A Decade of the Berkeley Math Circle – the American Experience

∞ What have we learned? ∞

by

Zvezdelina Stankova

Berkeley Math Circle Director
Mathematics and Computer Science Department
Mills College, Oakland, CA

April 16 2009, MSRI

For the bored die-hards:

Build a math circle session around each problem.

Extra credit

Solve each problem.

Three problems.

Questions to think about:

Math area(s)?

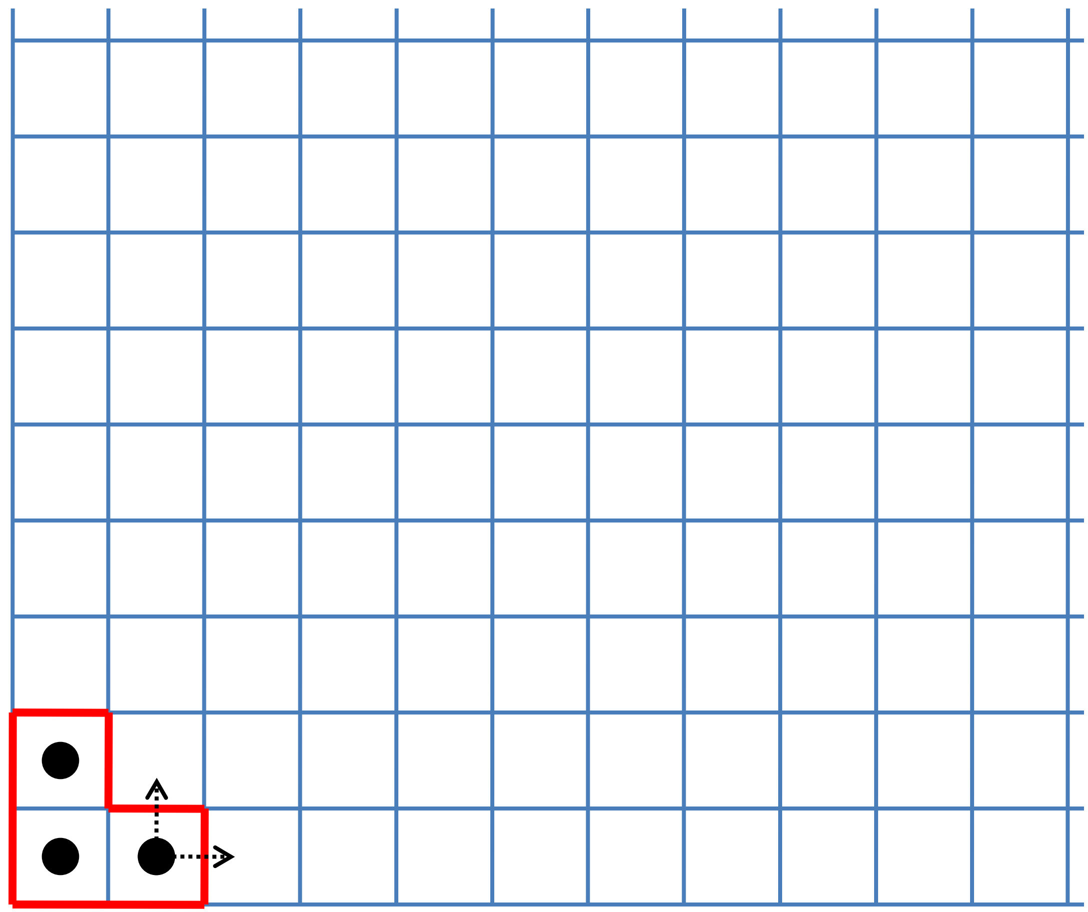
Math theory?

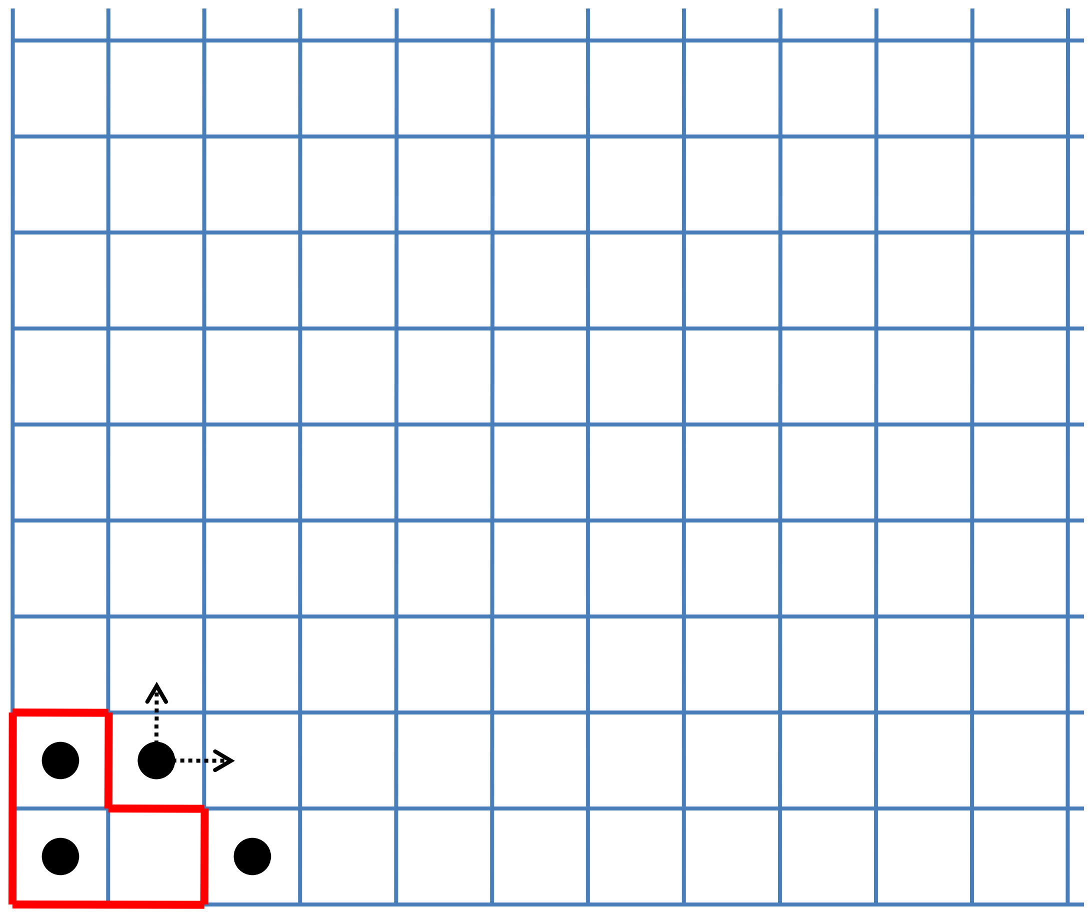
Problem Solving Techniques?

