APPLICATIONS OF GROUPS AND ISOMORPHIC GROUPS TO TOPICS IN THE STANDARD CURRICULUM, GRADES 9–11: PART I

Many relationships between groups and topics of secondary school mathematics are shown by the author, who proposes that the study of groups be included as standard fare in the mathematics curriculum of the average college-bound student.

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THE subfield of pure mathematics that has grown most significantly in the past few decades is that of algebra, by which is meant "higher" or "abstract" algebra and linear algebra. Twenty years ago courses in algebra were at the advanced undergraduate and graduate level, and it was easy to become a certified mathematics teacher without having any knowledge of groups, rings, fields, or vector spaces. Today virtually all prospective teachers take a course in which some of these structures are studied.

Yet, if we judge the situation from our textbooks, we find most students who take eleven years of school mathematics rarely explicitly encounter any of the common algebraic structures except by way of discussions of field properties. Even though such material often appears early in texts and is seemingly necessary for future work, the ideas are seldom used in any constructive way.

At the present time, in the United States, only books written for *bright* students have used structures as an integral part of the development. Two examples are *A Vector Approach to Euclidean Geometry,* by Vaughan and Szabo (New York: Macmillan, 1972) and *Unified Mathematics,* vols. 1–3, by Fehr, Fey, and Hill (Reading, Mass.: Addison-Wesley Publishing Co., 1972). Yet in foreign texts, groups and other structures are not uncommon even for average students. For example, see the texts of the School Mathematics Project (New York: Cuisenaire Co.) and *Modern Mathematics 1* and *2*, by Papy (New York: Macmillan Co., 1968, 1970).

The United States situation, in view of the importance that algebra now has in mathematics, may be serious. A recent survey found that 15% of the doctorates in mathematics in 1968–71 were in algebra, second only to analysis (21%) among the branches (*CBMS Newsletter,* vol. 7, no. 3, May 1972). However, it cannot be assumed that something ought to be studied by high school students just because it is important to mathematicians. But it can be assumed that algebraic structures should at least be given serious consideration. "Groups" are among the most fundamental of these structures and seem to be very amenable to early study by students. For example, several books by Dienes and Golding (1967 [a], [b], and [c]) contain activities for children in grades 3 through 6.

The major purpose of this article is to exhibit relationships between groups and various topics in secondary school mathematics. By presenting a wide variety of topics and approaches, it is hoped to add evidence to the increasingly strong case for the inclusion of some study using groups as standard fare in the mathematics curriculum of the average college-bound student. (Part II of this article will appear in next month's issue.)

Preliminary Remarks

The author has experimented with various ways of introducing groups to tenth and eleventh graders. For average students it seems best not to begin with a definition and examples. A more natural approach is to begin with a variety of situations in which the group properties appear but the groups are not identified and to use the identification of groups as a device to summarize and unify previously unjoined content. Similarly, it seems better to present numerous situations involving isomorphic groups before either the terminology or any formal definitions are given. Approaches using these strategies may be found in materials currently being tested (Usiskin 1972), and some similar ideas have been used by Dienes at the elementary level (Dienes and Golding 1967 [a], [b], and [c]). However, in this article, some definitions and examples are given before any applications are given to make it easier to identify specifically the mathematics involved in each application.

Here is a rather standard definition of "group."

DEFINITION: *Let * be a binary operation on a set S. Then S and * constitute the group ⟨S, *⟩ if and only if*

1. ** is closed in S*
2. ** is associative in S*
3. *there is an identity for * in S*
4. *each element in S has an inverse under *.*

The properties of closure, associativity, the existence of an identity, and the existence of inverses are called *group properties* here. It is notable that commutativity is not a group property.

Students in grades 10 and 11 seem to find it rather easy to check whether a set and an operation form a group, since the four properties have almost always been taught to them some time or another by the end of the ninth grade.

Here are some notable groups involving some or all of the real numbers. Groups 1–3 are *additive groups* because the operation (denoted by +) is addition. Groups 4–9 are *multiplicative groups* ("·" stands for multiplication). That the operation is just as important as the set will be seen from some of the applications.

