ARIZONA STATE UNIVERSITY

**Presentation to the National Mathematics Panel**
Aurora, IL, April 20, 2007

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Mr. Chairman, madam vice-chairman, panel members: Thank you very much for this opportunity to speak to you about the Panel's work. I am a researcher in the learning and teaching of mathematics at all levels of schooling, and a teacher of future and current mathematics teachers at elementary, secondary, and collegiate levels.

I will speak to five of the Panel's charges, specifically

1. The critical skills and skill progressions for students to acquire competence in algebra and readiness for higher levels of mathematics;
3. The processes by which students of various abilities and backgrounds learn mathematics;
4. Instructional practices, programs, and materials that are effective for improving mathematics learning;
5. The training, selection, placement, and professional development of teachers of mathematics in order to enhance students' learning of mathematics;
7. Needs for research in support of mathematics education;

**Charges 1 and 7: Critical Skills and Skill Progressions; Research in support of math education**

My first remark addresses Charge 1 and 7, but actually cuts across all of the above. It is that the Panel has the significant task of responding to a list of charges that take "skills" as the primary component of mathematics learning when the notion of skill itself is hardly well defined. Do you take "skill" to mean a child's ability to perform reliably a procedure when told to perform that procedure? Or, do you take "skill" to mean a child's ability to have developed sufficient knowledge and appropriate flexibility of thought to solve most problems of a particular genre of problems, even those that might have subtle and nuanced differences from any the students might have seen? The incommensurability of different viewpoints will be directly related to the distance between them on this spectrum. Where the Panel positions itself within these two extremes will affect the recommendations it issues.

For the same reason, one cannot simply look to "research" to answer the question of what policies the nation should follow in preparing students for algebra. Which algebra? Pushbutton algebra or, as Kaput calls it, the algebra of progressive generalization and symbolization? The two entail different philosophic and intellectual commitments for those who embrace them, and they entail different expectations for students' learning and teachers' knowledge at every grade.

Thus, it is incumbent upon the Panel to make clear where it stands with regard to what students should learn, and to justify that stance according to the pragmatic consequences that relative stances have regarding students' learning and preparation for future learning.

I have included several important references that address this tension.

**Charges 3 and 4: Processes of Learning; Effective instructional practices**
In regard to these I offer an example from a current research project on effective models for improving secondary mathematics teachers' instruction. Teachers in the project simply did not have an image of the kind intense intellectual interchange with students that we were attempting to help them create, and continued classroom-based efforts generated only small changes that still looked more like what they had been doing than what we hoped they would do. So, we created an implementation of Algebra I, in collaboration with one of the participating teachers, in which we hoped to create examples of instruction that supported students' engagement with significant mathematical ideas and issues. We also hoped that these students would display proficiency in the algebra that teachers were used to assessing, but display it as a consequence of understanding ideas well and not because of having memorized a prescribed procedure.

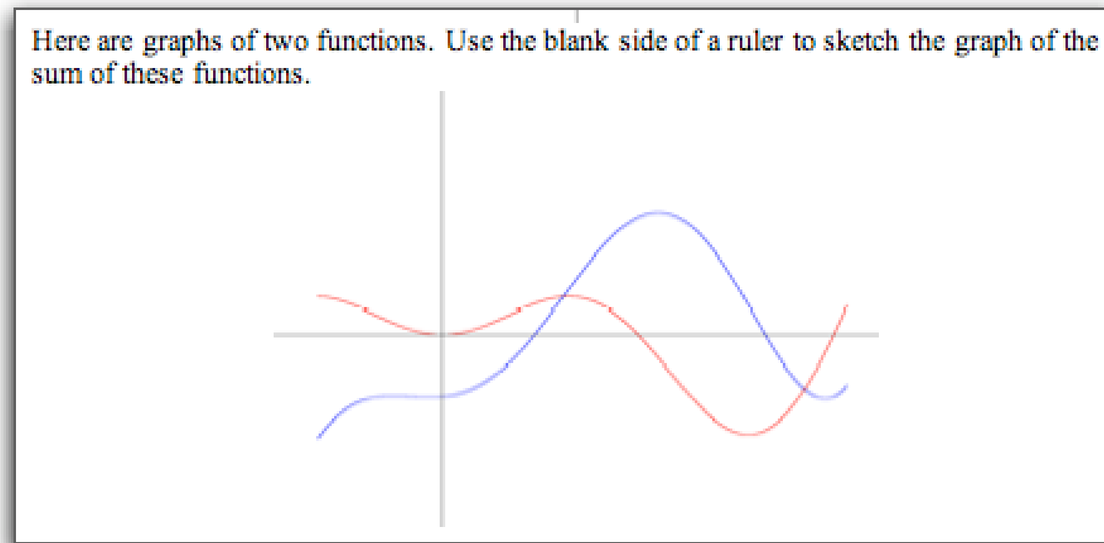
The students we taught were not in an honors program. Thus, they were students taking Algebra I in 9th grade. Their computational skills were atrocious. They had no understanding of fractions. Their experience of mathematics was that teachers showed them procedures they were supposed to remember until the next test. Their feelings about mathematics were that it was a dehumanizing experience that no one in their right mind would choose to experience had they the option not to.

So, our immediate question was what to do about their lack of skills given that our goal was to have them eventually engage with significant mathematical ideas. Do we re-teach what they had already not learned? Well, what they needed to know had already been re-taught several times. Thus, we decided to move on. We began the year with no review, and we designed instruction always guided by our goal of having them engage meaningfully with significant mathematical ideas at the same time as being able to pass the district's Algebra I final examination (which all students in the district take).

I will not describe what we did except to say that we focused on ideas of variation, covariation, rate of change, and functional relationship. As I said, our intent was not to test an intervention, but rather to create examples to use in our professional development research of how to deal with ill-prepared students in ways that gave *prima fascia* evidence that what students were doing was significant.

The appendices contain examples of the kind of work we have come to expect of students. Here I will give just one example to make a point.. The example is from a whole-class activity that occurred within a unit on quadratics. We wanted students to understand quadratics within a scheme of ways of thinking about polynomial functions:

- Monomial functions (like $f(x) = x^{44}$) have predictable and understandable properties.
- Polynomial functions are just sums of monomial functions
- Quadratic functions are a special case of polynomial functions
- It is possible to change the way a function is defined without affecting the underlying relationships that the definition expresses (as captured in the function's graph). Two definitions are equivalent if one is derivable from the other, which itself means that a definition's meaning remains the same despite changes in the way it appears.
- I emphasize that these students are *not* honors students




At the end of this activity the teacher showed the students, just for their information, the definitions of these functions. The definitions were

$$b(x) = \frac{\left(-\left(\frac{x}{2}-3\right)\right)(4x+2)(x+1)\sin\left(\frac{x}{2}\right)}{30} - 1, -2 \le x \le 6.5 \text{ (blue graph)}$$

$$r(x) = \frac{x\sin(x)}{3}, -2 \le x \le 6.5 \text{ (red graph)}$$

Half the class asked that she print the function definitions and the graphs so that they could show their friends and family. I share this not because I offer it as evidence of effectiveness. Rather, I offer it to illustrate what I mean by engaging students (I mean *all* students) in a way that legitimizes, in their worlds, intense engagement with significant mathematical ideas. I offer this example also to provide some insight into our quest that a majority of these students, all of whom are on a non-calculus track, take calculus while they are still in high school.