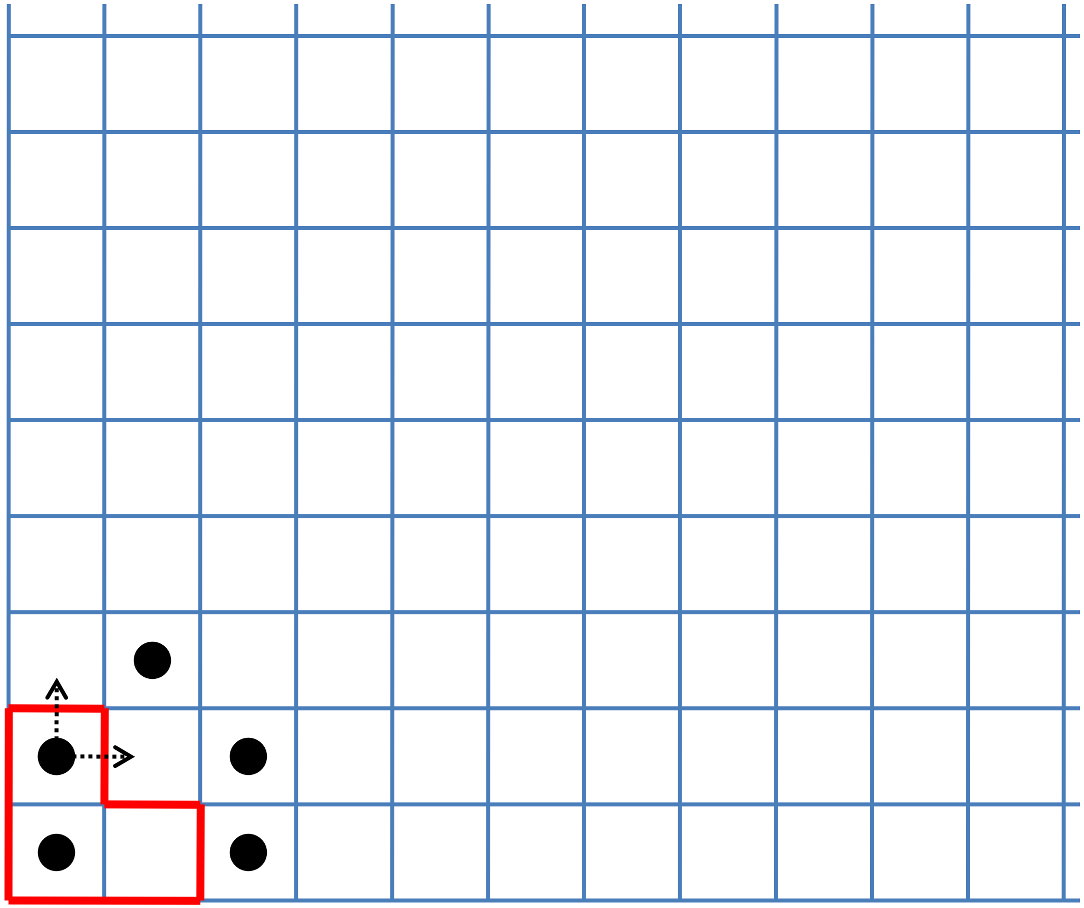
Other related problems?

Generalizations? Pushing on?

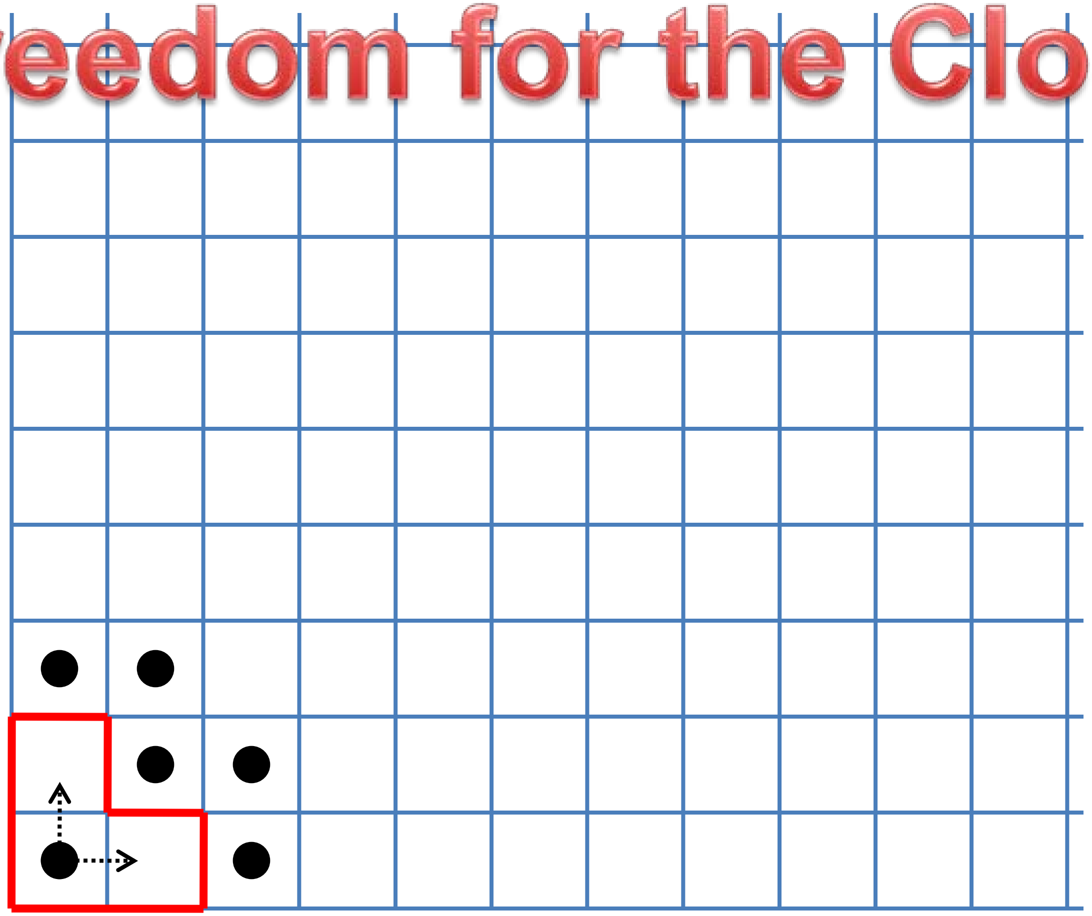
How many different math circles sessions do you envision per problem? On what did you base your decision on?







1. Freedom for the Clones



Question: Can you free all clones from the prison?

Introduced: by *Sam Vandervelde* at BMC

2. Infinite raffle

3	3	1	6	6	4	4	5	5	5	5	2

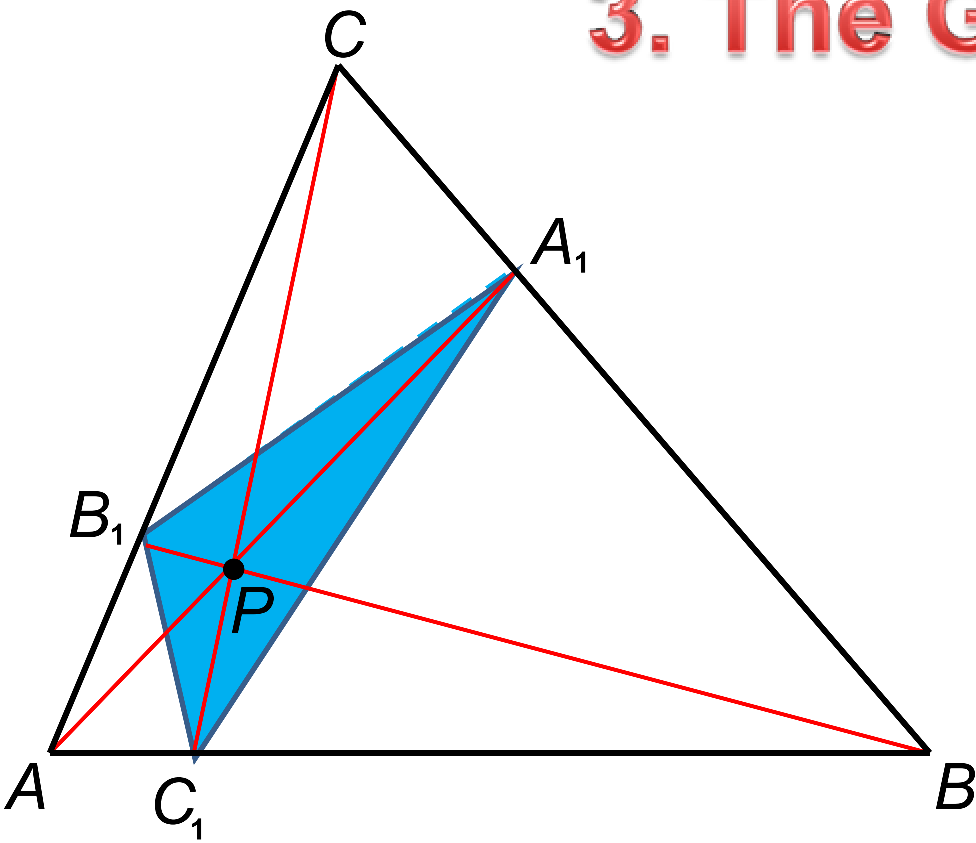
Rule: **For every natural number *n* the total number of tickets on which divisors of *n* are written is exactly *n*.**

Example: ***n* = 4,**
divisors are 1,2, and 4

Total of 4 tickets for divisors of 4.