1. ⟨set of integers, +⟩
2. ⟨set of rational numbers, +⟩
3. ⟨set of real numbers, +⟩
4. ⟨set of positive rationals, ·⟩
5. ⟨set of nonzero rationals, ·⟩
6. ⟨set of positive real numbers, ·⟩
7. ⟨set of nonzero real numbers, ·⟩
8. ⟨{0}, ·⟩
9. ⟨{1, −1}, ·⟩

Group 8 is notable in that the multiplicative identity for this group is 0. Group 8 is also the only possible multiplicative group whose set contains the real number 0. Groups 8 and 9 are *finite groups;* the other groups are *infinite.*

Isomorphic groups are best introduced by example. Here we list the integers and the integral powers of 2. The listing is arranged in such a way that m and 2^m are on the same horizontal line.

m	2^m
0	1
1	2
−1	.5
2	4
−2	.25
(3)	(8)
−3	.125
4	16
(−4)	(.0625)
5	32
−5	.03125
⋮	⋮
Add	Multiply

Two horizontal rows have been identified by drawing circles around the elements. Now, using the two marked rows, we *add* the elements in the left column and *multiply* the elements in the right column.

$$3 + -4 = -1 \qquad (8)(.0625) = .5$$

The answers (-1 at left, .5 at right) lie in the same row—the third row from the top. This is a characteristic of isomorphic groups—the answers correspond.

DEFINITION: *Two groups* $\langle S, * \rangle$ *and* $\langle T, \# \rangle$ *are isomorphic if and only if there is a one-to-one correspondence between the elements of* S *and* T *so that*
if a *in* S *corresponds to* x *in* T
and b *in* S *corresponds to* y *in* T
then $a * b$ *corresponds to* $x \# y$.

In this example, S is the set of integers and T is the set of integral powers of 2. $*$ is addition and $\#$ is multiplication. The one-to-one correspondence is given by the rule

$$m \longrightarrow 2^m.$$

That is, the one-to-one correspondence is the one-to-one function f, with $f(m) = 2^m$.

How do we know that $m \rightarrow 2^m$ is an isomorphism? This is easy to prove using the fundamental property of exponents.

Given

$$m \longrightarrow 2^m \qquad \text{for any integer } m,$$

then

$$n \longrightarrow 2^n$$

and certainly

$$m + n \longrightarrow 2^{m+n}.$$

But we know that

$$2^{m+n} = 2^m \cdot 2^n,$$

so

$$m + n \longrightarrow 2^m \cdot 2^n.$$

Thus, adding the integers and multiplying the corresponding powers of 2 yield corresponding answers. The correspondence is one-to-one because different integers give rise to different integral powers and vice versa.

There seem to be as many applications of isomorphic groups in elementary mathematics as there are of groups themselves. In particular, the isomorphism $m \rightarrow 2^m$ is of a type applied in Part II of this article.

Some properties of isomorphic groups are worth noting. (The reader should refer back to the columns of integers and powers.) In isomorphic groups, the identities correspond (0 and 1 in the top row). Inverses in one group correspond to inverses in the other group. (For example, 5 and -5 in the additive group of integers correspond to 32 and .03125 in the isomorphic multiplicative group of powers of 2.) And if one set and operation form a group and there is a one-to-one correspondence with another set and operation and answers correspond, then the second set and operation form a group.

Groups and Sentence Solving

The applications of groups to sentence solving are well known, found in some high school texts, yet important enough to deserve repeating here.

Essentially the idea is this: A set S and an operation $*$ constitute a group exactly when there is a *unique* solution in S to each of the equations

$$a * x = b \qquad y * a = b$$

regardless of the choice of a and b from S.

Application 1: It is exactly the group properties that are necessary to show that the equation $a + x = b$ has a unique solution when a and b are arbitrarily selected from a set S and x is to be in S.

Given

$$a + x = b$$

Existence of inverses in S
Well-definedness of $+$

$$-a + (a + x) = -a + b$$

Associativity of $+$

$$(-a + a) + x = -a + b$$

Property of $-a$

$$0 + x = -a + b$$

0 is identity for $+$

$$x = -a + b$$