^* of $N_G(Q)$, for some $Q \in \text{Syl}_q(G)$, with defect group D such that $\text{Irr}(B^*) = \{\chi^* | \chi \in \text{Irr}(B)\}$. Moreover, there exists a perfect isometry R such that $R(\chi) = \chi^*$ for $\chi \in \text{Irr}(B)$. This result is analogous to theorems of Watanabe and Horimoto establishing that the Glauberman correspondence affords a perfect isometry between p -blocks under suitable hypotheses.

Papers:

- [6] L. Sanus, *Induction and character correspondences in groups of odd order*, J. Algebra **249** (2002), no. 1, 186-195.
- [7] L. Sanus, *Perfect isometries and the Isaacs correspondence*, Osaka J. Math. **40** (2003), no. 2, 313-326.

References:

- [Is] I. M. Isaacs, *Characters of solvable and symplectic groups*, Amer. J. Math. **95** (1973), 594-635.

III. Recent Work.

Suppose that G is a finite group, let q be a prime number. In [IMN], I. M. Isaacs, G. Malle and G. Navarro studied the existence of non-trivial real-valued irreducible character of degree not divisible by q . More recently, G. Navarro and P. H. Tiep ([NT]) have also studied whether these characters could even be taken to be rational-valued. In [12], we turn our attention to p -Brauer characters of G . In [10] we fix a prime p and study groups with exactly two irreducible real Brauer characters, or in other words (by Brauer’s lemma on character tables) to groups with exactly two p -regular real conjugacy classes. We start by analyzing the case where p is odd. Contrary to the ordinary case in [Iw], we shall need the classification of finite simple groups. We see that groups with exactly two real p -regular classes are solvable for p odd (as in the ordinary case). The case $p = 2$ offers us a surprise: they need not be solvable.

In [DNT], S. Dolfi, G. Navarro and P. H. Tiep gave a ‘real version’ for $p = 2$ of the celebrated Ito-Michler theorem on character degrees: ‘all irreducible real-valued characters of G have odd degree if and only if a Sylow 2-subgroup of G is normal of *Chillag-Mann* type’ (studied in [CM].) It turns out that the real character degrees contain useful non-trivial information on certain aspects of the structure of a group, and that some classical theorems on degrees of characters admit a ‘real version’ for the prime 2. For instance, in [14] we show our version of Thompson’s theorem on character degrees.

Next we turn our attention to groups with a small number of degrees of real characters. An already classical theorem asserts that groups with at most three character degrees are solvable. We also prove in [14] that groups with at most three degrees of real-valued characters are solvable

When some theorems on character degrees are proven, to some extent it is natural to consider the corresponding problems on conjugacy class sizes. If there is some explanation that justifies the mysterious relationship between these two worlds, it has never been found. Sometimes (but not always), this analogy leads to interesting theorems on finite groups. In [14] we study if the corresponding “conjugacy class version” is true. We show a theorem on real conjugacy class sizes (and its character analog) that extends (and supplements) results by D. Chillag and A. Mann ([CM]).

One of the main problems in representation theory is to detect local properties of a finite group from global, and the other way around. In [13] and, in the poster that we present in this workshop, we characterize when a Sylow normalizer of a finite solvable group has a normal Sylow 2-subgroup from a global point of view.

In [15] we describe the structure of finite groups whose real-valued non-linear irreducible characters have all prime degree. The more general situation in which the real-valued irreducible characters of a finite group have all squarefree degree is also considered. Recently we have obtained some results in the corresponding “conjugacy class version”.

Papers: [10] G. Navarro, L. Sanus, P. H. Tiep, *Groups with two real Brauer characters*, J. Algebra **307** (2007), no 2, 891-898.

[11] I. M. Isaacs, G. Navarro, L. Sanus, *Field of values of Fong characters*, Archiv, der Mathematik **86** (2007), no 4, 305-309.

[12] L. Sanus, *Field of values and Brauer characters of q' -degree*, Journal of Algebra **311** (2007) 817-820.

[13] G. Navarro; L. Sanus, L. *Sylow normalizers with a normal Sylow 2-subgroup*, Proceedings of the Edinburgh Mathematical Society (accepted).(* POSTER)

[14] G. Navarro, L. Sanus, P. H. Tiep, *Real character and degrees* (submitted).

[15] S. Dolfi, E. Pacifici, L. Sanus, *Finite groups with real valued irreducible characters of prime degree* (submitted).

[16] G. Navarro, L. Sanus, *Rationality and normal 2-complements* (submitted).

References:

[CM] D. Chillag and A. Mann, *Nearly odd-order and nearly real finite groups*, *Comm. Algebra* **26** (1998), 2041 - 2064.

[DNT] S. Dolfi, G. Navarro, and P. H. Tiep, *Primes dividing the degrees of the real characters of finite groups*, *Math. Z.* (to appear).

[IMN] Isaacs, I. M.; Malle, G.; Navarro, G. *Real characters of p' -degree*, J. Algebra **278** (2004), no. 2, 611–620.

[Iw] S. Iwasaki, *On finite groups with exactly two real conjugate classes*, Arch. Math. **33** (1979) 512517.

[NT] G. Navarro and P. H. Tiep Proc. Amer. Math. Soc. 134 (2006), no. 10, 2833-2837.

Lucia Sanus
Universitat de Valencia (SPAIN)
e-mail: lucia.sanus@uv.es

SOPHIE MORIER-GENOUD

T.H.Hildebrandt Assistant Professor
University of Michigan, Ann Arbor
sophiemg@umich.edu

RESEARCH SUMMARY

My research interests are in Representation Theory of semisimple complex Lie groups and semisimple complex Lie algebras. More precisely I am interested in combinatorial models related to the Lusztig-Kashiwara canonical basis.

This basis was constructed in the 1990's independently by Kashiwara and Lusztig in the quantum groups and became an important object in the study of representations, [9], [10]. Given a semisimple complex Lie algebras \mathfrak{g} and a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, one knows that an irreducible finite dimensional \mathfrak{g} -module can be generated by one element, called highest weight vector, under the action of \mathfrak{n}^- . We denote a \mathfrak{g} -module by $V(\lambda) = U(\mathfrak{n}^-)v_\lambda$, where $U(\mathfrak{n}^-)$ is the universal enveloping algebra of \mathfrak{n}^- and v_λ a highest weight vector of weight $\lambda \in \mathfrak{h}^*$. The canonical basis \mathcal{B} of $U(\mathfrak{n}^-)$ has remarkable properties of compatibility with the \mathfrak{g} -modules. Indeed, the subset $\mathcal{B}_\lambda := \{b \in \mathcal{B}, bv_\lambda \neq 0\}$ is a basis of $V(\lambda)$ for any λ .

The constructions of the set \mathcal{B} lead to two different systems of parametrizations of the elements of \mathcal{B} with tuples of integers. Both parametrizations depend on a sequence of indices $\mathbf{i} = (i_1, i_2, \dots, i_N)$ encoding reduced decomposition of the element of maximal length, w_0 , in the Weyl group associated to \mathfrak{g} . One system is known as *Lusztig's parametrization*. The Lusztig parameters $b_i(b) = (l_1, l_2, \dots, l_N) \in \mathbb{Z}_{\geq 0}^N$ of an element b are given as the exponents of the monomial in a Poincaré-Birkhoff-Witt basis ordered by \mathbf{i} congruent to b modulo certain conditions. The other parametrization is known as *string parametrization*. The string parameters $c_i(b) = (l'_1, l'_2, \dots, l'_N) \in \mathbb{Z}_{\geq 0}^N$ of b , are recursively defined by applying a sequence of Kashiwara operators $\tilde{e}_{i_k}, 1 \leq k \leq N$. In the case $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, the combinatorics given by these parametrizations for the standard reduced word $(1, 2, 1, \dots, n, n-1, \dots, 1)$ coincide with the classical combinatorics of semistandard Young tableaux.

From algebro-geometric constructions of Lusztig, links between the canonical basis and the geometry of the Lie group G^\vee , Langlands dual of \mathfrak{g} , have been established. Berenstein and Zelevinsky, [4], introduced positive varieties in G^\vee whose parametrizations (that also depend on a sequence \mathbf{i}) are related to the parametrizations of the canonical basis. Formulas of reparametrizations concerning the varieties and the canonical basis are the same under a transformation map called *tropicalization*. This map consists in changing the operations \times to $+$, \div to $-$ and $+$ to \min .

MAIN RESULTS :

In my thesis I studied the generalized Schützenberger involution. This involution coincides (up to a scalar) with the action of w_0 on the modules $V(\lambda)$. Let us denote this involution by S_λ . Lusztig showed that S_λ acts by permutation on the set \mathcal{B}_λ . My problem was to find explicit formulas for the Lusztig/string parameters of the element $b' = S_\lambda(b)$ in terms of Lusztig/string parameters of b .

