APPLICATIONS OF GROUPS AND ISOMORPHIC GROUPS TO TOPICS IN THE STANDARD CURRICULUM, GRADES 9–11: PART II

Many relationships between groups and topics of secondary school mathematics are shown by the author, who proposes that the study of groups be included as standard fare in the mathematics curriculum of the average college-bound student.

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IT IS possible to form a set in an infinite additive group by beginning with any nonzero real a and adding it to itself over and over again, then including zero and the opposites of all numbers in the set. We call such a set the set of integral multiples of a. If $a = 3$, then here is such a set.

$$\{0, 3, -3, 6, -6, 9, -9, \cdots\}$$

With addition, this set forms a group, the group of *integral multiples* of 3.

In Part I of this article, the set of integral powers of 2 was seen to form a group with multiplication. This set can be thought of as beginning with 2, multiplying 2 by itself over and over again, then including 1 and the reciprocals of the numbers in the set. Thus the sets of integral powers and integral multiples are formed by analogous means, with the only differences being the number begun with (the *generator*) and the operation used. This hints at the existence of isomorphic groups. Listing a possible correspondence between the sets shows that this is the case.

⟨set of integral multiples of 3, +⟩	⟨set of integral powers of 2, ·⟩
0	1
3	2
−3	.5
6	4
−6	.25
9	8
−9	.125
⋮	⋮

Part I of this article, which appeared in the February issue of the MATHEMATICS TEACHER, contains applications of groups to sentence solving, systems, and real- and complex-number operations. Definitions and examples of groups and isomorphic groups are given there.

The reader is requested to add any two numbers in the left column, multiply the corresponding numbers in the right column, and check that the answers correspond, thus verifying the isomorphism. The reader should also *subtract* one number from another in the left column, *divide* the corresponding numbers in the right column, and again check that the answers correspond.

This particular example of isomorphic groups involving multiples and powers is very good for giving the idea of what is meant by an isomorphism, but the specific numbers can disguise the applications. It is easier to work in general terms.

Application 8: Isomorphic groups of multiples and powers can be used to help students understand fundamental properties of powers.

We begin by examining the additive group of multiples. Let us suppose that the group is generated by the number a. Then an element of the set of multiples will be of the form ma, where m is an integer. Then:

1. Closure of multiples is indicated by the distributive property, $ma + na = (m + n)a$.
2. The identity multiple occurs when m is 0, $0a = 0$.
3. Inverse multiples are of the form ma and $(-m)a$.

Most students are familiar with these three properties (even if not the present context) before they have worked much with powers. But now let us consider the corresponding properties for the multiplicative group of powers. If the group is generated by x, where $x \neq 1$ and $x \neq 0$, then an element of the set of powers will be of the form x^m, where m is an integer. Since the two groups are isomorphic, for each property of multiples there will be a corresponding property of powers.

1. Closure of powers is indicated by the power property: $x^m \cdot x^n = x^{m+n}$.
2. The identity power occurs when m is 0: $x^0 = 1$.
3. Inverse powers are of the form x^m and x^{-m}.

For most students, properties of powers seem unrelated to properties of multiples. But, in fact, *every* property of multiples has a corresponding property of powers.

4. The multiple of a multiple is a multiple:

$$n(ma) = (nm)a$$

5. The multiple of a sum is the sum of the multiples:

$$m(a + b) = ma + mb$$

4. The power of a power is a power:

$$(x^m)^n = x^{mn}$$

5. The power of a product is the product of the powers:

$$(xy)^m = x^m y^m$$

This approach to properties of powers can help the student realize that properties of powers are naturally connected with multiplication and that any connections of powers and addition are tenuous. (Thus $(x + y)^2 = x^2 + y^2$ is unreasonable.) If a student thinks that zero or negative powers are unnatural, he should be reminded that they are no more unnatural than zero or negative multiples. Advanced students can be shown that property 2 can be proved from property 1 using corresponding proofs.

Given: $ma + na = (m + n)a$ $\qquad x^m \cdot x^n = x^{m+n}$.

Let $m = 0$, $n = 1$. Then:

$$0a + 1a = (0 + 1)a \qquad x^0 \cdot x^1 = x^{0+1}$$
$$0a + 1a = 1a \qquad x^0 \cdot x^1 = x^1$$

Since in a group the identity is unique, $0a$ must be the additive identity and x^0 the multiplicative identity.

The correspondences can be used to generate new properties of powers. Let us consider our next property.

6. The multiple of a difference is the difference of the multiples.

What property of powers would correspond? *Answer:* The power of a quotient is the quotient of the powers. That is, $m(a - b) = ma - mb$ corresponds to

$$\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}.$$

A student can use this idea to check properties of powers by examining what would be the corresponding property of multiples. Also, the correspondences give meaning to zero and negative integral exponents. Even more is possible.

Application 9: Isomorphic groups of multiples and powers can be used to help students understand fractional powers.