I have another point in offering this example. It is to say that, in my opinion, this nation suffers from a lack of imagination more than a lack of research. And it suffers from lack of imagination at all levels, especially at the levels of policy and politics.

**Charge 5: Training, selection, professional development of teachers**

The prior discussion is also pertinent to training, selecting, and providing professional development for mathematics teachers. However, I would like to raise a matter not in this list that is important to us at Arizona State University. It is the recruitment and retention of secondary mathematics teachers (teachers certified with single-subject mathematics credentials). In Arizona, over the past 5 years, its state universities have produced a combined average of 55 secondary mathematics teachers per year. Just one school district in the Phoenix area is looking to hire 22 secondary mathematics teachers for next year. All the high-minded programs we can imagine are for naught if we have no teachers.

ASU is conducting a study, called the Freshman STEM Improvement Project, that is yielding some insight into the problem. We asked the question, “Where are we losing students?” In secondary mathematics education, and to a lesser extent in all disciplines, we are losing them in calculus. Secondary mathematics education requires 3 semesters of calculus. Less than 30% of secondary mathematics education students who enroll in Calculus I complete Calculus III. The attrition rate is about 50% per semester. Clearly, the problem is an interaction among student preparation, course content, and instructional practices. We are continuing our study to gain insight into the problem. But the solution will require imagination, perhaps of the kind I illustrated regarding Algebra I, where we punted with regard to students’ preparation and focused on how they might recover.

Another possible strategy with regard to increasing the rate at which school students persist in their mathematical study is to change the nature of teacher preparation so that we do not pretend to prepare undergraduate students for entry into graduate mathematics programs. Instead, we might focus the undergraduate preparation of mathematics teachers on their ability to help students understand the mathematics of middle and secondary school. As with our example in Algebra I, this might have the salutary effect that it results simultaneously in better-prepared middle and secondary students and better preparation of secondary-certified mathematics teachers for advanced study of mathematics.

# Appendices

## Tests and Quizzes from Math 1/2 (Algebra I)

Tempe CAMPUS
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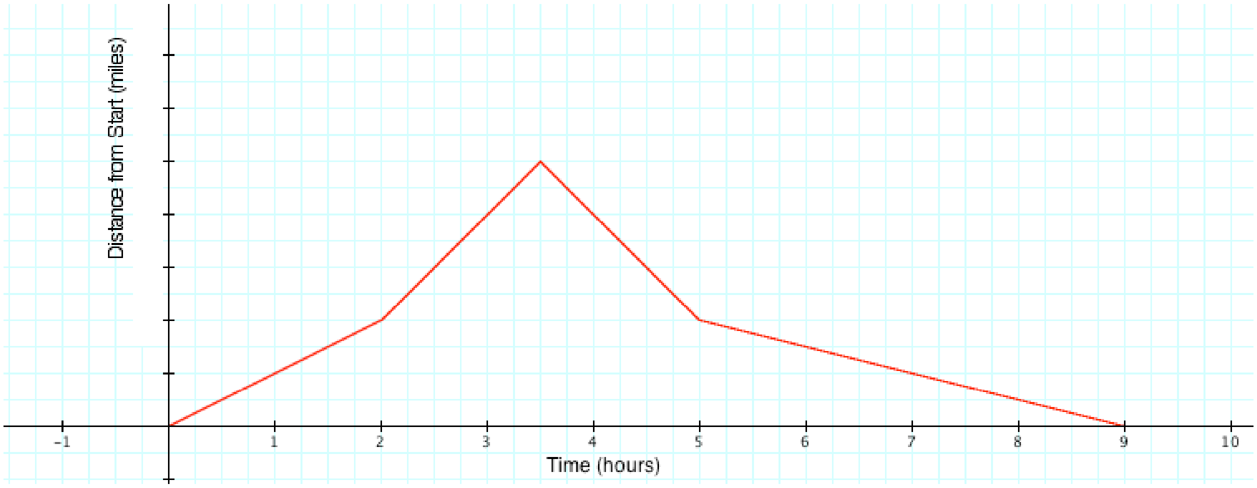
# Interpreting/Creating Graphs Quiz

1. Clown went on a bike trip. The following graph shows the number of miles that Clown was from the start relative to the number of hours since he started.




   a. In the first part of his trip Clown traveled at a speed of 5 mi/hr. Put numbers on the vertical axis so that the graph is accurate.

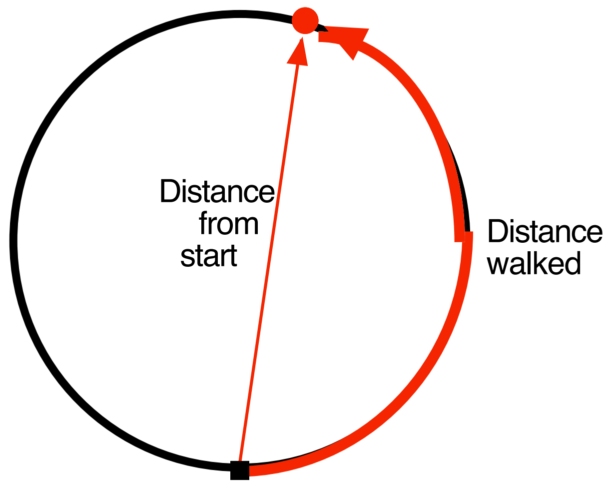
   b. How many miles did clown travel in the third part of his trip? How do you know?