In the case $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, Berenstein and Zelevinsky, [3], obtained the explicit formula for the particular reduced decomposition $\mathbf{i} = (1, 2, 1, \dots, n, n-1, \dots, 1)$ and showed that in terms of Young tableaux S_λ corresponds to the Schützenberger evacuation algorithm.

I obtained general formulas for any \mathfrak{g} and any reduced word \mathbf{i} , [11]. The key was to define a geometric analog of S_λ in the positive subvarieties of G^\vee and to use results about tropicalisation. It turns out that the formulas mixing the two systems of parametrizations, *e.g* the formula for $c_i(S_\lambda(b)) = (l'_1, \dots, l'_N)$ in terms of $b_i(b) = (l_1, \dots, l_N)$, have nice affine expressions with linear part independent of λ and constant term given by the parameters of the lowest weight element.

These results enabled us to establish a link between Caldero's adapted algebras and Lusztig cones, [7]. Adapted algebras are generated by subsets of \mathcal{B} stable under multiplication. Caldero gave a construction of *standard* adapted algebras A_i , associated to any reduced decomposition of w_0 . Lusztig cones were introduced to give a condition on the exponents of a Poincaré-Birkhoff-Witt monomial so that the monomial belongs to \mathcal{B} . In [7], we realized a Lusztig cone as the set of string parameters of elements in a twisted adapted algebra $\phi(A_i^\sigma)$ (ϕ corresponds to the Schützenberger involution).

I also used this formula to construct toric degenerations of Richardson varieties, [12]. Caldero, [6], gave a method to construct toric degenerations of flag varieties and Schubert varieties. The method involves the combinatorics of the canonical basis. The toric varieties obtained in the degenerations are encoded by lattice polytopes in \mathbb{N}^N given as sets of string parameters. I reused the method in the case of opposite Schubert varieties (that are images of regular Schubert varieties under the action of w_0) and Richardson varieties (that are intersections of Schubert varieties and opposite Schubert varieties). It required to understand how change the string parameters under the action of w_0 , what I could do knowing explicit formula for the generalized Schützenberger involution.

CURRENT RESEARCH :

One of my current research interest are the Mirkovic-Vilonen cycles. I am particularly interested in their descriptions involving the parametrizations of the canonical basis and in the crystal structures associated to them, [5], [8], [2].

A second current interest concerns the use of toric degenerations in mirror symmetry problems. Toric degenerations were used by Batyrev and al. [1] to construct mirror varieties of some Calabi-Yau hypersurfaces in flag varieties. Their constructions hold in the case of $SL_n(\mathbb{C})$ and involve very specific combinatorial models in type A. I am generalizing some of their results in other types by using more general combinatorics.

RÉFÉRENCES

- [1] V.V. Batyrev, I. Ciocan-Fontanine, B.Kim, D. van Straten. Mirror symmetry and toric degenerations of partial flag manifolds. *Acta. Math.* 184 (2000),1-39
- [2] P. Baumann, S. Gaussent. On Mirkovic-Vilonen cycles and crystals combinatorics. préprint arXiv :math.RT/0606711
- [3] A. Berenstein, A. Zelevinski. Canonical bases for the quantum group of type A_r , and piecewise-linear combinatorics. *Duke Math.*, 143 (1996), 473–502.
- [4] A. Berenstein and A. Zelevinsky. Tensor product multiplicities, Canonical bases and Totally positive varieties. *Invent. Math.*, 143 (2001), 77–128.
- [5] A. Braverman and D. Gaitsgory, Crystals via the affine Grassmannian, *Duke Math. J.* **107** (2001), 561–575.
- [6] P. Caldero. Toric degenerations of Schubert varieties. *Transf. Groups*, Vol. 7, No 1, 51-60, 2002.
- [7] P. Caldero, R. Marsh, S. Morier-Genoud. Realisation of Lusztig cones. *Represent. Theory* 8 (2004), 458-478.
- [8] J. Kamnitzer, The crystal structure on the set of Mirković-Vilonen polytopes, préprint arXiv :math.QA/0505398.
- [9] M. Kashiwara. On Crystal Bases. *Canad. Math. Soc., Conference Proceed.*, 16, 155-195, 1995.
- [10] G. Lusztig. Introduction to quantum groups. *Progress in Mathematics*, 110, Birkhäuser, 1993.
- [11] S. Morier-Genoud. Relèvement géométrique de la base canonique et involution de Schützenberger. *C.R. Acad. Sci. Paris. Ser. I*337, 371-374, 2003.
- [12] S. Morier-Genoud, Geometric lifting of the canonical basis and semitoric degenerations of Richardson varieties. *Trans. Amer. Math. Soc.* 360 (2008) 215-235.

RESEARCH INTERESTS AND ACCOMPLISHMENTS**Part I. Research Interests**

Algebraic and enumerative combinatorics.

Representation theory of finite-dimensional algebras.

Algebraic geometry.

Part II. Most significant contributions to research and development

[1] Nantel Bergeron, Thomas Lam and **Huilan Li** (2007) Combinatorial Hopf Algebras and Towers of Algebras, preprint (arXiv:0710.3744).

[2] Nantel Bergeron and **Huilan Li** (2006) Algebraic Structures of Grothendieck Groups of a Tower of Algebras, submitted (arXiv:math.RA/0612170, Ph.D.'s work).

[3] **Huilan Li** (2003) Δ -Directing Modules. Journal of Beijing Normal University (Natural Science), June 2003, Vol.39 No.3. 285- 290 (In Chinese, Master's work).

The paper [1] is concerned with the interplay between towers of associative algebras, pairs of dual combinatorial Hopf algebras, and dual graded graphs. Bergeron and I have introduced a set of axioms which guarantee that the Grothendieck groups of a tower of algebras $\bigoplus_{n \geq 0} A_n$ can be endowed with the structure of graded dual Hopf algebras. Hivert and Nzeutzhap, and independently Lam and Shimozono constructed dual graded graphs from primitive elements in Hopf algebras. In this paper we apply the composition of these constructions to towers of algebras. We show that if a tower $\bigoplus_{n \geq 0} A_n$ gives rise to graded dual Hopf algebras then we must have $\dim(A_n) = r^n n!$ where $r = \dim(A_1)$. This rigidity suggests that there may be a structure

theorem for towers of algebras which give rise to combinatorial Hopf algebras. In particular, it suggests that one should study algebras related to symmetric groups (or wreath products of symmetric groups).

In [2], we found a list of axioms on a tower of algebras which imply that their Grothendieck groups are graded Hopf algebras and dual to each other. We also checked some well-known towers of algebras satisfying these axioms. No formal study of this kind has been done so far. Up to this point it was not clear what were the right conditions to impose on a tower of algebras to get the desired algebraic structure on their Grothendieck groups. Moreover, there is an interesting result on it own. Given a self-injective algebra H , it gives a necessary and sufficient condition for a principal left ideal $H\nu$ to be isomorphic to a projective left ideal Hg .

Δ -directing modules play an important role in representation theory. In [3], the Δ -directing modules are studied by using quadratic form and the structure of Aslander-Reiten quiver of $\mathcal{F}(\Delta)$ the category of Δ -good modules of a quasi-hereditary algebra. It is shown that if every indecomposable module in $\mathcal{F}(\Delta)$ is Δ -directing then $\mathcal{F}(\Delta)$ is finite. Although this result can be derived from some paper, here the proof in [2] is much simpler. Moreover, it is shown that if $\mathcal{F}(\Delta)$ is finite then that all the indecomposable modules are Δ -directing if and only if all the indecomposable projective modules are Δ -directing.

A Very Brief and Personal¹ Abstract of Research

Submitted for the MSRI Workshop
“Connections for Women: Introduction to the Spring 2008 Programs”

Terrell L. Hodge

Department of Mathematics, Western Michigan University
Kalamazoo, MI 49008-5248 terrell.hodge@wmich.edu

Date: January 14, 2008

Since graduating with my Ph. D. in 1999 from the University of Virginia under the direction of Brian Parshall, my work has, until recently, been focused on two areas:

- (1) modular representations of algebraic groups
- (2) theory of Lie triple systems and their representations

These two topics were tied together by my dissertation research (see e.g., [3], [4]), which explored a modular version of Harish-Chandra modules for real Lie groups, in the form of (\mathfrak{g}, K) -modules for \mathfrak{g} the Lie algebra of a suitable algebraic group G over $k = \overline{\mathbb{F}}_p$, $p > 2$, and $K = G^\theta$ the fixed-point subgroup of an involution θ on G . In this setting, Lie triple systems (structures akin to Lie algebras, but with a ternary, rather than binary operation) arise naturally as tangent spaces \mathfrak{p} to the ‘symmetric space’ G/K , equivalently, as the -1 -eigenspaces of the differential of θ on \mathfrak{g} . I gave a classification for finite-dimensional irreducible ‘restricted’ (\mathfrak{g}, K) -modules (e.g., KG_1 -modules) akin to the known classification for irreducible finite-dimensional BG_1 modules for B a Borel subgroup of G and G_1 the Frobenius kernel. This work also introduced and studied the notion of a restricted Lie triple system, akin to a restricted Lie algebra.