Students should be taught that fractional powers are no more unnatural than fractional multiples. For instance, $\frac{1}{2}a$ results from solving an equation involving multiples ($2b = a$) in the same manner that $x^{1/2}$ results from solving an equation involving powers ($y^2 = x$). But in order to maintain the one-to-one correspondence necessary in an isomorphism, we must require that the group of rational powers have a positive generator and only positive elements. Thus, we now restrict x to be positive, and $x^{1/2}$ stands only for the positive square root of x.

Here are some corresponding properties of rational multiples and rational powers:

1. $\frac{1}{2}a$ is the number that, when added to itself, gives a.

2. Half of a sum is the sum of the halves:

$$\frac{1}{2}(a + b) = \frac{1}{2}a + \frac{1}{2}b.$$

(Each is the arithmetic mean of a and b.)

3. $$\underbrace{\frac{1}{n}a + \cdots + \frac{1}{n}a}_{n \text{ terms}} = \frac{n}{n}a = a.$$

4. $$\frac{m}{n}a = m\left(\frac{1}{n}a\right) = \frac{1}{n}(ma).$$

1. $x^{1/2}$ is the number that, when multiplied by itself, gives x.

2. The square root of a product is the product of the square roots:

$$(xy)^{1/2} = x^{1/2}y^{1/2},$$

or $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$.

(Each is the geometric mean of x and y.)

3. $$\underbrace{x^{1/n} \cdot \cdots \cdot x^{1/n}}_{n \text{ factors}} = x^{n/n} = x.$$

4. $$x^{m/n} = (x^{1/n})^m = (x^m)^{1/n},$$

or $x^{m/n} = (\sqrt[n]{x})^m = \sqrt[n]{x^m}$.

The properties given in (4) demonstrate how notations can disguise relationships between properties.

There are more than seventy such corresponding properties. Many of these are given in a different paper devoted to this idea (Usiskin, 1974). Here we give only one further pair of properties.

5. If p is between m and n, then pa is between ma and na.	5. If p is between m and n, then x^p is between x^m and x^n.

We might find a decimal approximation to $3\sqrt{2}$ by noting that this number is between 3(1.41) and 3(1.42). Similarly, $5^{\sqrt{2}}$ lies between $5^{1.41}$ and $5^{1.42}$. Thus the above properties allow irrational multiples and powers to be interpreted. These are necessary for the next application.

Application 10: Logarithms can be developed through a consideration of isomorphic groups of real numbers.

It is usually easier to work in a group of multiples than in a group of powers because addition is easier than multiplication. The largest possible isomorphic groups of real multiples and powers are these:

the additive group of real multiples of a, $a \neq 0$	the multiplicative group of real powers of x, $x > 0$, $x \neq 1$

Subgroups of these have been studied in this article: The additive group of *integral* multiples of a and the multiplicative group of *integral* powers of x in application 8, and the corresponding groups involving *rational* multiples and powers in application 9. Actually, the names for these largest groups can be misleading. Since any real number is a real multiple of any other nonzero real and any positive real is a real power of any non-"one" positive real, the groups may be renamed as the following:

the additive group of reals	the multiplicative group of positive reals

What does isomorphism between these two groups mean? It means that if a and b from the additive group correspond to x and y from the multiplicative group, then the following correspond:

	Add reals	*Multiply positive reals*
1.	$a + b$	xy
2.	$-b$	$\frac{1}{y}$
3.	$a + -b = a - b$	$x \cdot \frac{1}{y} = \frac{x}{y}$
4.	na	x^n

In words, addition corresponds to multiplication, opposites to reciprocals, subtraction to division, and multiples to powers. Furthermore, the identities correspond.

5.	0	1

This is the only correspondence involving two known numbers that always holds. If one knew how to make the other elements in the groups correspond, then any multiplication could be converted to an addition problem, any calculation of reciprocals to a calculation of opposites, and so on. This is the fundamental idea behind logarithms. The numbers at the left will be the logarithms of the corresponding numbers at the right. Correspondences involving other numbers are determined by what corresponds to 1. Here we let B correspond to 1 and call B the *base* of the correspondence:

Logarithm	*Number*
1	B

Notice that B can be any positive number but 1. (If $B = 1$, the correspondence is not one to one.) Because multiples correspond to powers, we obtain the following:

	$1 + 1 = 2$	$B \cdot B = B^2$
6.	n	B^n
	$.5$	$\sqrt{B}$
	$.1$	$\sqrt[10]{B}$

This correspondence $n \to B^n$ (domain at left) is the exponential function with base B. The inverse correspondence (domain at right) is the logarithm function with base B. We commonly choose $B = 10$ in first meetings with logarithms, although for applications $B = e$ is most natural. When $B = 10$,

m	is the logarithm of	10^m
1		10
2		100
-1		.1
-2		.01
.		.
.		.
.		.

The advantage of 10 is well known. If the logarithm of a number in decimal notation is known, then others can be derived from it. Suppose

.3010	is the logarithm of	2.

Then multiplication at right corresponds to addition at left, and any number with scientific notation involving 2 has a logarithm whose decimal part is .3010. For example:

$.3010 + m$	is the logarithm of	$2 \cdot 10^m$
$.3010 - 4$	is the logarithm of	.0002