Show: **Every such *n* appears on at least 1 ticket.**

3. The Golden Centroid



Show: ***P*** **is the centroid of triangle** ***ABC*** **iff** ***P*** **is the centroid of triangle** $A_1B_1C_1$**.**

Introduced: at the *Bay Area Mathematical Olympiad 2006*

For the bored die-hards:

Build a math circle session around each problem.

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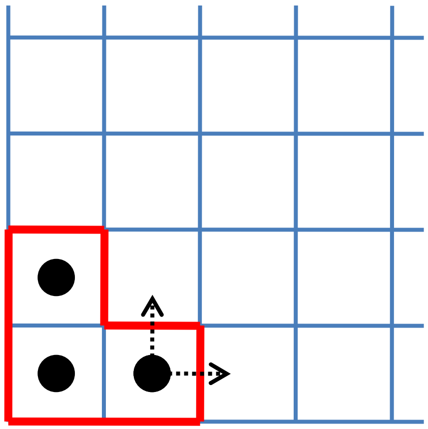

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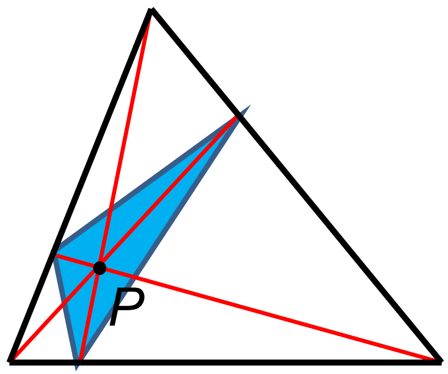
3. The Golden Centroid



Show:
P is the centroid of the big triangle iff *P* is the centroid of the small triangle.

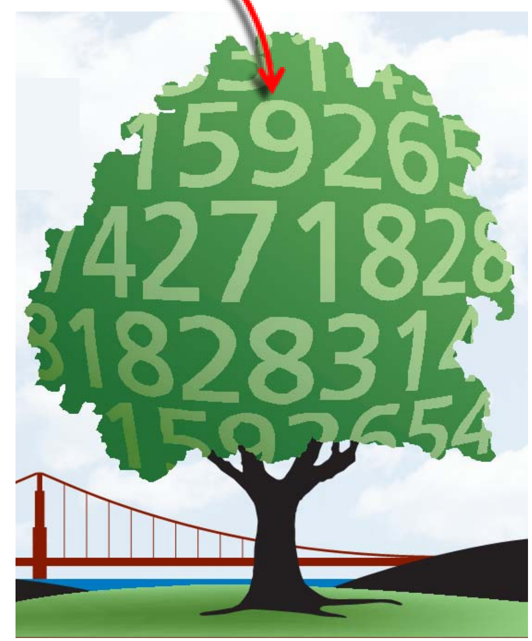
Questions to think about:

- **Math area(s)?**
  - **Math theory?**
    - **Problem Solving Techniques?**
      - **Other related problems?**
        - **Generalizations? Pushing on?**

- **How many different math circles sessions do you envision per problem? On what did you base your decision?**