In later joint work with Brian Parshall [5], we delved further into homological properties of Lie triple system modules (restricted and non-restricted) and laid out a new categorical framework for comparing module categories for Lie triple systems and related Lie and associative algebras. A significant, as yet unrealized aim, was to find a cohomological interpretation for the restricted nullcone $\mathcal{N}_1(\mathfrak{p})$ inside the full nullcone $\mathcal{N}(\mathfrak{p})$ of a restricted Lie triple system; these varieties arise via intersection with \mathfrak{p} by the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$ and the full nullcone $\mathcal{N}(\mathfrak{g})$ of \mathfrak{g} , where it is known, e.g., that the variety $\mathcal{N}_1(\mathfrak{g})$ is the spectrum $\text{Spec}(H_{res}^{2\bullet}(\mathfrak{g}, k))$ for the ring of restricted cohomology for the restricted Lie algebra \mathfrak{g} . The related geometric theory of the nilpotent orbits of G on $\mathcal{N}(\mathfrak{g})$ (generalizing, e.g., the classification of nilpotent $n \times n$ matrices $M_n(k)$ by their Jordan canonical forms) is a very rich subject with a long history. Related to this study, my Ph. D. student, Joe Fox (now at Salem State near Boston), created a Bala-Carter type classification for the naturally defined nilpotent K -orbits $\mathcal{N}(\mathfrak{p})/K$ when p is a good prime [2], following ideas suggested by Premet’s non-computational proof

¹That is, for the sake of brevity, key references to all but the most self-promotional papers, i.e., my own, are missing!

of the Bala-Carter classification of the nilpotent G -orbits $\mathcal{N}(\mathfrak{g})/G$ for good primes. For the special cases of orbits $\mathcal{N}(\mathfrak{p})/K$ arising from involutions on classical groups G , Brian, Joe, and I have given a new proof of a ‘partition-type’ classification of these orbits, valid for both char. 0 and char. p , that relies mostly on a little linear algebra with just a bit of algebraic group theory. We consider also some applications, e.g., to the Zariski partial order on orbits $\mathcal{N}(\mathfrak{p})/K$ and to the structure of $\mathcal{N}_1(\mathfrak{p})$. In a related vein, David Murphy (now at Hillsdale College in MI) and I have looked at some questions associated to desingularizations of nilpotent orbit closures for $\mathcal{N}(\mathfrak{p})/K$, where the most well-known analogue in the case of $\mathcal{N}(\mathfrak{g})/G$ is Springer’s resolution of $\mathcal{N}(\mathfrak{g})$.

On sabbatical last year I began work on several projects with closer ties to (1) the representation theory of quantum groups and finite groups, and to (2) combinatorial representation theory, both key topics at the MSRI workshops. In the former case (1), for example, last June I co-directed an AIM Workshop on computational methods related to the modular representation theory and cohomology of finite groups [1], and began a project with UVA’s Leonard Scott and Paramasamy Karapuchamy to give new proofs of some fundamental results on quantum group representations in work of Arkhipov, Bezrukavnikov, and Ginzburg relating quantum groups, the loop Grassmannian, and the Springer resolution. In the latter case (2), connections take the form of (i) some current work with David Murphy on generalized RSK-type algorithms, and (ii) investigations into algebraic statistics, a subject which is one focal point for an NSF-CCLI award [6] I recently received (with co-PI and mathematician Raina Robeva of Sweet Briar College and two biologists from our respective institutions) to produce materials for a broad spectrum of mathematics courses that will integrate applications of algebraic statistics, algebraic geometry, and other topics in modern discrete mathematics to molecular biology. To approach this material not only from a pedagogical perspective, but from a research one as well, is one goal, as I am interested in potential applications of representation theory to molecular biology via algebraic statistics and combinatorial representation theory.

REFERENCES

- [1] C. Bendel, T. Hodge, B. Parshall, and C. Pillen, “Cohomology and representation theory for finite groups of Lie type”, AIM Workshop, June 25-29, 2007. <http://www.aimath.org/pastworkshops/finiteliegps.html>
- [2] J. Fox, A classification of nilpotent orbits in infinitesimal symmetric spaces (preprint).
- [3] T. L. Hodge, Lie triple systems, restricted Lie triple systems, and algebraic groups, *J. Algebra* **244** (2001), 533-580.
- [4] T. L. Hodge, Representations associated to involutions on algebraic groups and modular Harish-Chandra modules, *J. Pure Appl. Alg.* **198** (2005), 197-223.
- [5] T. L. Hodge and B. J. Parshall, On the representation theory of Lie triple systems, *Trans. Amer. Math. Soc.* **354** (2002), 4359-4391.
- [6] “Modern Biology, Modern Mathematics, and Modern Solutions: Moving Biomathematics Education Beyond Calculus,” NSF-CCLI grant DUE-0737467, is under the direction of Raina S. Robeva, Terrell L. Hodge, Alexander J. Enyedi, and Robin L. Davies.

APPLICATIONS OF TOPOLOGICAL COMBINATORICS TO COMBINATORIAL REPRESENTATION THEORY

PATRICIA HERSH

I am primarily interested in topological combinatorics, but recently have begun working on applications to combinatorial representation theory. One very recent such endeavor is described below. Specifically, included below is the introduction of a paper involving both areas

1. INTRODUCTION AND TERMINOLOGY

This paper gives the following new characterization of which finite CW complexes are regular, followed by the proof of a conjecture of Fomin and Shapiro from [FS] regarding stratified, totally positive spaces that model Bruhat intervals.

Theorem 1.1. *Let K be a finite CW complex with characteristic maps $f_\alpha : B^{\dim \alpha} \rightarrow \overline{e_\alpha}$. Then K is regular with respect to the characteristic maps $\{f_\alpha\}$ if and only if the following conditions hold:*

- (1) *For each α , $f_\alpha(S^{\dim \alpha - 1})$ is a union of open cells.*
- (2) *For each f_α , the preimages of open cells of dimension exactly one less than e_α form a dense subset of the boundary of $B^{\dim \alpha}$.*
- (3) *The closure poset of K is thin. Additionally, each open interval (u, v) with $rk(v) - rk(u) > 2$ is connected.*
- (4) *For each α , the restriction of f_α to the preimages of the open cells of dimension exactly one less than e_α is an injective map.*
- (5) *For each $e_\sigma \subseteq \overline{e_\alpha}$, f_σ factors as a continuous inclusion map $\iota : B^{\dim \sigma} \rightarrow B^{\dim \alpha}$ followed by f_α .*

Condition 2 implies that the closure poset is graded by cell dimension. Section 2 gives examples demonstrating that each of conditions 2, 3, 4, and 5 is not redundant, then proves Theorem 1.1. The fairly technical conditions of Theorem 1.1 seem to capture how the combinatorics (encoded in condition 3) substantially reduces what one must check topologically. Notably absent is the requirement that f_α is bijective between the entire boundary of $B^{\dim \alpha}$ and a union of open cells.

Björner proved in [Bj] that any finite poset which has a unique minimal element and is thin and shellable (i.e. stronger conditions than condition 3 above) is the closure poset of a finite regular CW complex.

However, this by no means guarantees that any particular CW complex with this closure poset will be regular. One goal of this paper is to explore how the combinatorial data of the closure poset may be used in conjunction with limited topological information (namely information about the codimension one cell incidences) to prove that a CW complex is regular; this in turn enables determination of its homeomorphism type directly from the combinatorics of its closure poset.

Björner asked in [Bj] for a naturally arising family of regular CW complexes whose closure posets are the intervals of Bruhat order, since Bruhat order was proven to be thin and shellable in [BW]. To this end, Fomin and Shapiro introduced stratifications of links of open cells within bigger closed cells, all within the totally positive part of the unipotent radical of a semisimple, simply connected algebraic group. In [FS], they showed these had the Bruhat intervals as their closure posets and proved quite a bit about their topological structure (especially in type A). They also conjectured that these decompositions were regular CW decompositions. In Section 3, we prove this conjecture:

Theorem 1.2. *These combinatorial decompositions from [FS] are regular CW decompositions, implying the spaces are homeomorphic to balls.*

A simple consequence of the exchange axiom for Coxeter groups will allow us to confirm condition 4 of Theorem 1.1, using an argument that cannot possibly generalize to higher codimension cell incidences.