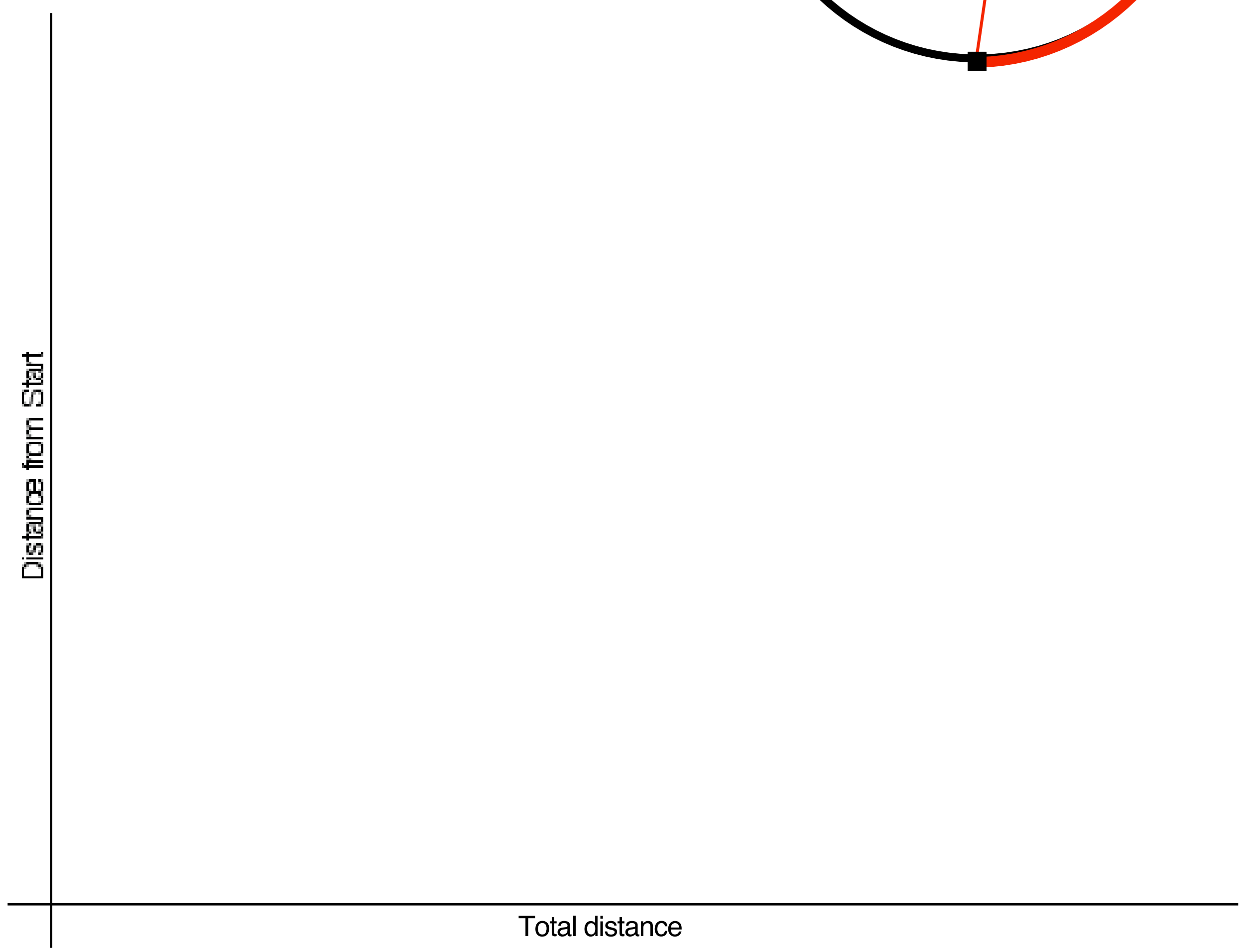
   c. How fast did Clown travel in the fourth part of his trip? How do you know?

   d. In one part of his trip, Clown blew a tire and had to walk. When did this happen? How do you know?

2. Matt walked abound a circular track (see diagram). The black square shows where he began. The arrows show that we are tracking Matt's distance from his current location to the starting point and the distance he had walked to get to his current location.

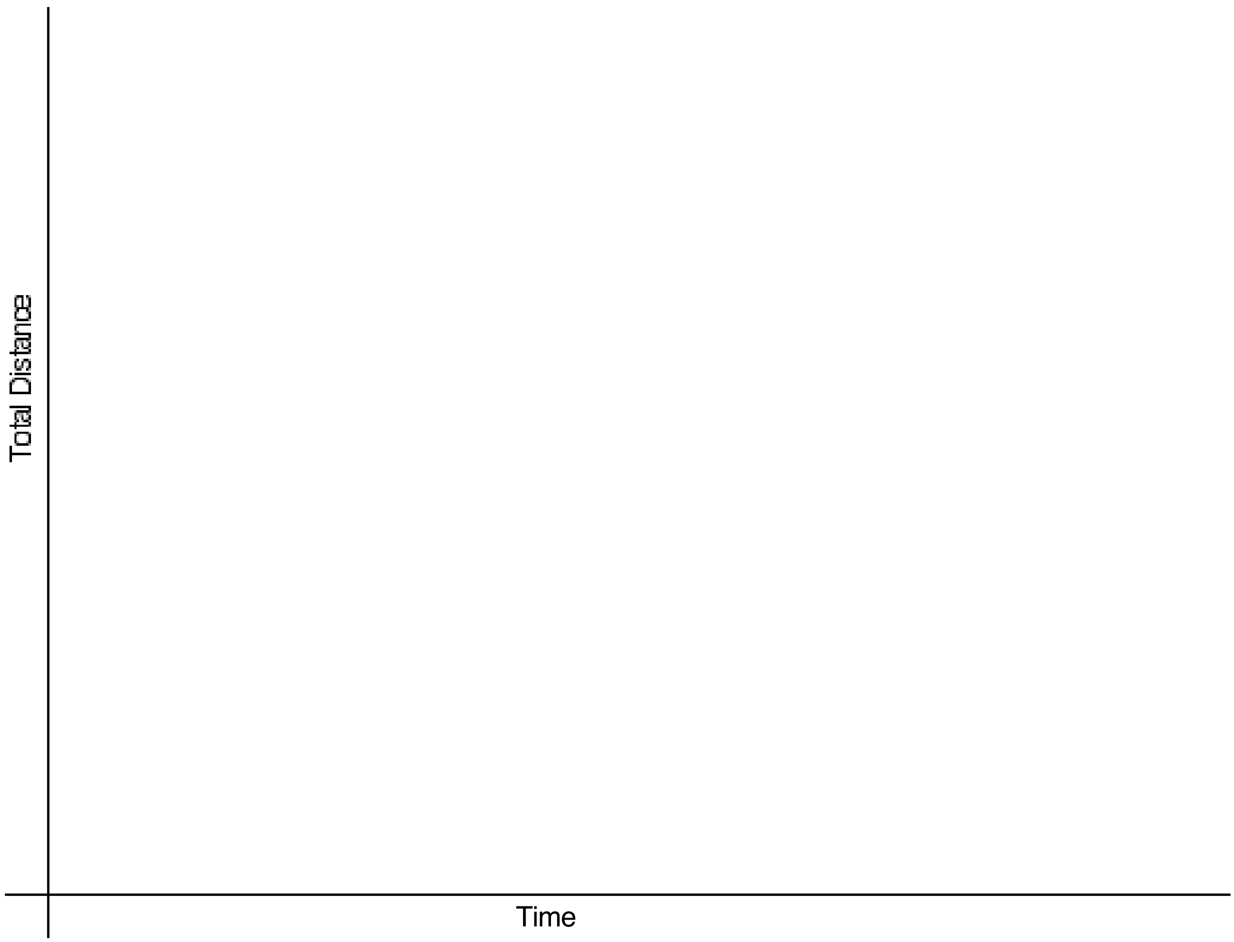
   Imagine Matt walking around the track once. <u>Use your finger tool</u> to sketch a graph of Matt's walk. Keep track of how far he had walked with your *horizontal* finger and his distance from the start with your *vertical* finger. (Hint: Think of playing Matt's walk a frame at a time to keep track of both.)



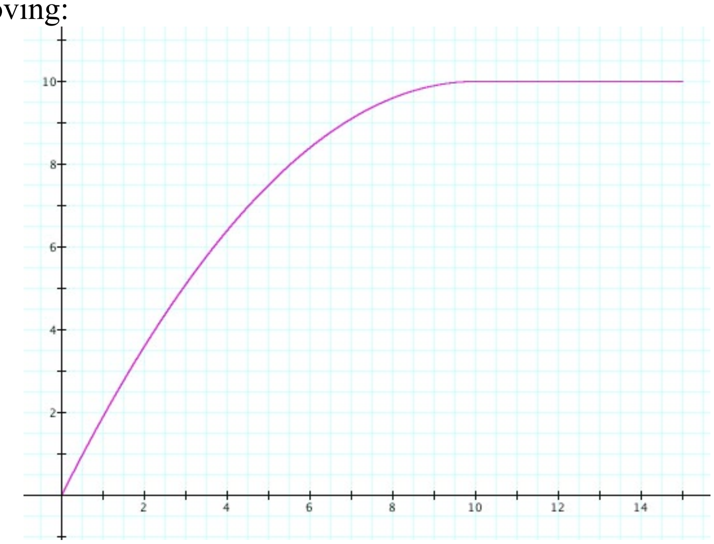



3. Miss Coombs' new years resolution is to run more. Sketch a graph of her **TOTAL** distance relative to the number of minutes she's been jogging while she was out on the following run:

   - She began at a steady jog for 5 minutes (warming up, you know).
   - Then, she picked up the pace to a run, running top speed for 3 minutes.
   - She got a huge cramp and had to slow way down. She went at this slower speed for 4 minutes.
   - The cramp would not go away so she stopped for one minute
   - After that, she decided to toughen up and she sprinted all the way back home



4. The following graph tracks an object's SPEED relative to the number of seconds it had been moving:




a. Describe this object's motion over the 15 seconds shown in the above graph.