Math Circle

Organization



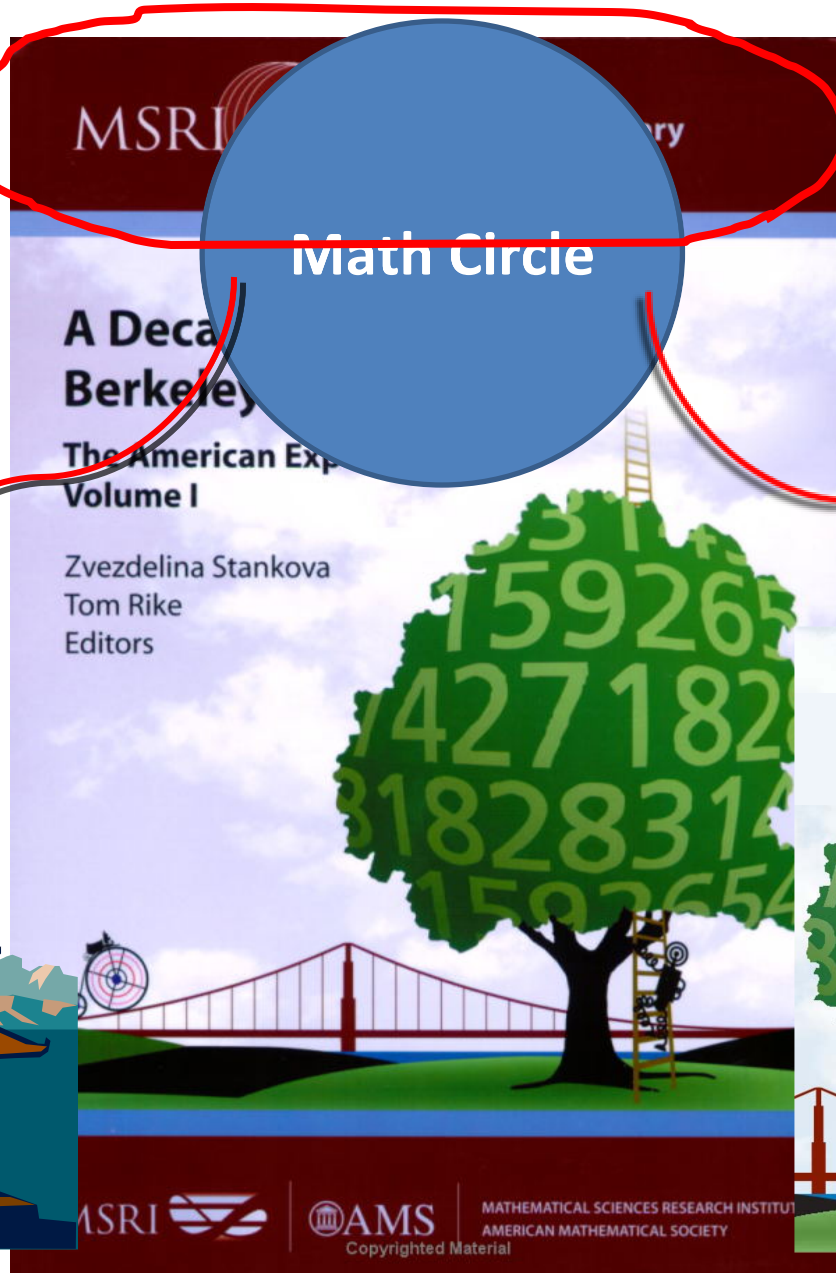
Mathematics

- Jointly published by

AMS & MSRI

- First book in
- 12 articles

adapted from actual live BMC sessions in last decade



- Edited by

Zvezdelina Stankova & Tom Rike

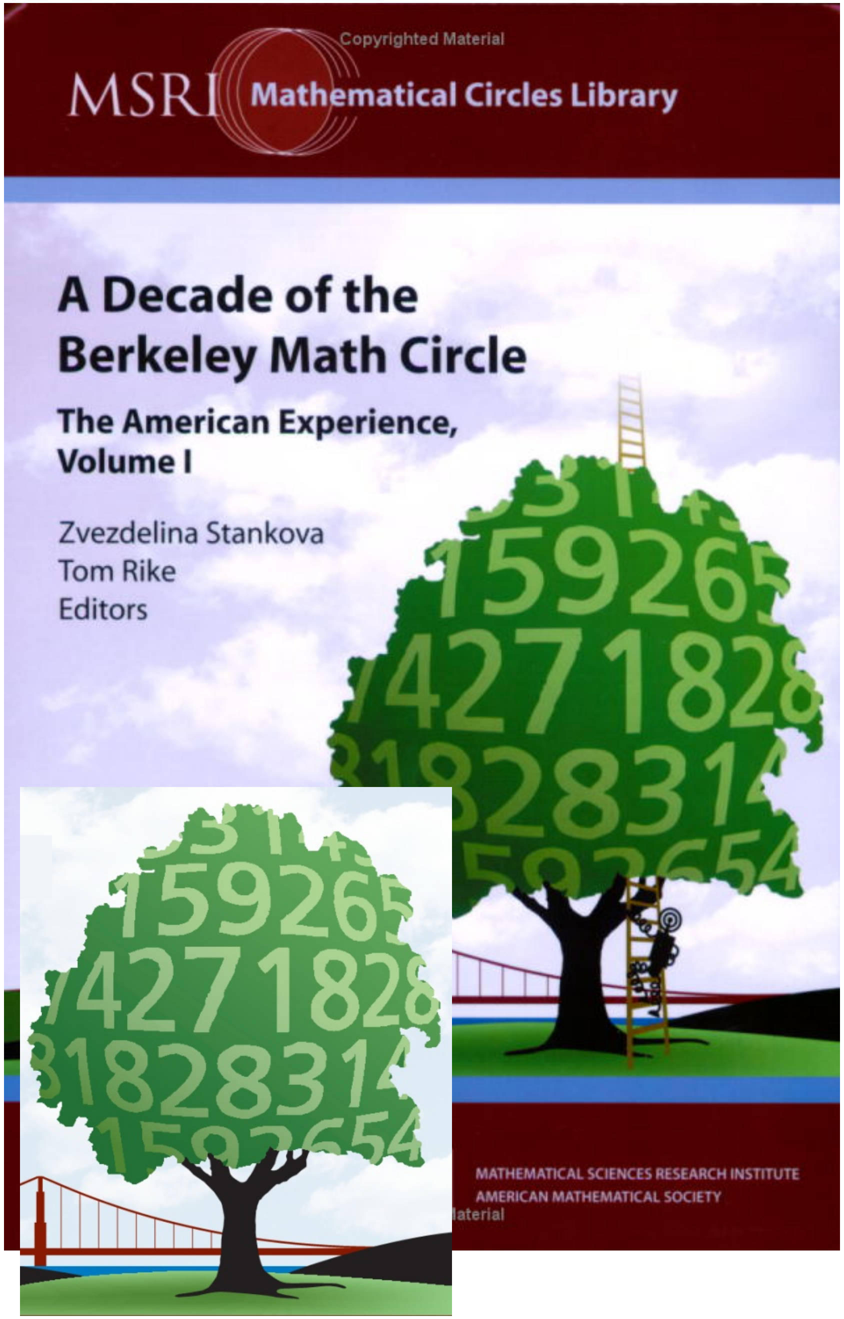
- Volume I

2/3 for beginners

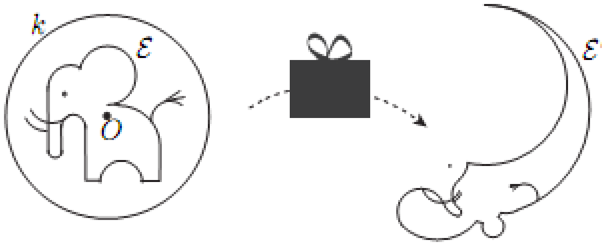
Organization

Mathematics

- **12 articles** **adapted from actual live BMC sessions in last decade**



Inversion in the Plane. Part I



Zvezdelina Stankova

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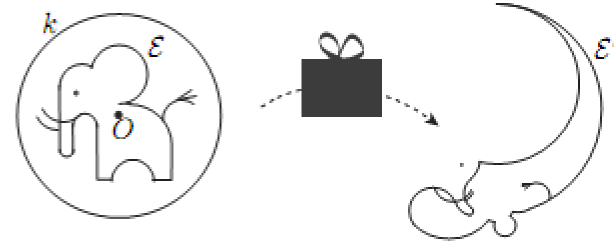
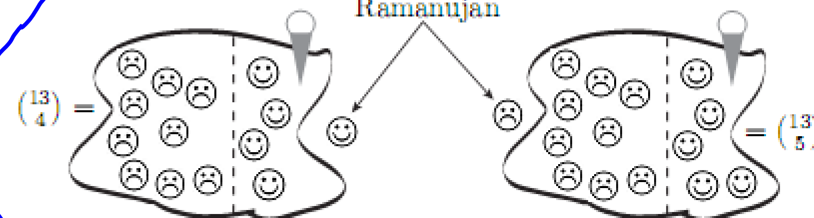


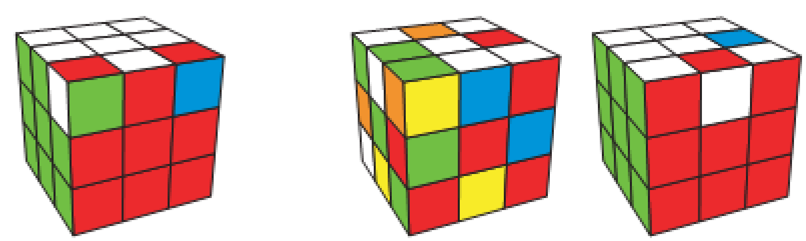
Zvezdelina Stankova

Combinatorics. Part I



Paul Zeitz

Rubik's Cube. Part I



Tom Davis

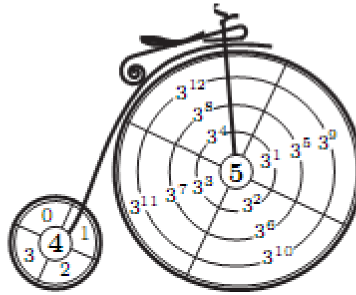
ANGELA CARA PANCRAZIO — MERCURY NEWS

Tom Davis gets a laugh by "morphing" pictures of George Bush and Dan Quayle into pictures of Clinton and Gore.

Number Theory. Part I

Remainders, Divisibility, Congruences and More





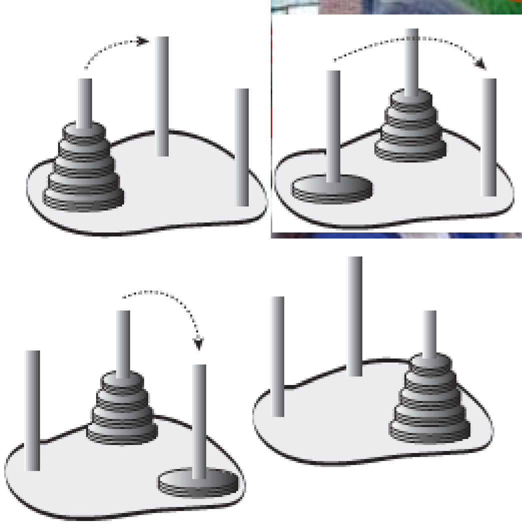
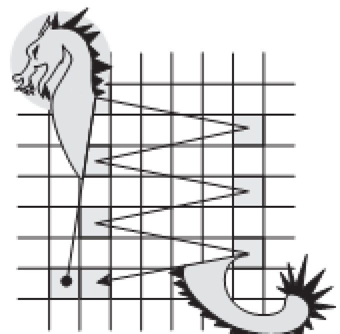
Zvezdelina Stankova

- **12 articles**

A Few Words About Proofs. Part I

Mira Bernstein

actual live


Quan Lam

Mathematical Induction

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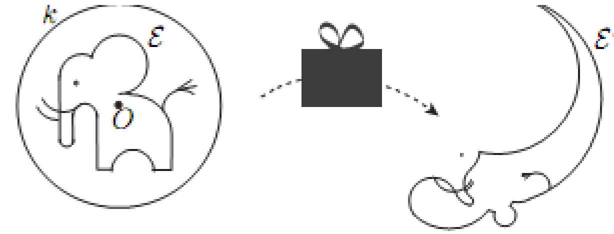

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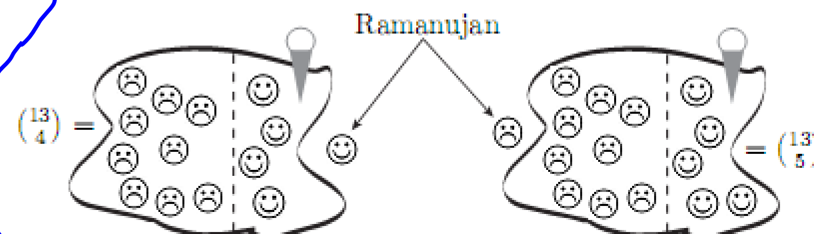

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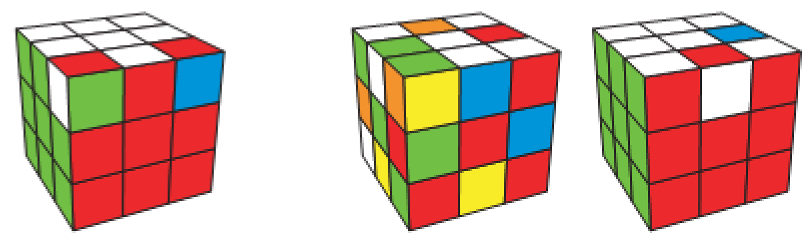