Definition 1.3. A *CW complex* is a space X and a collection of disjoint open cells e_α whose union is X such that:

- (1) X is Hausdorff.
- (2) For each open m -cell e_α of the collection, there exists a continuous map $f_\alpha : B^m \rightarrow X$ that maps the interior of B^m homeomorphically onto e_α and carries the boundary of B^m into a finite union of open cells, each of dimension less than m .
- (3) A set A is closed in X if $A \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each α .

An *open cell* is any space which is homeomorphic to the interior of a ball. Denote the closure of a cell α by $\overline{\alpha}$. A *finite CW complex* is a CW complex with finitely many open cells. A CW complex is *regular* if additionally each of the maps f_α restricts to a homeomorphism from the boundary of B^m onto a finite union of lower dimensional open cells.

The *closure poset* of a finite CW complex is the partially ordered set (or poset) of open cells with $\sigma \leq \tau$ iff $\sigma \subseteq \overline{\tau}$. By convention, we adjoin a unique minimal element $\hat{0}$ which is covered by all the 0-cells. Let $[\sigma, \tau]$ denote the subposet consisting of elements z such that $\sigma \leq z \leq \tau$, called the *closed interval* from σ to τ . Likewise, the *open interval* from

σ to τ , denoted (σ, τ) , consists of those z with $\sigma < z < \tau$. A cell σ covers a cell ρ , denoted $\rho \prec \sigma$, if $\rho < \sigma$ and $\rho \leq z \leq \sigma$ implies $z = \rho$ or $z = \sigma$. A finite, graded poset is *thin* if each rank two closed interval $[u, v]$ has exactly four elements. The *order complex* of a finite partially set is the simplicial complex whose i -dimensional faces are the chains $u_0 < \cdots < u_i$ of $i+1$ comparable poset elements. The order complex of the closure poset of a finite regular CW complex K (with $\hat{0}$ removed) is the first barycentric subdivision of K , hence is homeomorphic to K . In particular, this implies that the order complex for any open interval (u, v) will be homeomorphic to a sphere $S^{rk(v)-rk(u)-2}$.

Remark 1.4. Lusztig and Rietsch have also introduced a combinatorial decomposition for the totally positive part of a flag variety (cf. [Lu] and [Ri]). Lauren Williams conjectured in [Wi] that this is a regular CW complex. It seems quite plausible that Theorem 1.1 could also be a useful ingredient for proving that conjecture.

Rietsch determined the closure poset of this decomposition in [Ri]. Williams proved in [Wi] that this poset is shellable and thin, hence meets condition 3 of Theorem 1.1. Recently, Postnikov, Speyer and Williams proved in [PSW] for the special case of the Grassmannian that its decomposition is a CW decomposition; Rietsch and Williams subsequently generalized this to all flag varieties in [RW]. In each case, it remains open whether these CW complexes are regular.

REFERENCES

- [Bj] A. Björner, Posets, Regular CW complexes and Bruhat order, *European J. Combin.* **5** (1984), no. 1, 7–16.
- [BW] A. Björner and M. Wachs, Bruhat order of Coxeter groups and shellability, *Adv. Math.* **43** (1982), no. 1, 87–100.
- [FS] S. Fomin and M. Shapiro, Stratified spaces formed by totally positive varieties. Dedicated to William Fulton on the occasion of his 60th birthday. *Michigan Math. J.* **48** (2000), 253–270.
- [Lu] G. Lusztig, Total positivity in reductive groups, in: *Lie theory and geometry: in honor of Bertram Kostant*, *Progress in Mathematics* **123**, Birkhäuser, 1994.
- [PSW] A. Postnikov, D. Speyer and L. Williams, Matching polytopes, toric geometry and the nonnegative part of the Grassmannian, preprint.
- [Ri] K. Rietsch, Closure relations for totally nonnegative cells in G/P , *Math. Res. Lett.* **13** (2006), no. 5-6, 775–786.
- [RW] K. Rietsch and L. Williams, in preparation.
- [Wi] L. Williams, Shelling totally nonnegative flag varieties, *J. Reine Angew. Math.*, Volume 2007, Issue 609 (2007), 1–22.

DEPT. OF MATHEMATICS, RAWLES HALL, INDIANA UNIVERSITY-BLOOMINGTON,
BLOOMINGTON, IN 47405

E-mail address: phersh@indiana.edu

IMAGINARY WHITTAKER MODULES FOR AFFINE LIE ALGEBRAS AND REALIZATIONS OF n -POINT AFFINE LIE ALGEBRAS

KONSTANTINA CHRISTODOULOPOULOU
UNIVERSITY OF WINDSOR, CANADA
dinachris@gmail.com

Block '81: The simple complex \mathfrak{sl}_2 -modules fall into three families:

- Highest (or lowest) weight modules.
- Whittaker modules.

• Modules obtained by localization
Let \mathfrak{g} be a Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and let $\eta : \mathfrak{n}_+ \rightarrow \mathbb{C}$ be a **non-zero** Lie algebra homomorphism. A \mathfrak{g} -module V is called a **Whittaker module of type η** if V is generated by an eigenvector for \mathfrak{n}_+ with eigenvalues given by η . The excluded case $\eta|_{\mathfrak{n}_+} = 0$ leads to the well-known highest weight modules.

- B. Kostant (1978) classified the Whittaker modules for semisimple Lie algebras.
- E. McDowell, D. Miličić and W. Soergel studied a category of modules for semisimple Lie algebras which includes \mathcal{O} and Whittaker modules.
- Generalized Whittaker modules have connections with finite W -algebras in the work of A. Premet, J. Brundan, A. Kleshchev.
- M. Ondrus classified the Whittaker modules for $\mathcal{U}_q(\mathfrak{sl}_2)$.

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be an affine Lie algebra. A Lie subalgebra \mathfrak{p} is called **parabolic** if $\mathfrak{h} \subset \mathfrak{p}$ and $\mathfrak{p} + \sigma(\mathfrak{p}) = \mathfrak{g}$, where $\sigma : \mathfrak{n}_+ \rightarrow \mathfrak{n}_-$ is the Chevalley involution.

V. Futorny classified the parabolic subalgebras of affine Lie algebras and showed that every parabolic

subalgebra \mathfrak{p} contains either a **reductive Lie algebra** or an **infinite-dimensional Heisenberg algebra**. If $\mathfrak{p} \supset \mathfrak{h} \oplus \mathfrak{n}_+$, then \mathfrak{p} is called **standard**.

Whittaker modules induced from standard parabolic subalgebras

Let $\eta : \mathfrak{n}_+ \rightarrow \mathbb{C}$ be a **non-zero** Lie algebra homomorphism such that η is zero on at least one of the generators of \mathfrak{n}_+ .

- In [1], I associate to η a standard parabolic subalgebra \mathfrak{p} of \mathfrak{g} and I construct **Whittaker modules** $M(W)$ of type η for \mathfrak{g} by inducing over $\mathcal{U}(\mathfrak{p})$ starting from simple Whittaker modules W for the reductive subalgebra of \mathfrak{p} . I show that $M(W)$ has a unique simple quotient $L(W)$.

The following result classifies certain Whittaker modules for $\widehat{\mathfrak{sl}}_2 = (\mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$:

Theorem 1 [1] *Let V be an simple Whittaker module of type η for $\widehat{\mathfrak{sl}}_2$. Then there exists a simple Whittaker module W for the reductive subalgebra of \mathfrak{p} such that $V \cong L(W)$.*

- In [1], I also establish an irreducibility criterion for the modules $M(W)$ for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$.

Imaginary Whittaker modules for affine Lie algebras:

Let $\mathfrak{g} = (\hat{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$ be a non-twisted affine Lie algebra where

- $\hat{\mathfrak{g}}$ is a simple Lie algebra.
- c is central.
- d is a derivation.

Fix $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+$ a triangular decomposition, where $\hat{\mathfrak{h}}$ is a Cartan subalgebra of $\hat{\mathfrak{g}}$. The **homogeneous Heisenberg subalgebra** of \mathfrak{g} is $\mathfrak{t} = (\hat{\mathfrak{h}} \otimes t^{-1}\mathbb{C}[t^{-1}]) \oplus \mathbb{C}c \oplus (\hat{\mathfrak{h}} \otimes t\mathbb{C}[t])$. Set $\tilde{\mathfrak{t}} = \mathfrak{t} \oplus \mathbb{C}d$.

- In [2] I classify the simple Whittaker modules for \mathfrak{t} and $\tilde{\mathfrak{t}}$.

The subalgebra $\mathfrak{p} = \tilde{\mathfrak{t}} \oplus (\hat{\mathfrak{h}} \otimes 1) \oplus (\hat{\mathfrak{n}}_+ \otimes \mathbb{C}[t, t^{-1}])$ is a parabolic subalgebra of \mathfrak{g} which contains the **imaginary Borel subalgebra** $\mathfrak{t}^+ \oplus (\hat{\mathfrak{h}} \otimes 1) \oplus \mathbb{C}c \oplus \mathbb{C}d \oplus (\hat{\mathfrak{n}}_+ \otimes \mathbb{C}[t, t^{-1}])$, where $\mathfrak{t}^+ = \hat{\mathfrak{h}} \otimes t\mathbb{C}[t]$.