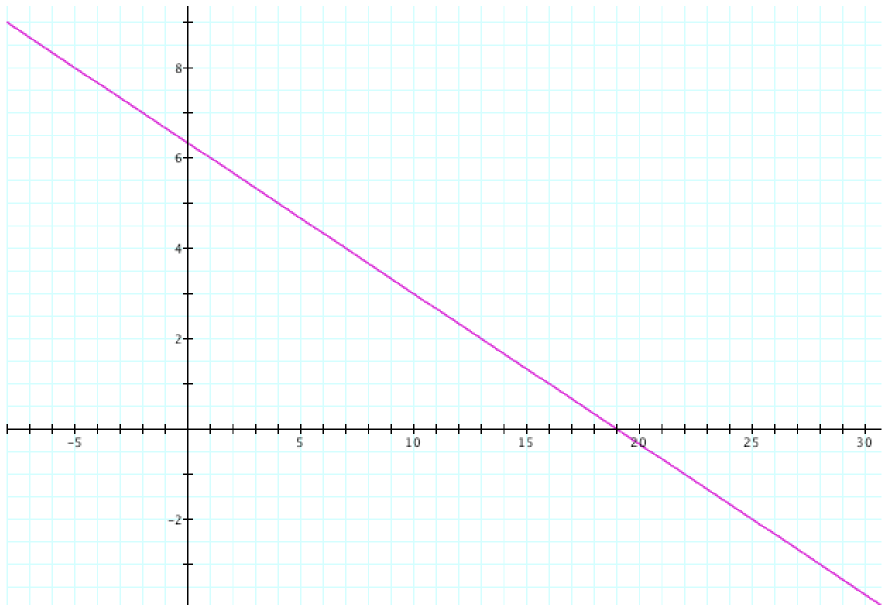
b. Sketch a graph (don't worry about total accuracy) of the object's *distance from start* in relation to the number of seconds it has been moving. Explain your graph (use the back of this sheet if necessary). ☺

1st Hour Topics 2 Name: ___________________________________

# Linear Function Test ☺

February 5, 2007

1) Use the graph below of the linear function that passes through (4,5) and has a rate of change m = -1/3.




a) Circle the initial value (b)

b) Draw the "calculus triangle" that takes the given point to the initial value

c) Label the side lengths of the triangle you just drew. Label the change in x and the change in y. Describe how you would calculate those changes

2) During the Superbowl yesterday, the Bears were really trying to cover some yards. After 11 minutes the Bears had a total of 44 yards. After 23 minutes the Bears had a total of 101 yards. Assuming they were gaining yards at a constant rate, find the rate of change.

3) Convert the function y = -4x + 5 into ax + by = c form

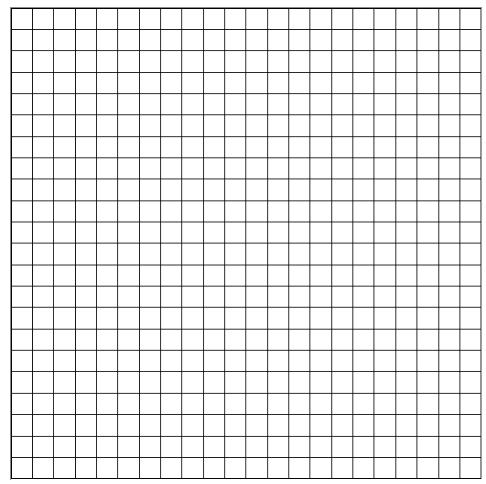
4) Convert the function -2x + 4y = 12 into y = mx + b form

$1^{st}$ Hour Topics 2 Name: ___________________________________

5) My fiancé had a bet with one of his buddies. They wanted to see who could eat the most chicken wings. I showed up a little late, but when I got there, my fiancé was winning by 3 wings. I estimated he was eating 2.5 wings per minute.

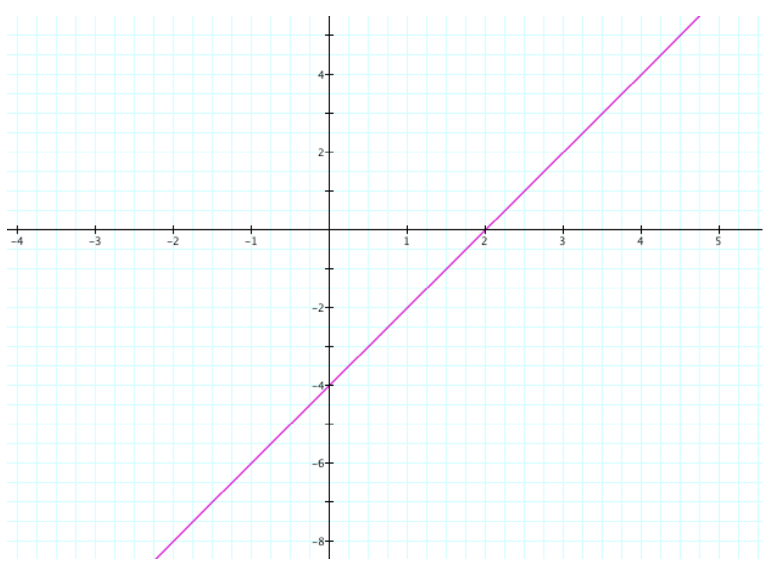
a) Define a linear function in the form y = mx + b that relates the number of wings my fiancé has eaten and the number of minutes I've been there

b) Graph the linear function in part a



6) Define a linear function in the form y = mx + b with a constant rate of change that has a graph that passes through the points (3, -4) and (-1, 9)

7) Write the function definition that would produce the following graph:



1st Hour Topics 2 Name: ___________________________________

**Systems Test**
February 23, 2007

1) Spring break is almost here and I think I need a vacation! While I'm there, I plan on renting a car. Hertz charges a $100 insurance deposit plus $35 per day. Enterprise charges a $60 insurance deposit and $50 per day.

a) Write the system of linear functions that would model this situation

b) Solve the above system

c) What does the solution mean with regards to which company I should go with?

2) The physics department at MHS is planning a field trip to Six Flags Magic Mountain. There are some students going, and some adult chaperones. The park charges a special rate for these occasions: $32 for a student and $40 for an adult chaperone. A total of 65 people are going on this trip and it's going to cost them $2200.

Set up AND SOLVE a system of equations that finds how many students there are and how many adult chaperones there are.