Tom Davis


MSRI
Mathematical Sciences
Research Institute

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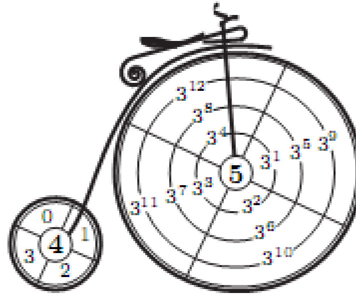
Remainders, Divisibility, Congruences and More



Zvezdelina Stankova

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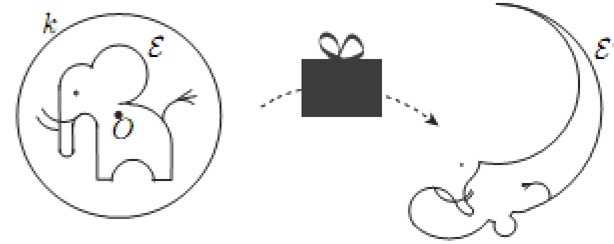

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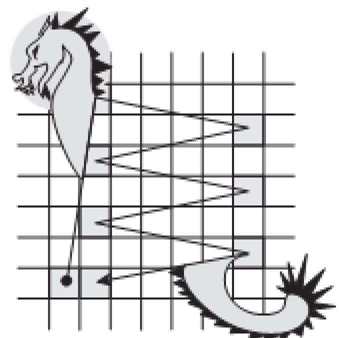

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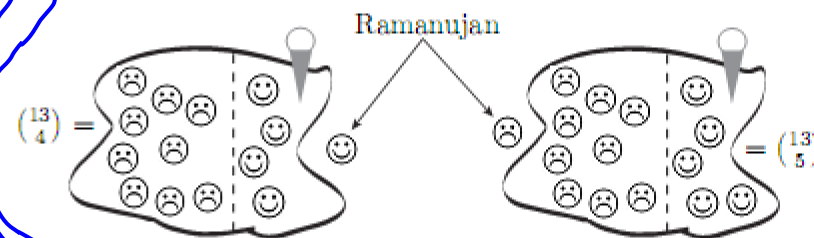
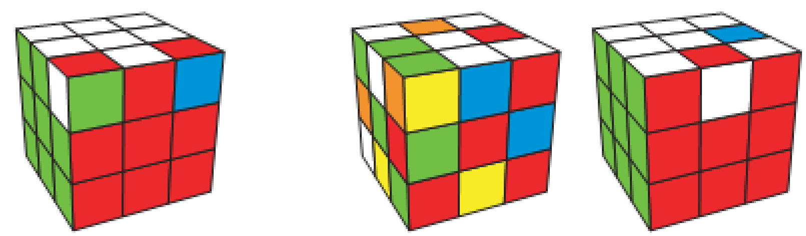


Paul Zeitz

Rubik's Cube. Part I



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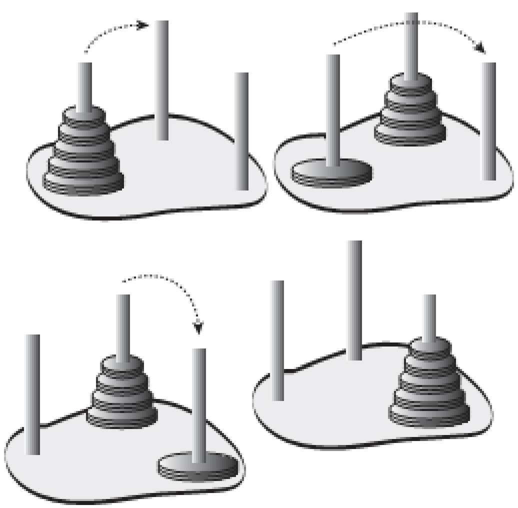

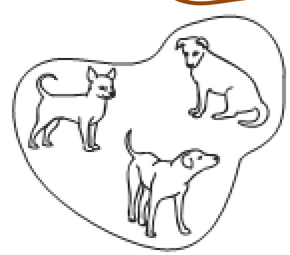

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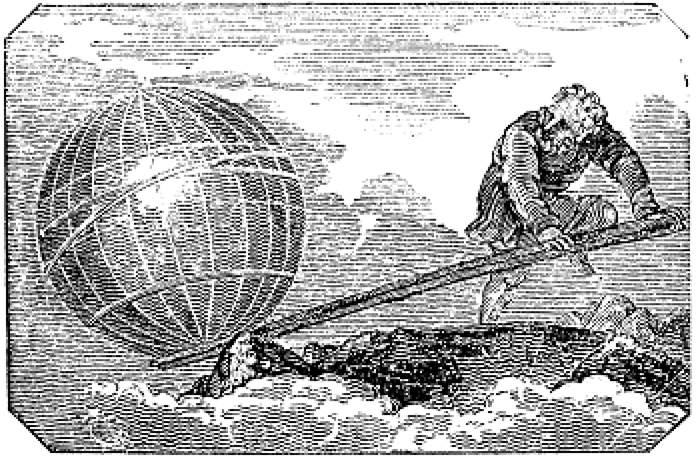

Tom Rike

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Rubik's Cube. Part I

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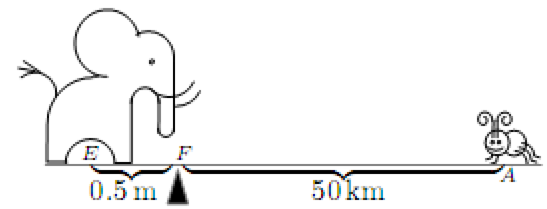
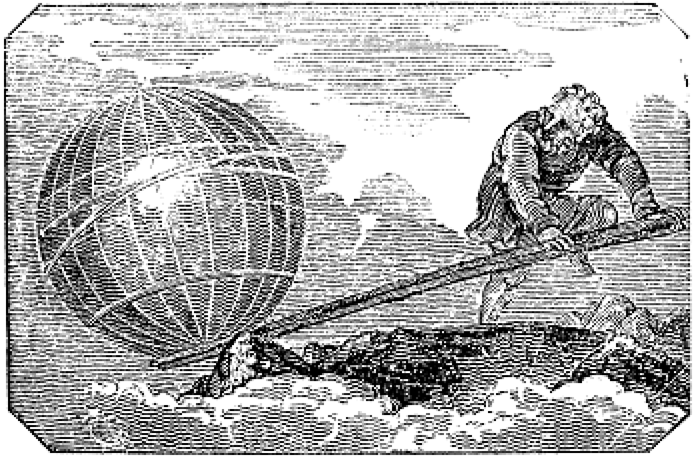
Mass Point Geometry





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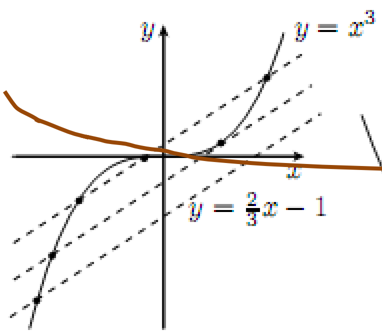
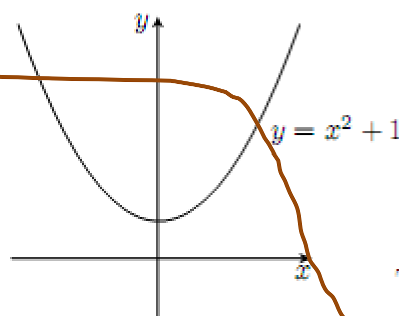
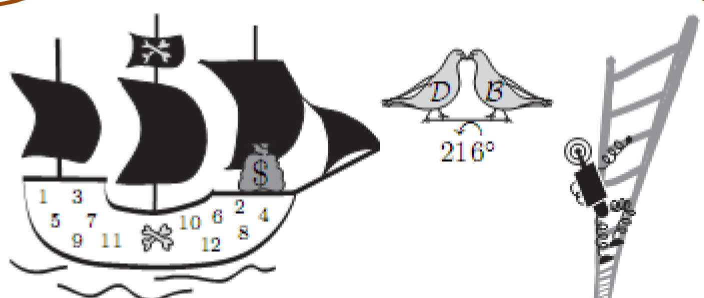
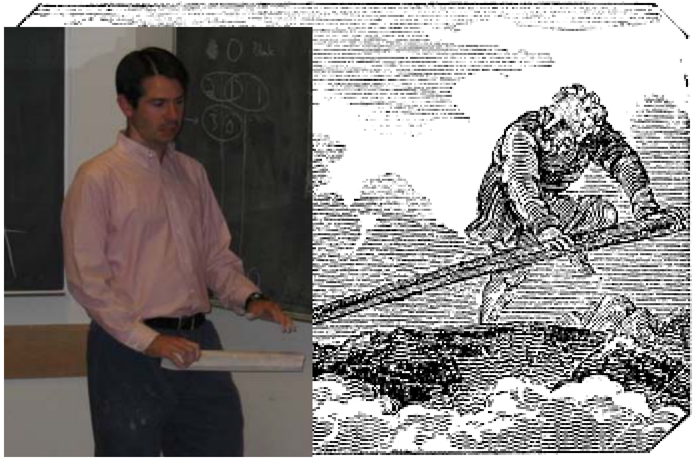