- V. Futorny constructed a new class of simple weight \mathfrak{g} -modules by inducing over $\mathcal{U}(\mathfrak{p})$ starting from simple *highest weight modules of non-zero level* for $\tilde{\mathfrak{t}}$. He called these \mathfrak{g} -modules **imaginary Verma modules**.
- V. Chari constructed the class of level zero simple integrable weight \mathfrak{g} -modules with finite-dimensional weight spaces by inducing over $\mathcal{U}(\mathfrak{p})$ starting from simple \mathbb{Z} -graded \mathfrak{t} -modules of zero level.

Let $\lambda \in ((\hat{\mathfrak{h}} \otimes 1) \oplus \mathbb{C}c)^*$ and let $\eta : \mathfrak{t}^+ \rightarrow \mathbb{C}$ be a Lie algebra homomor-

phism such that $\eta|_{\hat{\mathfrak{h}} \otimes t^i} \neq 0$ for infinitely many $i \in \mathbb{Z}_{>0}$ if $\lambda(c) \neq 0$.

- In [2] I construct a new class of \mathfrak{g} -modules $V_{\eta, \lambda}$ by inducing over $\mathcal{U}(\mathfrak{p})$ starting from *simple Whittaker \mathfrak{t} -modules of type η* . I call these modules **imaginary Whittaker modules**. $V_{\eta, \lambda}$ is generated by an **eigenvector** for the root vectors corresponding to the **positive imaginary roots** with **eigenvalues given by η** .

Theorem 2 [2]

- (i) If $\lambda(c) \neq 0$, then $V_{\eta, \lambda}$ is simple as a $\mathcal{U}(\mathfrak{g})$ -module.
- (ii) If $\lambda(c) = 0$, then $V_{\eta, \lambda}$ has a unique simple quotient.

Part (i) is an analogue of a similar result for imaginary Verma modules due to V. Futorny.

Future Directions

- Describe the simple Whittaker modules for η non-zero on all generators of \mathfrak{n}_+ in the affine case and thus completely classify all simple Whittaker modules for \mathfrak{sl}_2 .
- Examine analogous constructions of Whittaker type modules for quantum enveloping algebras.
- Describe the submodule structure and the simple quotients of imaginary Whittaker modules of zero level.
- Construct analogues of imaginary Whittaker modules for twisted affine Lie algebras and extended affine Lie algebras.

n -point affine Lie algebras

Let $\hat{\mathfrak{g}}$ be a simple Lie algebra. M. Bremner showed that the **universal central extension** of the loop algebra $\hat{\mathfrak{g}} \otimes \mathbb{C}[t, (t - a_1)^{-1}, \dots, (t - a_{n-1})^{-1}]$, where $a_i \in \mathbb{C}$, has a $n - 1$ -dimensional center. The Lie algebras obtained in this way are called **n -point affine Lie algebras**. In ongoing joint work with Michael Lau, we are studying **bosonic** and **fermionic** representations for n -point affine Lie algebras. B. Cox recently obtained bosonic realizations of the four-point affine \mathfrak{sl}_2 Lie algebra.

- We have constructed a fermionic realization of this Lie algebra and in the future we hope to generalize our construction to the n -point case. We are also interested in studying connections with **vertex operators** and **combinatorial identities**.

References

- [1] K. Christodoulou, On a class of Whittaker modules for affine Lie algebras, in preparation.
- [2] K. Christodoulou, Whittaker modules for Heisenberg Lie algebras and imaginary Whittaker modules for affine Lie algebras, 31 pages, submitted.

On the cyclotomic Hecke algebras of complex reflection groups

Maria Chlouveraki

The classification of complex reflection groups was realized by Shepard and Todd in 1954. If W is an (irreducible) complex reflection group, then

- either there exist positive integers d, e, r such that W is isomorphic to $G(de, e, r)$, where $G(de, e, r)$ is the group of all $r \times r$ monomial matrices such that the non-zero coefficients are de -th roots of unity whose product is a d -th root of unity.
- or W is isomorphic to an exceptional group G_n ($n = 4, \dots, 37$).

Every complex reflection group has a “field of realization” K . If $K \subseteq \mathbb{R}$, then W is a (finite) Coxeter group. If $K = \mathbb{Q}$, then W is a Weyl group.

The work of G.Lusztig on the irreducible characters of reductive groups over finite fields has displayed the important role of the “families of characters” of the Weyl groups concerned. However, only recently was it realized that it would be of great interest to generalize the notion of families of characters to the complex reflection groups, or more precisely to some types of Hecke algebras associated with complex reflection groups.

On one hand, the complex reflection groups and their associated “cyclotomic” Hecke algebras appear naturally in the classification of the “cyclotomic Harish-Chandra series” of the characters of the finite reductive groups, generalizing the role of the Weyl group and its traditional Hecke algebra in the principal series. Since the character families of the Weyl group play an essential role in the definition of the families of unipotent characters of the corresponding finite reductive group, we can hope that the families of characters of the cyclotomic Hecke algebras play a key role in the organization of families of unipotent characters more generally.

On the other hand, for some complex reflection groups (non-Coxeter) W , some data have been gathered which seem to indicate that behind the group W , there exists another mysterious object - the *Spets*- that could play the role of the “series of finite reductive groups of Weyl group W ”. In some cases, one can define the unipotent characters of the *Spets*, which are controlled by the “spetsial” Hecke algebra of W , a generalization of the classical Hecke algebra of the Weyl groups.

The main obstacle for this generalization is the lack of Kazhdan-Lusztig bases for the non-Coxeter complex reflection groups. However, more recent results by Gyoja and Rouquier have made possible the definition of a substitute for families of characters which can be applied to all complex reflection groups. Gyoja has shown (case by case) that the partition into “ p -blocks” of the Iwahori-Hecke algebra of a Weyl group W coincides with the partition into families, when p is the unique bad prime number for W . Later, Rouquier showed that the families of characters of a Weyl group W are exactly the blocks of characters of the Iwahori-Hecke algebra of W over a suitable coefficient ring, the *Rouquier ring*. This definition generalizes without problem to all the cyclotomic Hecke algebras of complex reflection groups. The blocks of a cyclotomic Hecke algebra over the Rouquier ring are its *Rouquier blocks*. Thus, the Rouquier blocks play an essential role in the program “Spets” whose ambition is to give to complex reflection groups the role of Weyl groups of as yet mysterious structures.

As far as the calculation of the Rouquier blocks is concerned, the case of the infinite series has already been treated by Broué and Kim. Moreover, the Rouquier blocks of the “spetsial” cyclotomic Hecke algebra of many exceptional complex reflection groups have been determined by Malle and Rouquier. Generalizing the methods that they used, we have been able to calculate the Rouquier blocks of all cyclotomic Hecke algebras of all exceptional complex reflection groups. Moreover, we discovered that the Rouquier blocks of a cyclotomic Hecke algebra associated to a complex reflection group depend on a numerical datum of the group, its *essential hyperplanes*.

More precisely, for every symmetric algebra, we associate a “Schur element” to each irreducible character. The Schur elements of the generic Hecke algebras of all complex reflection groups have already been calculated and, in my thesis, I show that they are of a specific form (products of cyclotomic polynomials evaluated on monomials). This form is unique and defines the *essential monomials* and the *essential hyperplanes* for the group. A cyclotomic Hecke algebra is the algebra obtained as the specialization of the generic Hecke algebra via a “cyclotomic specialization”. Every cyclotomic specialization is determined by a family of integers. I’ve proven that the Rouquier blocks of the corresponding cyclotomic Hecke algebra depend only on the essential hyperplanes these integers belong to. More specifically, to each essential hyperplane H , we can associate a partition into blocks \mathcal{B}_H . The partition into Rouquier blocks of a cyclotomic Hecke algebra is generated by the partitions \mathcal{B}_H , where H runs over the set of the essential hyperplanes that the integers of the corresponding cyclotomic specialization belong to.

In order to determine the blocks of the cyclotomic Hecke algebra of a group W which appears as a symmetric sub-algebra of the cyclotomic Hecke algebra of another group W' (that contains W), I’ve had to generalize some classic results, known as “Clifford theory”, to the “twisted” symmetric algebras of finite groups and more precisely of finite cyclic groups.

Using GAP and the package CHEVIE, I’ve constructed an algorithm which determines the partitions \mathcal{B}_H for all exceptional complex reflection groups. I’ve created GAP functions that use these data and calculate the Rouquier blocks of any cyclotomic Hecke algebra associated to an exceptional complex reflection group. These functions are available on my website <http://www.math.jussieu.fr/~chlouweraki>.