3) Explain when a system of linear functions will have NO solution. Explain what it means both graphically and how the function definitions will be related

4) Explain when a system of linear functions will have INFINITE solutions. Explain what it means both graphically and how the function definitions will be related.

 Name: ______________________________

6) Which of the following points would be a solution to the following system of inequalities? (Circle all that apply)

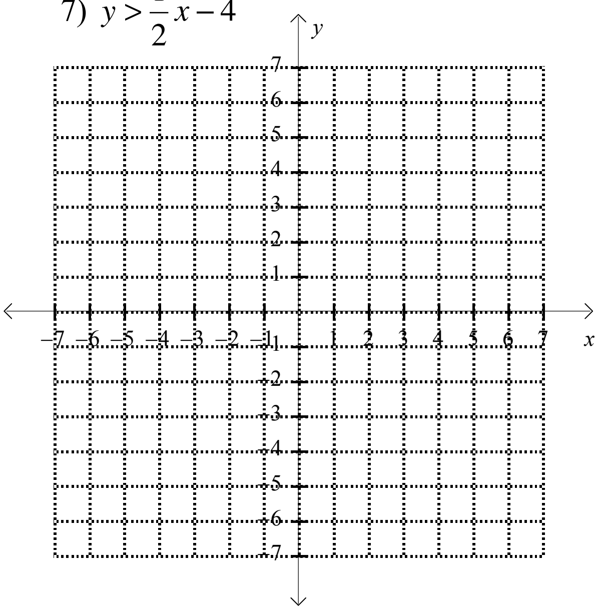
$y > -x + 2$
$y \leq 2x - 1$

a) (0, 0) b) (2, 4)

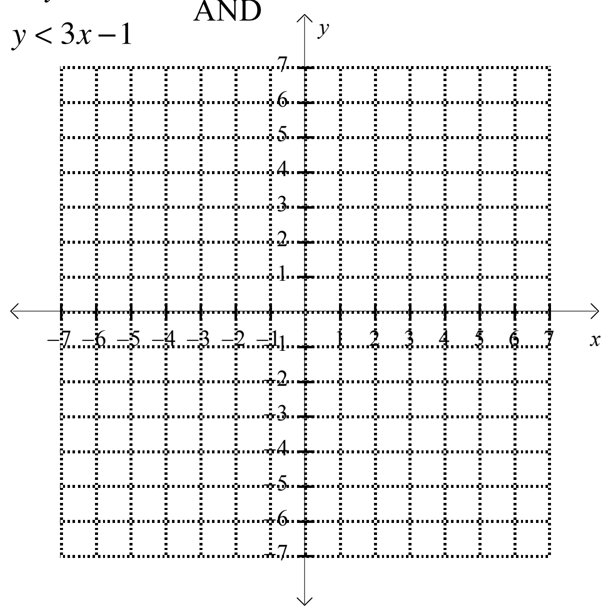
c) (3, -4) d) (5, 0)

Find all of the points that satisfy the following inequalities:

7) $y > \frac{1}{2}x - 4$



8) $-2y \leq 4x - 6$ AND $y < 3x - 1$



9) The graph of a linear function, we'll call it "F", passes through the points (3,7) and (5,1).

a) What is the rate of change of a function whose graph is perpendicular to the graph of F?

b) What is the rate of change of a function whose graph is parallel to the graph of F?

**Solve the following systems linear functions:**

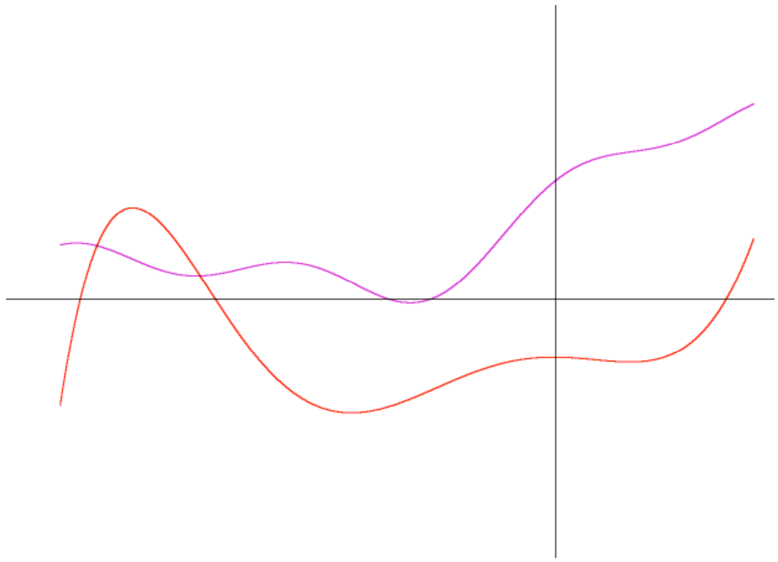
11) $y = -2x + 11$
$y = x - 7$

12) $3x - 2y = 10$
$5x + 5y = 0$

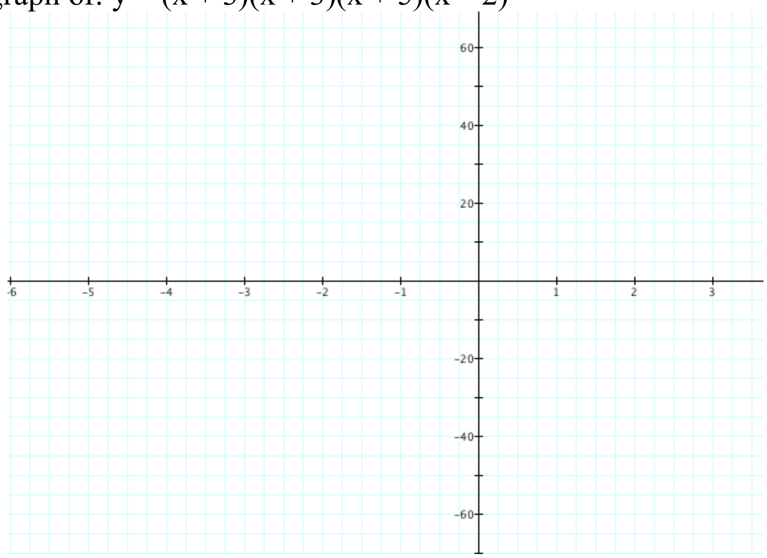
Coombs 1st Hour topics 2 Quiz                    Name: ______________________________________
April 5, 2007

Graphs, Sums, and Factored Form Quiz
April 5, 2007

1) Two functions, *f* and *g*, have the graphs given below. Sketch a graph of the sum of *f* and *g*:

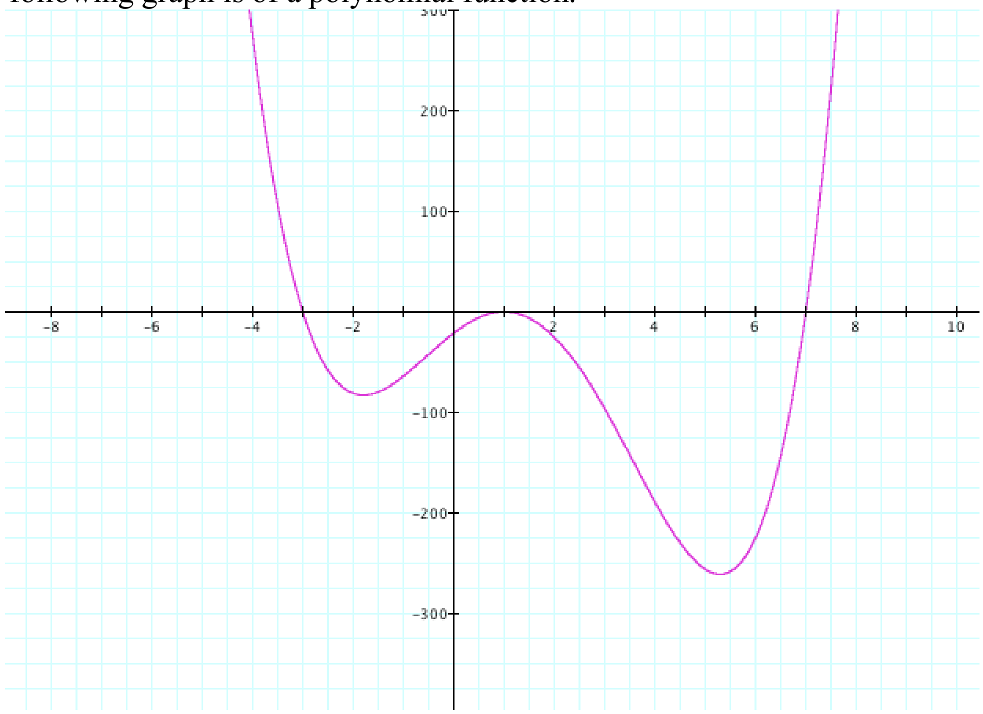


2) Sketch the graph of: y = (x + 3)(x + 3)(x + 5)(x – 2)



Coombs 1st Hour topics 2 Quiz Name: ______________________________________
April 5, 2007

3) The following graph is of a polynomial function.




a. Write the function's definition in factored form.

b. Answer this question without expanding the factored form you wrote in part (a).

When the function's definition is written in standard form, the largest exponent of *x* is _____ because

Expand the following function definitions:

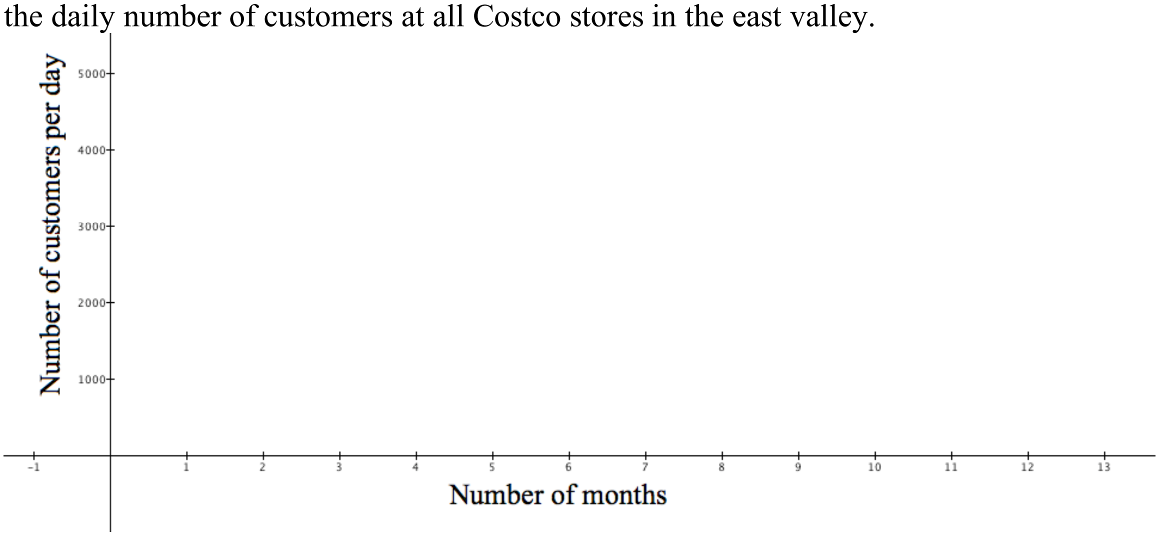
4) $(2x - 5)(x + 3)$ 5) $(x - 9)(2x - 1)$ 6) $(x + 4)(x^2+5x+7)$

Coombs 1st Hour topics 2 Quiz Name: ______________________________
April 5, 2007

7) Costco has two stores in the east valley. Corporate headquarters has asked them to estimate the number of employees they will need over the upcoming year. Each store manager produced a graph (based on historical data) of the anticipated number of customers per day at that store. Their two graphs are shown below on the same coordinate system. The *x* axis represents the number of months in the upcoming year. The *y* axis represents the number of people expected at a store each day.


a.). What does the point (6.25,1854) represent on Store A's graph?

b) The corporate office does all the hiring for these stores. So it only needs to have an estimate of the stores' *total* number of customers. Using the axes below, sketch a graph of the daily number of customers at all Costco stores in the east valley.



Coombs 1st Hour topics 2 Quiz Name: ______________________________________
April 5, 2007

8) Explain the following about the graph of the function $y = x^{62}$


a) Why are the y-values always positive?

b) Why does the graph look so "flat" between -1 and 1?

c) Describe the function's behavior as *x* varies from 0.5 to 1.5.

1st Hour Topics 2 Name: ___________________________________

**Quadratics Quiz**
April 17, 2007

1) Gina expanded a product of two factors. This was one of the lines in Gina's work.

$$\vdots$$
$$2x(x+4)+3(x+4)$$
$$\vdots$$

Write the product that Gina was expanding. _________________________

2) Write the following function definitions in factored form:

a) $y = x^2 - 2x + 12$ b) $y = x^2 - 11x + 18$

c) $y = -x^2 - 4x + 12$ d) $y = 6x^2 + 17x + 5$

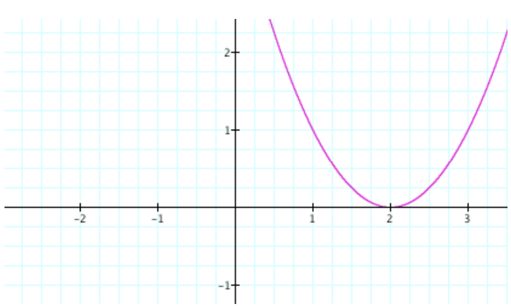
3) $y = x^2 - 10x - 24$ can be rewritten as $(rx + s)(tx + u)$ for certain numbers r, s, t, and u.

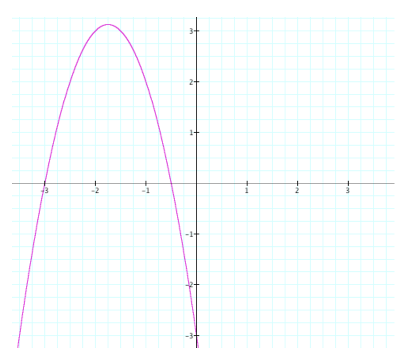
a) What is the product of s and u? ________ b) What is the product of r and t? ________

4) Expand each function definition. Show your work:

a) $y = (2x + 7)(x - 3)$ b) $y = (x + 2)(x + 4)$ c) $y = (x - 1)(x - 8)$

5) Here are graphs of two quadratic functions. Write the definition of each function in expanded form.