Tatiana Shubin



Mira Bernstein and Zvezdelina Stankova



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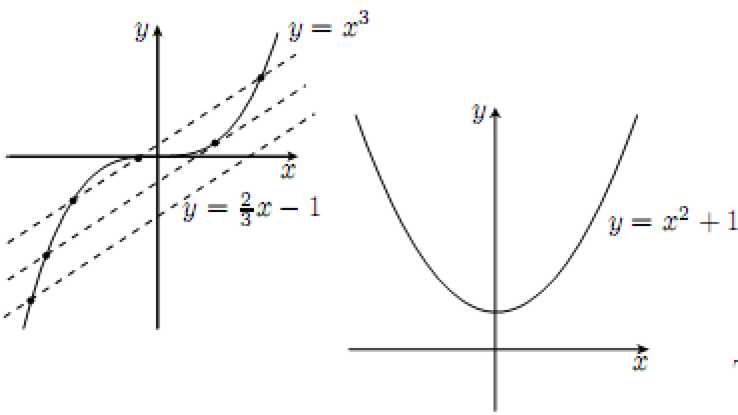
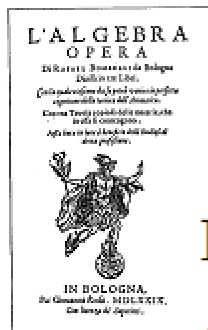
Circle Geometry





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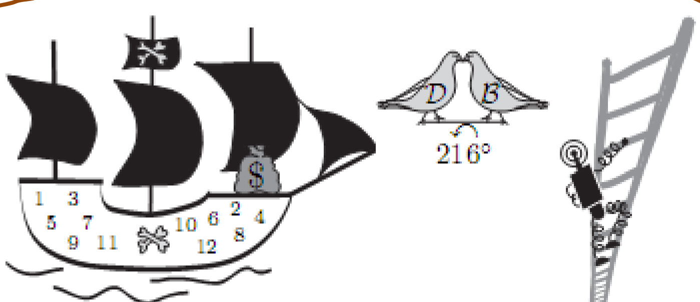

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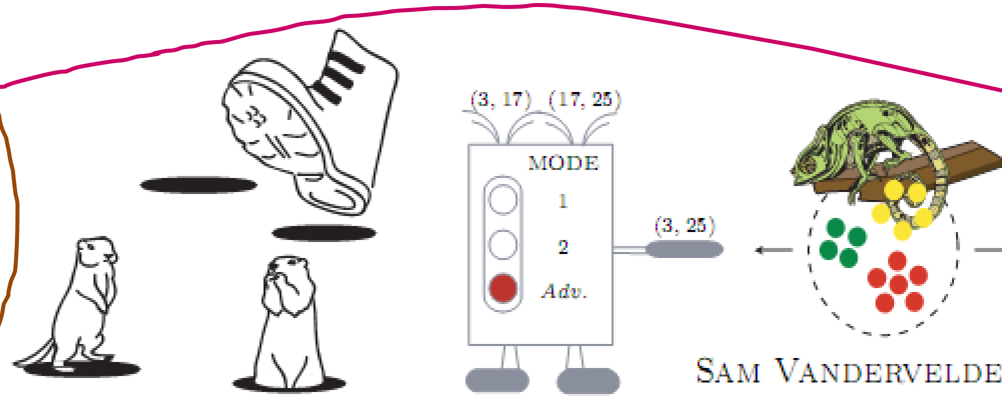

Sam Vandervelde

Ivan Matić

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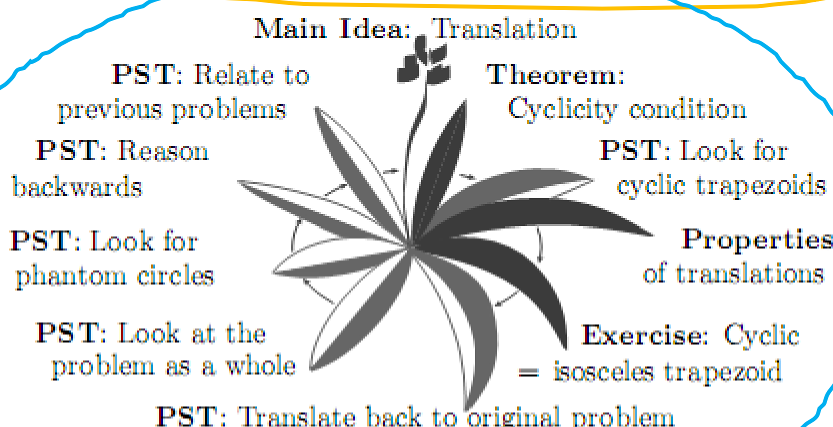
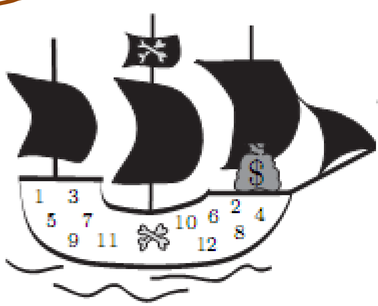
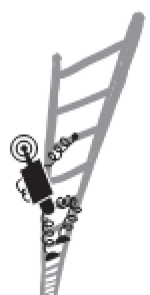
Monovariants. Part I

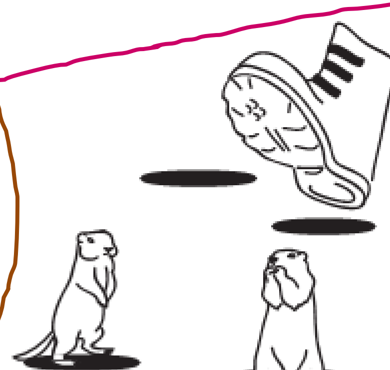
Mansion Walks and Frog Migrations

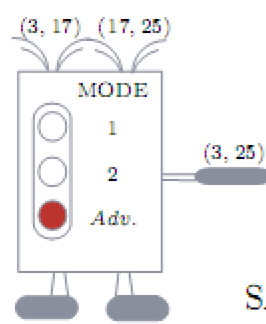


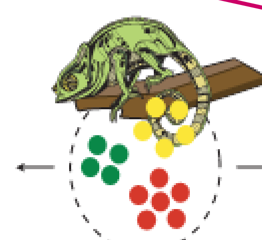




Mira Bernstein and Zvezdelina Stankova







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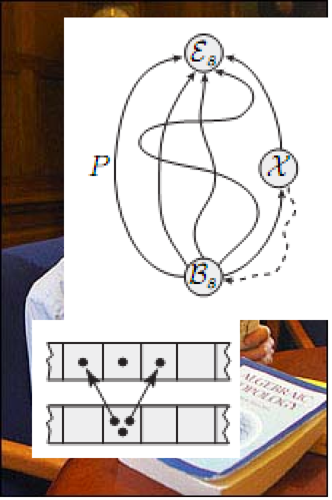
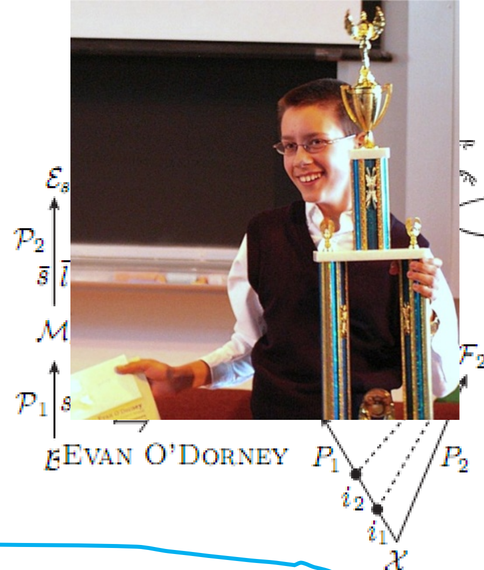