More recently, using the theory of “essential hyperplanes” and GAP functions that I’ve created, I was able to prove that the valuation and the degree of the Schur elements (functions a and A) are constant on the Rouquier blocks of the cyclotomic Hecke algebras associated to the exceptional complex reflection groups. In the case of the Weyl groups and their usual Hecke algebra, the families of characters can be defined using the existence of Kazhdan-Lusztig bases. Lusztig attaches to every irreducible character two integers, denoted by a and A , and shows that they are constant on the families. In an analogue way, we define the integers a and A attached to every irreducible character of a cyclotomic Hecke algebra of a complex reflection group. For the groups of the infinite series, it has been shown by Broué and Kim that a and A are constant on the Rouquier blocks. Having proved the same result for the exceptional complex reflection groups, we have one more indication of the possible existence of “Kazhdan-Lusztig bases” for other reflection groups than the Coxeter groups, and for all their cyclotomic algebras.

THE DIMENSIONS OF $LU(3, q)$ CODES

Ogul Arslan, ogul@math.ufl.edu

Advisor: Prof. Peter Sin

UNIVERSITY OF FLORIDA

ABSTRACT:

A family of LDPC codes, called $LU(3, q)$ was constructed using q -regular bipartite graphs. The dimension of these codes for q is a power of an odd prime was proven by P. Sin and Q. Xiang recently. For the case q is a power of 2, the lower bound for the dimension of $LU(3, q)$ code was given also. In this paper we show that the given lower bound is actually the exact dimension of the $LU(3, q)$ code when q is a power of 2. Hence we complete the proof for the dimension of $LU(3, q)$ code for q is a prime power. The proof involves the geometry of symplectic generalized quadrangle, representation theory of $Sp(4, q)$, and the ring of polynomials.

SUMMARY

This is a very short summary of the previous work, the techniques we use, and some of the lemmas we prove in the paper.

Let V be a 4 dimensional vector space over the field \mathbb{F}_q of q elements. We assume that V has an alternating bilinear form (v, v') . Let $G = Sp(4, q)$ be the symplectic group of linear automorphisms preserving this form.

We denote by P , the projective space $P(V)$, the space of one dimensional subspaces of V . Let L be the set of totally isotropic 2-dimensional subspaces of V , called the lines in P . The pair (P, L) , with the natural relation of incidence between the lines and points is called the symplectic generalized quadrangle.

We fix a point $p_0 \in P$ and $\ell_0 \in L$. For a point $p \in P$, we define p^\perp to be the set of points on all the lines that passes through p . Let P_1 be the set of points not in p_0^\perp and L_1 be the set of lines which does not intersect ℓ_0 . We denote by $M(P, L)$ the incidence matrix whose rows indexed by P , and the columns by L . Similarly, we get the incidence matrix $M(P_1, L_1)$, which can be thought as a submatrix of $M(P, L)$.

P. Sin and Q. Xiang proved in their paper that $M(P_1, L_1)$ is a parity check matrix of the $LU(3, q)$ code.

It was proven by N.S.N. Sastry and P. Sin that the 2-rank of $M(P, L)$ is $1 + (\frac{1+\sqrt{17}}{2})^{2t} + (\frac{1-\sqrt{17}}{2})^{2t}$ for the case where q is a power of 2. In this paper we prove the following theorem.

Theorem 1. *Assume $q = 2^t$ for some positive integer t . The 2-rank of $M(P_1, L_1)$ equals $1 + (\frac{1+\sqrt{17}}{2})^{2t} + (\frac{1-\sqrt{17}}{2})^{2t} - 2^{t+1}$.*

The above formula was conjectured by P. Sin and Q. Xiang based on the computer calculations of J.-L. Kim.

Corollary 2. *If $q = 2^t$, then the dimension of $LU(3, q)$ is $2^{3t} + 2^{t+1} - 1 - (\frac{1+\sqrt{17}}{2})^{2t} - (\frac{1-\sqrt{17}}{2})^{2t}$.*

For the rest of the section we can assume that q is an arbitrary prime power.

We denote by $\mathbb{F}_2[P]$ the space of \mathbb{F}_2 valued functions on P . The characteristic function χ_p for a point $p \in P$ is the function whose value is 1 at p , and zero at any other point. Thus, χ_p can be thought as the $q^3 + q^2 + q + 1$ component vector whose entry that corresponds to p is 1, and all the other entries are zero. For any line $\ell \in L$, the characteristic function χ_ℓ is the function given by the sum of $q + 1$ characteristic functions of the points of ℓ . The subspace of $\mathbb{F}_2[P]$ spanned by all the χ_ℓ is the \mathbb{F}_2 code of (P, L) , denoted by $C(P, L)$. For brevity we do not make a distinction between the lines and the characteristic functions of the lines. So, let $C(P, L_1)$ be the subspace of $\mathbb{F}_2[P]$ spanned by the lines of L_1 . Let $C(P_1, L_1)$ denote the code of (P_1, L_1) viewed as a subspace of $\mathbb{F}_2[P_1]$, and let $C(P_1, L)$ be the larger subspace of $\mathbb{F}_2[P_1]$ spanned by the restrictions to P_1 of the characteristic functions of all lines of L .

We consider the natural projection map $\pi_{P_1} : \mathbb{F}_2[P] \rightarrow \mathbb{F}_2[P_1]$ given by the restriction of functions to P_1 . We denote its kernel by $\ker \pi_{P_1}$.

Let $Z \subset C(P, L_1)$ be a set of characteristic functions of lines in L_1 which maps bijectively under π_{P_1} to a basis of $C(P_1, L_1)$. Let X be the set of characteristic functions of the $q + 1$ lines passing

through p_0 , and let $X_0 = X \setminus \ell_0$. Furthermore, we pick q lines that intersect ℓ_0 at q distinct points except p_0 , and call the set of these lines as Y . These sets X, Y , and Z are disjoint, also note that $X \subset \ker \pi_{P_1}$.

The following lemma and corollary were proven in a paper by P. Sin and Q. Xiang.

Lemma 3. $Z \cup X_0 \cup Y$ is linearly independent over F_2 .

Hence, $|X_0 \cup Y| = 2q$, while $|Z| = \dim_{\mathbb{F}_2} C(P_1, L_1)$.

Corollary 4. Let q be an arbitrary prime power. Then $\dim_{\mathbb{F}_2} LU(3, q) \geq q^3 - \dim_{\mathbb{F}_2} C(P, L) + 2q$.

The proof of Theorem 1 will be completed if we can show that $Z \cup X_0 \cup Y$ spans $C(P, L)$ as a vector space over \mathbb{F}_2 .

Lemma 5. Let ℓ and ℓ' be two lines passing through $p \in \ell_0$. Then $\chi_\ell - \chi_{\ell'} \in C(P, L_1)$.

In order to prove this lemma we use the geometry of symplectic generalized quadrangle. We first show that there is a grid of lines in L_1 that intersect ℓ and ℓ' , and we add these lines to obtain $\chi_\ell - \chi_{\ell'}$.

Then using Lemma 5 we prove the following two lemmas.

Lemma 6. For any choice of $Y, \ell \in L \setminus \{\ell_0\}$ and $\mathbf{1}$ are in the span of $X_0 \cup Y \cup L_1$.

Lemma 7. For $q = 2^t$, ℓ_0 is in the span of $X_0 \cup Y \cup L_1$.

Thus any line $\ell \in L$ is in the span of $X_0 \cup Y \cup L_1$. It remains to show the span of $X_0 \cup Y \cup L_1$ is the same as the span of $X_0 \cup Y \cup Z$. We need a corollary to prove this.

Corollary 8. $\dim(\ker \pi_{P_1} \cap C(P, L)) = q + 1$.

In order to prove the above corollary we use some classes of polynomials in $\mathbb{F}_q[x_0, x_1, x_2, x_3] / \langle (x_0^q - x_0), (x_1^q - x_1), (x_2^q - x_2), (x_3^q - x_3) \rangle$ to represent lines. By the action of $Sp(4, q)$ on the lines of L we show that $C(P, L)$ is spanned by the classes of polynomials of the form $(1 + (\sum_{i=0}^3 a_i x_i)^{q-1})(1 + (\sum_{i=0}^3 b_i x_i)^{q-1}) + I$, for some $a_i, b_i \in k$ so that $\langle (a_0 : a_1 : a_2 : a_3), (b_0 : b_1 : b_2 : b_3) \rangle$ is a line in L . Then we describe another way to represent the polynomials.

Definition: We call a polynomial $f \in R^*$ *digitizable* if it is possible to find square free homogeneous polynomials, f_i , called *digits* of f , so that $f = f_0 f_1^2 f_2^{2^2} \dots f_{t-1}^{2^{t-1}}$. In this case, we denote f as $[f_0, f_1, \dots, f_{t-1}]$, and call this notation as 2-adic t-tuple of f .