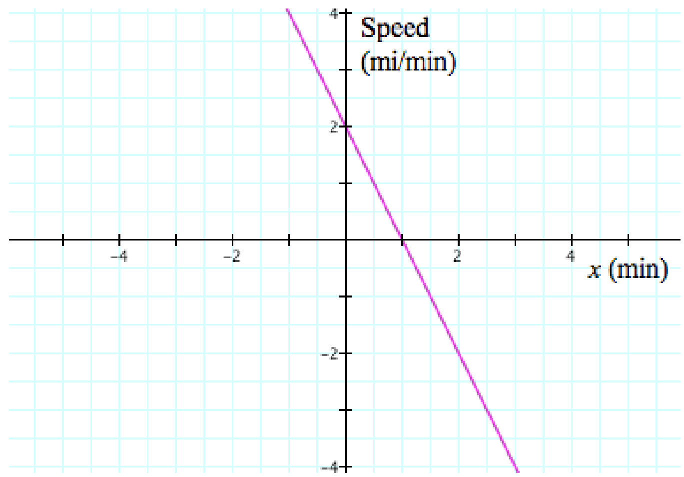


1st Period Math 2 Name: ______________________________

Quadratics Quiz
May 4, 2007

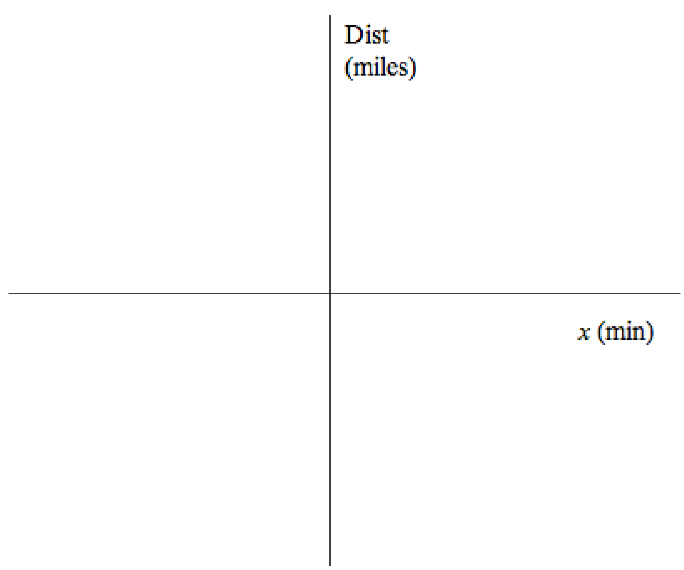
1. This graph gives Spiderman's speed in miles per minute as he chased Venom through the skies. (The graph doesn't start at 0 because *x* stands for the number of minutes since the director said "action", but Spidey was chasing Venom even before that.) Spidey's distance was measured in miles from the director's station.




   a) The point (0.6, 0.8) is on Spidey's speed graph. What does this point represent?

   b) The point (1.5, -1) is on Spidey's speed graph. What does this tell you about Spidey's chase after 1.5 minutes?

   c) Sketch a graph of Spidey's distance from the director's station given that Spidey was 1 mile in front of the director when he called "action".



1st Period Math 2 Name: ____________________________

2. These questions are about functions and quadratics.

   a) What does it mean when a function’s rate of change is 0?

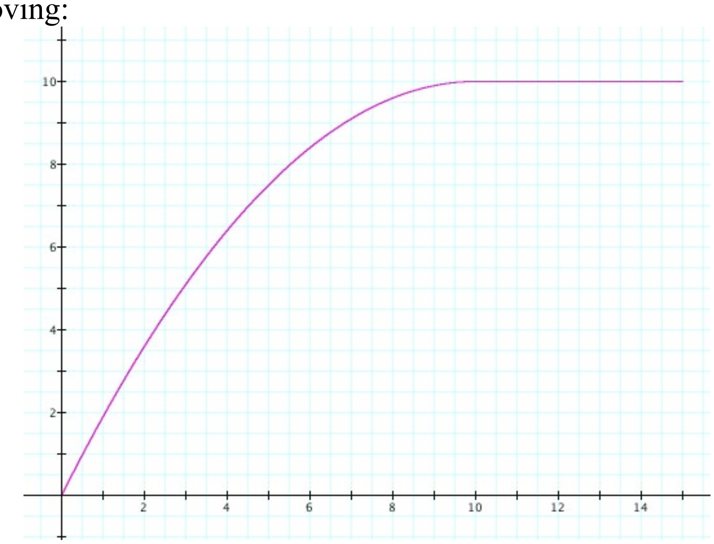
   b) What does it mean about a quadratic when its rate of change is 0?

   c) How do you know whether the vertex of the graph of a parabola is a maximum or a minimum?

   d) What happens to a function’s rate of change nearby a maximum or a minimum?

1st Period Math 2 Name: ______________________________

3. The following graph tracks an object's SPEED relative to the number of seconds it had been moving:



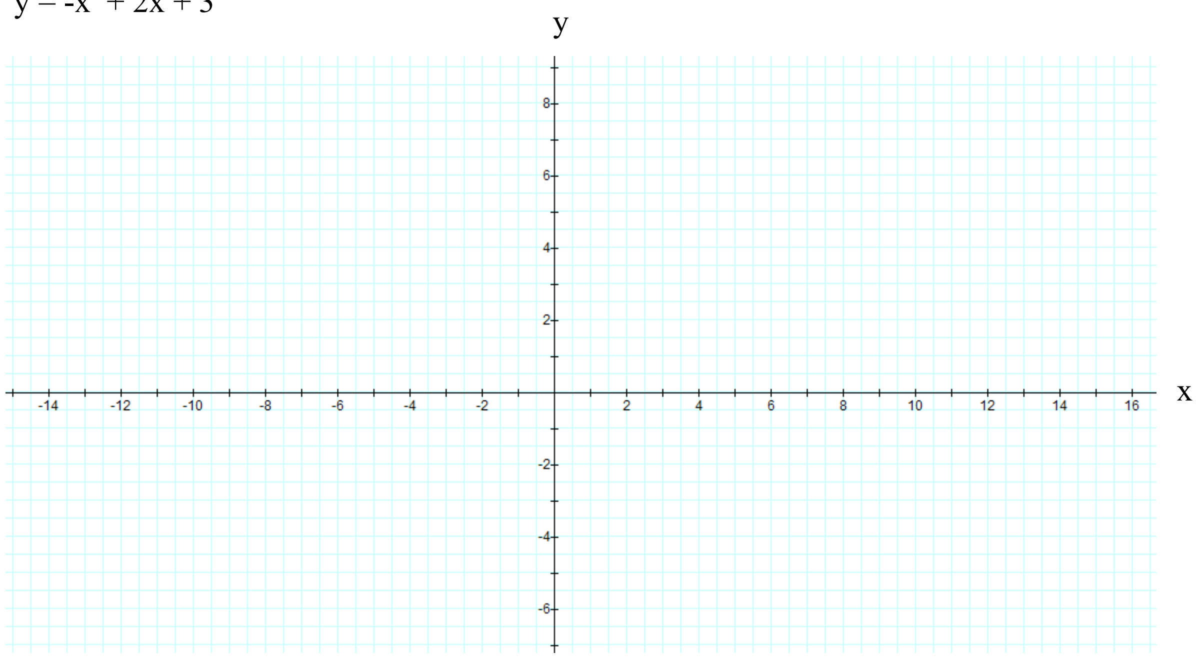

a) Describe this object's motion over the 15 seconds shown in the above graph.

b) Sketch a graph (don't worry about total accuracy) of the object's *distance from start* in relation to the number of seconds it has been moving. Explain your graph (use the back of this sheet if necessary). ☺

4. If a quadratic function had a rate of change equation of $y = 6x - 10$, what would the original quadratic function definition be?