GABRIEL CARROLL



EVAN O'DORNEY

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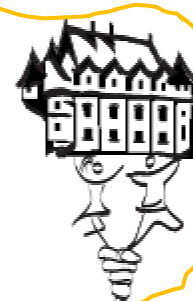
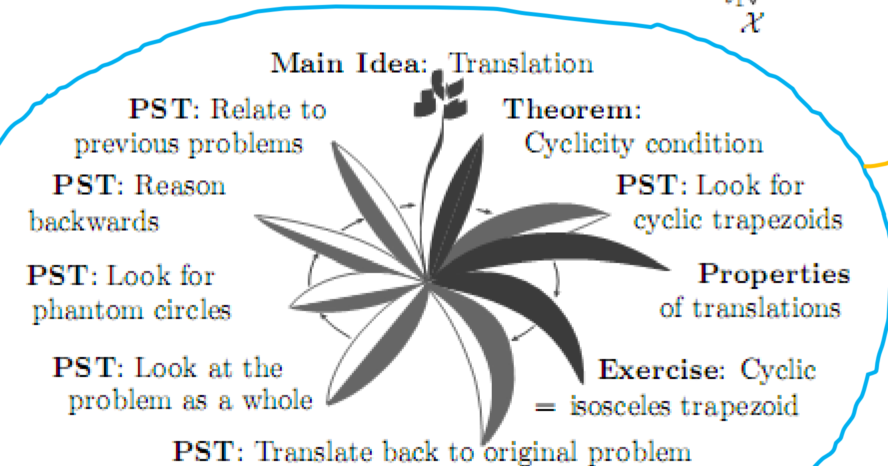
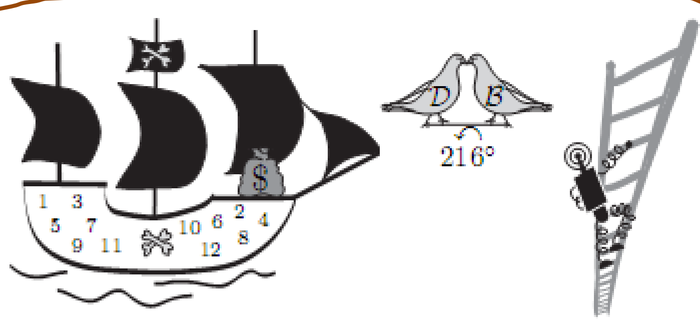
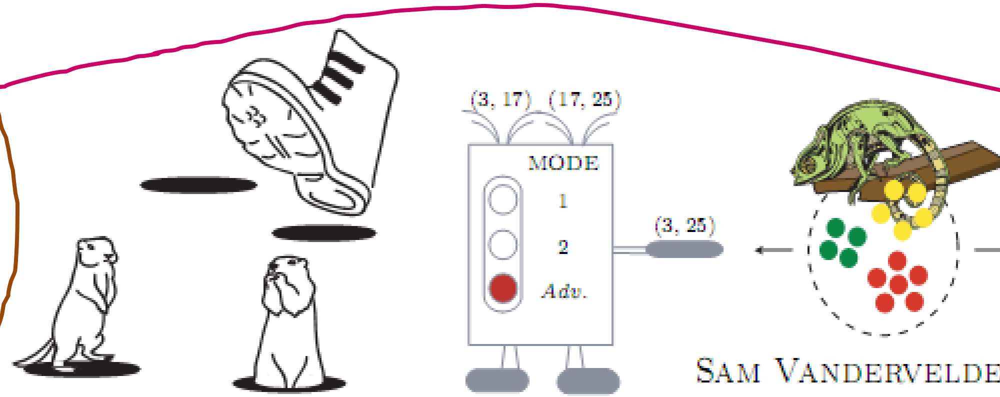
Mansion Appendix for the Advanced Reader







MIRA BERNSTEIN and ZVEZDELINA STANKOVA



SAM VANDERVELDE

MSRI Mathematical Circles Library

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The American Experience, Volume I

Zvezdelina Stankova
Tom Rike
Editors

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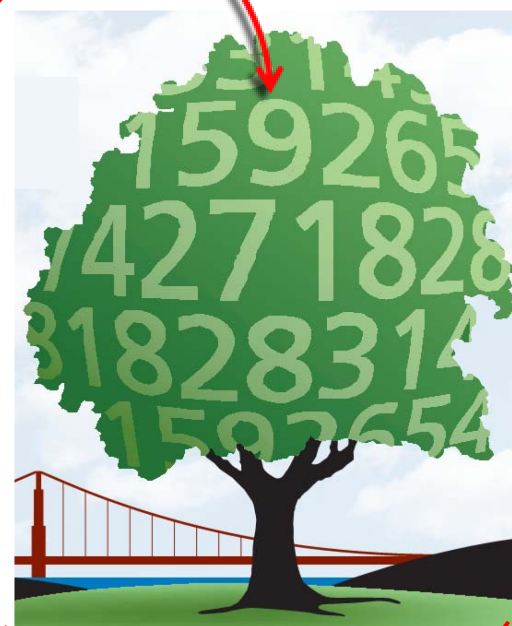
AMERICAN MATHEMATICAL SOCIETY

Introduction

Epilogue √

Math Circle

Organization



Mathematics

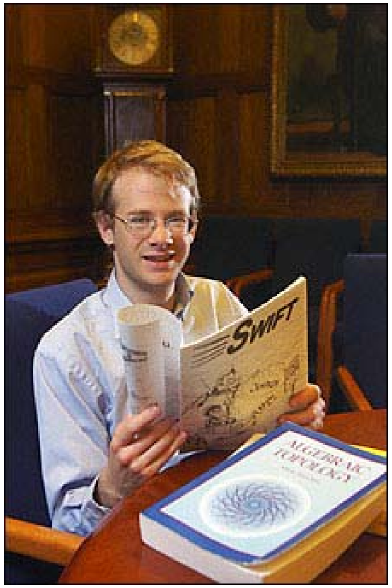
Introduction

Definition of Math Circle

- critical in student development
- get a flow of mathematical ideas and problems to think about
- meet other interested students and professional mathematicians
- get stimulation from exchanging ideas with other people that you don't get from reading books at home.
- applications to careers even outside mathematics: law, policy analysis, philosophy, economics, computer science
- develops logical, abstract thinking
- attracts students whose lifelong passion is for mathematics
- prepares for others careers along the way.

"The Berkeley Math Circle was really critical in my development. It was the best method available not only to get a flow of mathematical ideas and problems to think about each week but also to meet other interested students and professional mathematicians from all over the Bay Area. You get stimulation from exchanging ideas with other people that you don't get from reading books at home.

I can also testify to the usefulness of studying mathematics even for students who don't plan on doing it as a career. For someone who wants to go into, say, law, policy analysis, philosophy, economics, or computer science, the kind of logical, abstract thinking that mathematics develops is really the best preparation. I realize that the Circle is most interested in attracting students whose lifelong passion is for mathematics, but it also helps others along the way."



Gabriel Carroll, BMC alumnus
Perfect IMO '01 score
Four-time Putnam Fellow
Ph.D. student in economics, MIT

Introduction

Definition of Math Circle

By whom is a Math Circle to be run?