Using this new way of writing polynomials and some linear algebra we show that $\dim(\ker \pi_{P_1} \cap C(P, L)) = q + 1$.

Lemma 9. $\ker \pi_{P_1} \cap C(P_1, L_1)$ has dimension $q - 1$ and basis the set of functions $\chi_\ell - \chi_{\ell'}$ where $\ell \neq \ell_0$ is an arbitrary but fixed line through p_0 and ℓ' varies over the $q - 1$ lines through p_0 different from ℓ_0 and ℓ .

Corollary 10. The span of $Z \cup X_0$ and $L_1 \cup X_0$ are the same.

Therefore, $Z \cup X_0 \cup Y$ spans $C(P, L)$ as a vector space. So, $\dim(C(P, L)) \leq \dim(C(P_1, L_1)) + 2q$ and this implies $\dim LU(3, q) = q^3 - \dim(C(P, L)) + 2q$.

Summary of Research

Emilie Wiesner, Ithaca College

My research is in the representation theory of Lie algebras and related algebras. In particular, I have considered various questions in representation theory as they apply to the Virasoro algebra and similar Lie algebras.

1. THE VIRASORO ALGEBRA

The Virasoro algebra is the Lie algebra $\mathcal{V} = \text{span}_{\mathbb{C}}\{z, d_k | k \in \mathbb{Z}\}$ and defining relations

$$[d_j, d_k] = (j - k)d_{j+k} + \delta_{j,-k} \frac{j^3 - j}{12} z, \quad [z, z] = 0 = [z, d_k].$$

The Virasoro algebra is the central extension of the Witt algebra, which can be realized as the Lie algebra of derivations of $\mathbb{C}[t^{\pm}]$.

The Virasoro algebra plays a role in mathematical physics and is connected to the representation theory of affine Kac-Moody Lie algebras. The Virasoro algebra is also an example of a Lie algebra with triangular decomposition: it decomposes as $Vir = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where

$$\mathfrak{n}_- = \text{span}_{\mathbb{C}}\{d_k | k \in \mathbb{Z}_{>0}\}, \quad \mathfrak{h} = \text{span}_{\mathbb{C}}\{z, d_0\}, \quad \mathfrak{n}_+ = \text{span}_{\mathbb{C}}\{d_k | k \in \mathbb{Z}_{<0}\}.$$

Lie algebras with triangular decomposition generalize many important properties of (finite-dimensional) complex semisimple Lie algebras. Because of this, it is useful to consider the extension of various questions from the representation theory of complex semisimple Lie algebras to the Virasoro algebra. For example, the category of modules \mathcal{O} and Whittaker modules come from the world of complex semisimple Lie algebras but can be easily defined for Lie algebras with triangular decomposition.

1.1. Category \mathcal{O} . The category \mathcal{O} of \mathcal{V} -modules closely reflects the triangular decomposition of the Virasoro algebra (see [7]). Within Category \mathcal{O} , Verma modules $M(\lambda)$ and simple modules $L(\lambda)$ ($\lambda \in \mathfrak{h}^*$) are of special interest. The set of weights \mathfrak{h}^* can be partitioned into classes, or *blocks*, by the equivalence relation $\lambda \sim \gamma$ if $L(\gamma)$ is a subquotient of $M(\lambda)$. Category \mathcal{O} has the nice property that it decomposes by blocks into subcategories: $\mathcal{O} = \bigoplus \mathcal{O}^{[\gamma]}$, where $\mathcal{O}^{[\gamma]}$ is the full subcategory containing all indecomposable modules M which have $L(\gamma')$, $\gamma' \in [\gamma]$, as a subquotient.

The work of Feigin and Fuchs [3] provides a description of the blocks of Vir . Their work uses the Shapovalov form $\langle, \rangle : M(\lambda) \times M(\lambda) \rightarrow \mathbb{C}$, a particular bilinear form on $M(\lambda)$, and the associated Shapovalov determinant. Degeneracies in \langle, \rangle are tied to the submodule structure of $M(\lambda)$, and thus they shed light on the block structure of \mathcal{O} .

In [10], I consider the decomposition of $M(\lambda) \otimes L(\mu)$ by blocks. This is a special case of the translation functor $F : \mathcal{O}^{[\lambda]} \rightarrow \mathcal{O}^{[\gamma]}$ given by $M \mapsto (M(\lambda) \otimes L(\mu))^{[\gamma]}$, where $(M(\lambda) \otimes L(\mu))^{[\gamma]}$ is the maximal submodule of $M(\lambda) \otimes L(\mu)$ contained in $\mathcal{O}^{[\gamma]}$. Motivated by Jantzen's work on translation functors for finite-dimensional semisimple Lie algebras [5], I use the Shapovalov determinant to study $(M(\lambda) \otimes L(\mu))^{[\gamma]}$ for the Virasoro algebra.

In joint work with Boe and Nakano [1], we study the cohomology of category \mathcal{O} for the Virasoro algebra. Beginning with the description of blocks given by Feigin and Fuchs, we compute $\text{Ext}^i(M(\lambda), L(\mu))$ and $\text{Ext}^i(L(\lambda), L(\mu))$ in the block $\mathcal{O}^{[\gamma]}$. These results give a more complete picture of the structure of the blocks of category \mathcal{O} and also provide an application of the theory of highest weights developed by Cline, Parshall, and Scott (cf. [2]).

1.2. Whittaker Modules. Whittaker modules are a class of modules not contained in the category \mathcal{O} . They were first defined for complex semisimple finite-dimensional Lie algebras by Kostant [6], in connection to Whittaker models for complex semisimple Lie groups. However, it is natural to extend their definition to other Lie algebras with triangular decomposition.

In ongoing joint work with Matthew Ondrus, we are studying Whittaker modules for the Virasoro algebra. Recall that Vir has a triangular decomposition $Vir = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. Note that \mathfrak{n}_+ is generated by d_1 and d_2 . We define a *Whittaker module* for Vir as a module M generated by some $w \in M$ such that $d_1 w = \xi_1 w$ and $d_2 w = \xi_2 w$ for some $0 \neq \xi_1, \xi_2 \in \mathbb{C}$. For fixed ξ_1, ξ_2 , we show that simple Whittaker modules are in one-to-one correspondence with the central characters of \mathcal{V} . We also provide descriptions of general Whittaker modules. (See the poster session for more details.)

2. WITT ALGEBRAS IN PRIME CHARACTERISTIC

In ongoing joint work with Boe and Nakano, we study the cohomology of the Witt algebra $W(1, 1)$ over a field k of characteristic $p > 3$. For a fixed k , $W(1, 1) = \text{span}_k\{e_{-1}, e_0, e_1, \dots, e_{p-2}\}$ with bracket $[e_i, e_j] = (i - j)e_{i+j}$ for $i + j < p - 1$ and $[e_i, e_j] = 0$ otherwise. These algebras are simple and constitute the smallest examples of the family of Cartan Lie algebras in characteristic p .

The simple restricted modules for $W(1, 1)$ have been classified, and it has been shown that there is exactly one block for the restricted modules ([4], [9], [8]). We compute $\text{Ext}^i(Z^+(\lambda), L(\mu))$ and $\text{Ext}^i(L(\lambda), L(\mu))$. (Here, $Z^+(\lambda)$ is the "baby" Verma module and $L(\lambda)$ is the simple module of weight $\lambda \in k$.) We are also able to compute extensions for several classes of nonrestricted modules.

REFERENCES

- [1] B. Boe, D. Nakano, and E. Wiesner. Category \mathcal{O} for the Virasoro algebra: Cohomology and Koszulity. *Pacific J. Math.*, To appear.
- [2] E. Cline, B. Parshall, and L. Scott. Abstract Kazhdan-Lusztig theories. *Tohoku Math. J. (2)*, 45(4):511–534, 1993.
- [3] B. L. Feigin and D. B. Fuchs. Representations of the Virasoro algebra. In *Representation of Lie groups and related topics*, volume 7 of *Adv. Stud. Contemp. Math.*, pages 465–554. Gordon and Breach, New York, 1990.
- [4] Randall R. Holmes and Daniel K. Nakano. Block degeneracy and Cartan invariants for graded Lie algebras of Cartan type. *J. Algebra*, 161(1):155–170, 1993.
- [5] J. C. Jantzen. Darstellungen halbeinfacher algebraischer gruppen und zugeordnete kontravariante formen. Thesis, 1973.
- [6] Bertram Kostant. On Whittaker vectors and representation theory. *Invent. Math.*, 48(2):101–184, 1978.
- [7] Robert V. Moody and Arturo Pianzola. *Lie algebras with triangular decompositions*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons Inc., New York, 1995. A Wiley-Interscience Publication.
- [8] Daniel K. Nakano. Projective modules over Lie algebras of Cartan type. *Mem. Amer. Math. Soc.*, 98(470):vi+84, 1992.
- [9] Guang Yu Shen. Graded modules of graded Lie algebras of Cartan type. III. Irreducible modules. *Chinese Ann. Math. Ser. B*, 9(4):404–417, 1988. A Chinese summary appears in *Chinese Ann. Math. Ser. A* **9** (1988), no. 5, 634.
- [10] E. Wiesner. The Shapovalov Determinant and Translation Functors for the Virasoro Algebra. *J. Algebra*, 1.