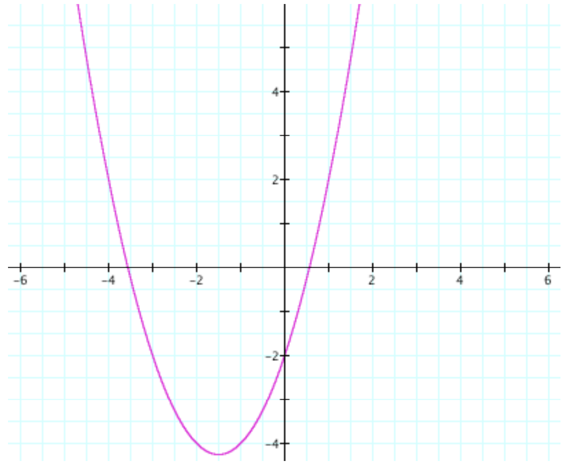
5. Graph the following function. Determine the zeroes, the vertex, and the initial values.

$y = -x^2 + 2x + 3$



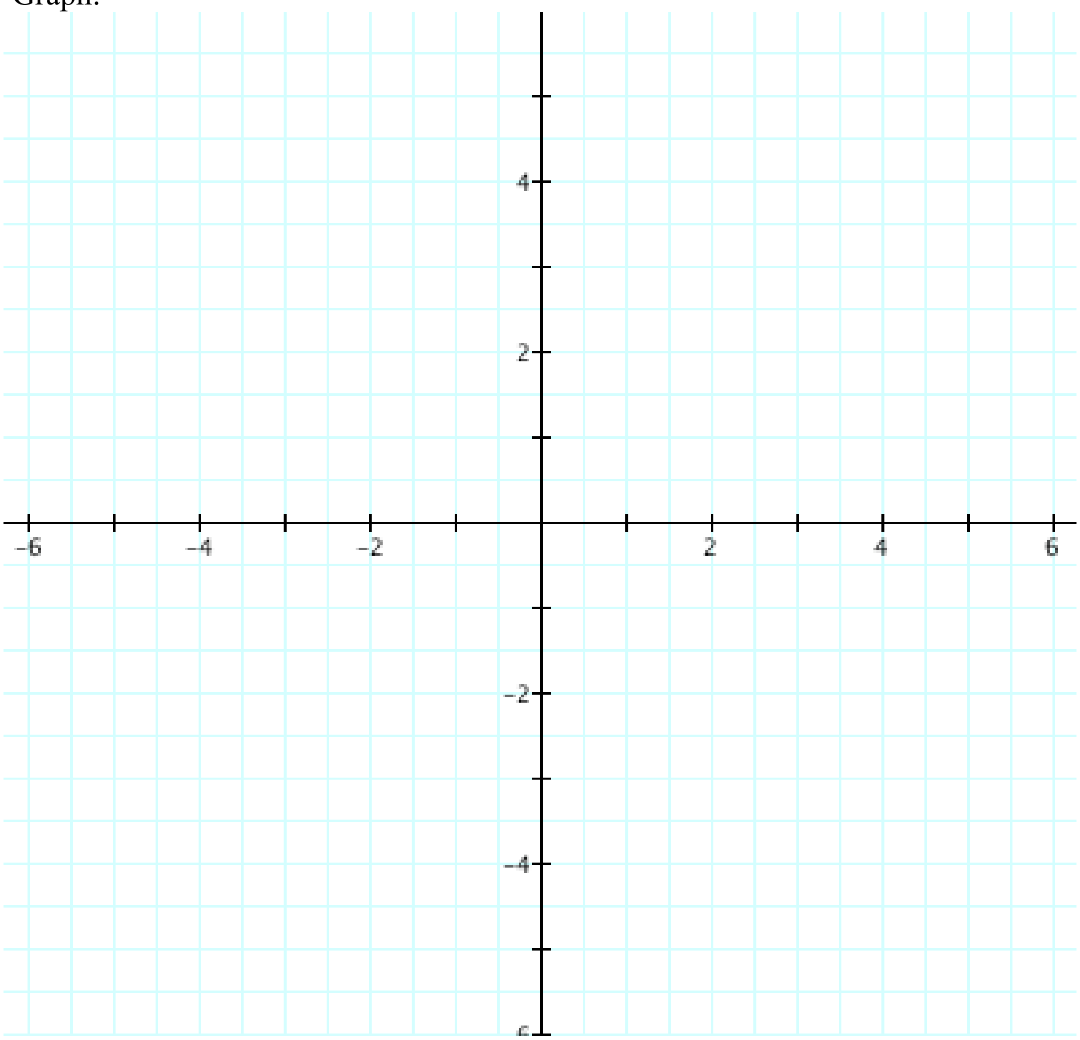
1st Period Math 2 Name: ______________________________

6. This graph is created by the function $y = x^2 + 3x - 2$. Write the definition of its ROC function and then sketch a graph relating the function's rate of change to values of x (make a table of values if necessary).



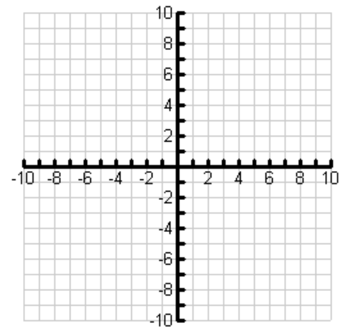

Definition: __________________________

Graph:

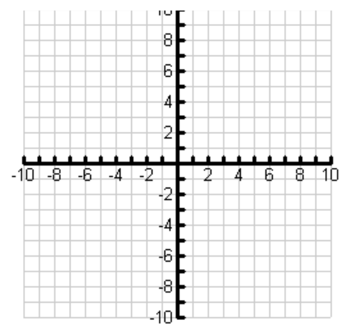


1st Period Math 2 Name: ______________________________

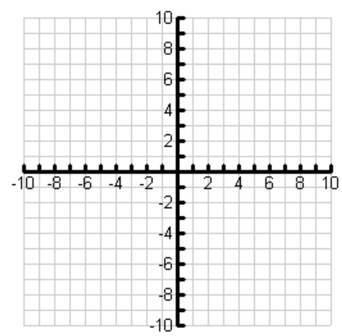
7. Given the following characteristics, **sketch** what the quadratic graph would look like…



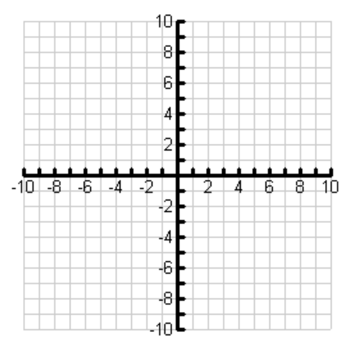
a) The vertex is a maximum and the function has no zeroes:



b) The vertex is a minimum and the function has one zero:



c) The ROCs change is positive and the function has two zeroes.



d) The ROCs change is negative and the function has no zeroes.