RESEARCH ABSTRACT

Shona Yu

The University of Sydney

Email: shonayu@maths.usyd.edu.au

I have been interested in the study of algebras associated with the Artin braid groups, the symmetric group and link invariants, alongside diagram algebras and cellular algebras.

My Ph.D. thesis [14] focussed in particular on the *cyclotomic Birman-Murakami-Wenzl (BMW) algebras*. It has led me to study the Iwahori-Hecke algebras and its cyclotomic and affine variants, in addition to many *diagram algebras*; that is, algebras with basis a given set of diagrams where multiplication is described by a simple diagram calculus which is both intuitive and computationally effective. These algebras feature in a vast variety of mathematical subjects, including combinatorial and geometric representation theory, knot theory, homological algebra, the study of subfactors, topological quantum field theory, and the theory of quantum groups. Furthermore, many diagram algebras originated from important problems in *statistical mechanics* and studying their representation theory not only has helped solve these problems but has often led to further questions and fruitful interplay between both subjects. They also help provide deeper insight into the representation theory of affine Hecke algebras. The algebras pertaining to my thesis are also examples of *cellular algebras*, in the sense of Graham and Lehrer [5].

The motivation behind the definition of the cyclotomic BMW algebras [6] can be traced back to an important problem in *knot theory*; namely, that of classifying knots (and links) up to isotopy, which leads to the study of *link invariants*.

Link invariants can be constructed by looking at representations of braid groups and related algebras. For example, the discovery of the famous Jones polynomial link invariant [7, 8] emerged from Jones' study of certain quotient algebras of the group algebra of the braid group called the *Temperley-Lieb algebras* [12], which in fact originally arose in the context of *statistical mechanics*! Using the existence of another link invariant, the *Kauffman polynomial* [9], Birman and Wenzl [1] and Murakami [11] *reversed* Jones' process to abstractly construct a new family of algebras. Morton and Wassermann [10] then proved these BMW algebras may be diagrammatically realised as the *Kauffman tangle algebra*, an algebra of (regular isotopy equivalence classes of) tangles on n strands in a solid cylinder involving a relation that encodes the Kauffman polynomial.

The BMW algebras are closely related to the Artin braid group of type A and Iwahori-Hecke algebras of type A . In view of these relationships, several authors have since generalised the BMW algebras for other types of Artin groups (e.g. [3]). Motivated by knot theory associated with the Artin braid group of type B and the cyclotomic Hecke algebras of type $G(k, 1, n)$, Häring-Oldenburg introduced the "cyclotomic BMW algebras" \mathcal{B}_n^k in [6].

My thesis [14] was primarily concerned with the question of finding a basis and a geometric realisation of these algebras. In my thesis, it is shown they have a diagrammatic interpretation as a certain cyclotomic analogue of the Kauffman tangle algebras and have a basis which one can describe both algebraically and diagrammatically. In order to achieve these results, complications unseen in previously mentioned algebras force an explicit investigation of the representation theory of \mathcal{B}_n^k at the $n = 2$ level (see [13]). Furthermore, by employing a known cellular structure of the cyclotomic Hecke algebras, it is also proven that \mathcal{B}_n^k is a cellular algebra.

Other research/future directions:

Consequences of cellularity. Given a cellular algebra, one may obtain a general description of its irreducible representations and block theory. Also, important questions, for example, regarding the semisimplicity and quasi-heredity of the algebra, are reduced to linear algebra problems involving a bilinear form.

By using this theory, the cellularity results in my thesis allows me to deduce further results on the representation theory and structure of \mathcal{B}_n^k , including a complete description of its irreducible representations over a field and a criterion for semisimplicity, which would generalise previously established results on the cellularity of the ordinary BMW algebras. Another problem one can consider is when

these algebras are quasi-hereditary, in the sense of Cline, Parshall and Scott [2], leading to interesting cohomological properties. Combinatorics (for example, Young tableaux and up-down tableaux) will feature prominently in this investigation.

Schur-Weyl duality. Many authors have studied versions of “classical Schur-Weyl Duality” for the BMW algebras and its related algebras. In view of existing results, a natural question to address is whether analogous results hold for these newer cyclotomic BMW algebras.

Link invariants. Another application to explore is in knot theory, specifically the study of link invariants. Ordinary braids (of type A) close to give links in S^3 . Elements of the Artin braid group of type B , “affine” or “cylindrical braids”, close to yield links in a solid torus. This leads to the study of Markov traces on the Artin braid group of type B and, moreover, invariants of links in the solid torus. Various solid torus link invariants, analogous to the Jones and HOMFLY-PT invariants (for links in S^3), have been discovered using Markov traces on the cyclotomic Hecke algebras. Moreover, Kauffman-type invariants for links in the solid torus can be recovered from the Markov trace on the affine BMW algebra, as discussed by Goodman and Hauschild in [4]. It seems plausible that one can use an analogous process to analyse the invariants which would arise from the algebras of various types.

References

- [1] J. S. Birman and H. Wenzl. Braids, link polynomials and a new algebra. *Transactions of the American Mathematical Society*, 313:249–273, 1989.
- [2] E. Cline, B. Parshall, and L. Scott. Finite-dimensional algebras and highest weight categories. *Journal für die Reine und Angewandte Mathematik*, 391:85–99, 1988.
- [3] A.M. Cohen, D.A.H. Gijssbers, and D.B. Wales. The BMW algebras of type D_n . Preprint, arXiv:0704.2743v1, 2007.
- [4] F. Goodman and H. Hauschild. Affine Birman-Wenzl-Murakami algebras and tangles in the solid torus. *Fundamenta Mathematicae*, 190:77–137, 2006.
- [5] J.J. Graham and G.I. Lehrer. Cellular algebras. *Inventiones Mathematicae*, 123(1):1–34, 1996.
- [6] R. Häring-Oldenburg. Cyclotomic Birman-Murakami-Wenzl algebras. *Journal of Pure and Applied Algebra*, 161:113–144, 2001.
- [7] V. F. R. Jones. Index for subfactors. *Inventiones Mathematicae*, 72:1–25, 1983.
- [8] V. F. R. Jones. A polynomial invariant for knots via von Neumann algebras. *American Mathematical Society. Bulletin. New Series*, 12(1):103–111, 1985.
- [9] L. H. Kauffman. An invariant of regular isotopy. *Transactions of the American Mathematical Society*, 318(2):417–471, 1990.
- [10] H. Morton and A. Wassermann. A basis for the Birman-Wenzl algebra. Unpublished manuscript, 29 pp., 1989.
- [11] J. Murakami. The Kauffman polynomial of links and representation theory. *Osaka Journal of Mathematics*, 24:745–758, 1987.
- [12] H. N. V. Temperley and E. H. Lieb. Relations between the “percolation” and “colouring” problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the “percolation” problem. *Proceedings of the Royal Society. London. Series A. Mathematical, Physical and Engineering Sciences*, 322(1549):251–280, 1971.
- [13] S. Wilcox and S. Yu. The cyclotomic BMW algebra associated with the two string type B braid group. Preprint, arXiv:math/0611518, 2006.
- [14] S. Yu. *The Cyclotomic Birman-Murakami-Wenzl algebras*. PhD thesis, The University of Sydney, December 2007.

Julie Beier

Graduate Program: Mathematics, North Carolina State University

Advisor: Dr. Kailash C. Misra

Combinatorics of Crystal Bases for Some Simple Quantum Affine Algebra Modules

Representations of quantum affine algebras play an important role in conformal field theory, quantum field theory and statistical mechanics in physics. Crystal bases provide a useful tool for studying the integrable representations of quantum affine algebras. To study the combinatorics of these representations, it is important to have concrete realizations of their crystal bases. Demazure modules form a finite dimensional subspace of the integrable modules of quantum affine algebras.

The aim of this project is to give concrete realizations of certain Demazure modules of the quantum affine algebra $U_q(\widehat{sl}(n))$ in terms of some combinatorial objects called extended Young diagrams.

The realizations of the crystal bases for integrable modules of the quantum affine algebra in terms of extended Young diagrams are already known. We utilize this realization to find concrete realizations of the corresponding Demazure modules.

Here we look at the extended Young diagrammatic realizations of certain Demazure modules with highest weight $k\Lambda_0$. Recently, I extended this work for any dominant weight.