Running a math circle, especially for a teacher, is a hard task. But it is possible. In the 1960's, a veteran high school math teacher) was working on his master's degree. While browsing in the library one day, he ran across *The USSR Olympiad Problem Book* (cf. [**82**]). It contained problems written for talented 7^{th}–10^{th} graders; yet, he could not solve any of these "elementary" problems. In his own words:

> *"My abstract algebra had been too* abstract, *and I did not have the concrete examples that I needed. I never took a class in number theory because it sounded too elementary. I had developed the real number system starting from the Peano axioms, but I didn't really understand the fundamentals of the natural numbers, prime numbers. This was an epiphany for me. I felt as though I had been challenged by some force outside me and did not know how to respond."*

For the next 30 years studied olympiad problem solving, first on his own, then through workshops and math circles in the SF Bay Area. He ran his own math circle at High School and gave talks at just about all other circles around. Even though at times he was only "a few pages" ahead of the students, he kept on learning and teaching problem solving because working on math circles had come to be a large part of his life:

> *"Although I have not attained my goal of becoming a true olympiad problem solver, the journey I have made in pursuit of this goal has been one of the most rewarding endeavors in my life."*

- **to run a math circle is a hard task, especially for a teacher**
- **could not solve any of these "elementary" problems**
- **insufficient mathematical background and/or preparation in problem solving**
- **an epiphany; challenged by an outside force and did not know how to respond.**
- **for 30 years kept on learning and teaching problem solving b/c working on math circles was a large part of his life**
- **to become a true olympiad problem solver has been one of the most rewarding endeavors in his life**

Introduction

Definition of Math Circle

By whom is a Math Circle to be run?

Tom Rike

Hence, a word to the middle and high school teachers: keep on reading the book, despite moments of difficulty or confusion. For the motivated, persevering, and caring teacher, there will come a time when he/she will look back at the material here, smile, and effortlessly deliver it to the students at his/her own math circle. Truly gratifying.

- to run a math circle is a hard task, especially for a teacher
- could not solve any of these "elementary" problems
- insufficient mathematical background and/or preparation in problem solving
- an epiphany; challenged by an outside force and did not know how to respond.
- for 30 years kept on learning and teaching problem solving b/c working on math circles was a large part of his life
- to become a true olympiad problem solver has been one of the most rewarding endeavors in his life
- the persevering and caring teacher will succeed in running a math circle

Introduction

Definition of Math Circle

By whom is a Math Circle to be run?

Math Circles and learning to be a mathematician?

"Perhaps I can best describe my experience of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture, but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were."

Sir Andrew John Wiles

There are no manuals on how to become a mathematician. This book will give you tips and point to possible paths, but the "art of being a mathematician" can be mastered only through personal experience. With every problem solved and every new definition or theorem learned, you will move closer to this goal. The two most important skills that you will acquire along the way are

- to think creatively while still "obeying the rules" and
- to make connections between problems, ideas, and even theories.

"*The best way to learn is to learn from your own mistakes.*"

Joe Harris

Nothing good comes "for free": you will have to work hard, always with a pencil and paper in hand. Keep in mind that the math world is huge: you'll never know everything, but you'll learn *where* to find things, how to connect and use them. The rewards will be substantial.

- **no manuals on how to become a mathematician**
- **to think creatively while still obeying the rules**
- **to make connections between problems, ideas and theories**
- **learn from your own mistakes**
- **work hard, with pencil & paper**
- **the math world is huge: you'll never know everything; use this to your advantage!**

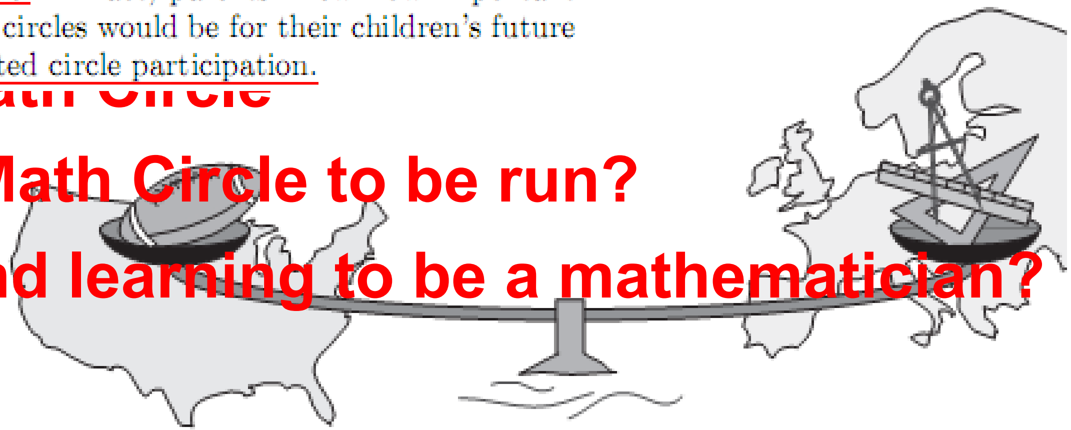
The math circles were only part of a large net of pre-college circles created to draw children and discover their talents. It was no more prestigious or "cool" to attend a soccer club or take music lessons than to be a member of, say, a high school physics circle. In fact, parents knew how important the advanced knowledge gained in circles would be for their children's future and hence enthusiastically supported circle participation.

- parents enthusiastically supported circle participation

Definition of Math Circle

By whom is a Math Circle to be run?

Math Circles and learning to be a mathematician?



2.2. Worthy of a circle. Mathematics was not the only subject "worthy of a circle". Starting in late middle or early high school, there were circles in chemistry, physics, and biology; in English, poetry, and literature. I participated in just about all of them at one time or another. I tried many fields because the opportunities were there for me to explore.

- circles were as "cool" as soccer clubs
- circles other than math circles

2.1. All you need is love. There is more than one way to fall in love with mathematics. Many Eastern European mathematicians have come along the path of math circles, where they have learned for the first time that the world of math is larger than one could imagine, more interesting, and more diverse. The math circle culture is ingrained in the societies in these countries. During the communist era, established mathematicians and pre-college teachers considered it their duty to expose the younger generation to the wonders of mathematics. And so they teamed together to found and run math circles.

- mathematicians and teachers teamed together to found and run math circles
- math circle culture is ingrained in the societies of these countries

The Culture of Math Circle

Epilogue

- East European mathematicians are "recruited" via math circles

Solution?

3.3. Get to the point. One way to resolve most of the problems associated with math circles in the U.S. is . . . (OK, start dreaming!) . . . to have *a math circle at every college and university*.

> The professor organizing and running the math circle will receive *a one- or two-course release* from the math department, depending on the frequency, length, and intensity of the circle sessions. This will compensate for the huge effort involved in directing a math circle and will hopefully encourage more mathematicians to get involved in educating the talented youth of the U.S.
>
> The math circle can be formally *organized as a math course* and, thus, be open also to undergraduates.
>
> Undergraduate and graduate students, as well as interested post-docs and tenured faculty, can be *vertically integrated* in this model.

- vertical integration
- math course for undergrads
- faculty 1 or 2 course release for running math circle
- start dreaming: math circle at every college and university

3.2.6. Popularity. Everyone in Eastern Europe knew about the math circles; children and parents alike were well aware of the opportunity to enroll and of the possibilities which successful participation might open in the students' future. What portion of the U.S. population has an inkling that math circles exist? Negligible. What status do math circles have in U.S. society and its educational system? Unclear. Can they compare in popularity to membership of a high school football or debate team? No, they can't.

- comparison of math circles in US vs. football/debate teams?
- status of math circles in US society & educational system?
- popularity of math circles in US society?

Eastern European vs. US Math Circles

The Culture of Math Circle

Epilogue

Solution?

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>
> The math circle can be formally *organized as a math course* and, thus, be open also to undergraduates.
>
> Undergraduate and graduate students, as well as interested post-docs and tenured faculty, can be *vertically integrated* in this model.
>
> A *reasonable modest semester fee* for non-university participants (pre-college students, teachers and others) will provide honoraria to the session leaders.
>
> The math department can provide secretarial and computing support and office supplies, as well as a work-study *student assistant and web administrator*.

The math circle will be an invaluable math program offered to the local community and can be viewed as part of the math department's *outreach activities*. This network model will resolve transportation problems at least for the urban and suburban areas (i.e., areas with an institution of higher education), will mobilize previously disinterested math faculty, and will give some *tangible and formal recognition to the work of math circle leaders*.

cles

Epilogue

Is this solution viable?

- **mobilize and give recognition to the most valuable resource: interested mathematics faculty**
- **math circle as outreach activity**
- **dept provides support**
- **modest fee for non-university participants**
- **vertical integration**
- **math course for undergrads**
- **faculty 1 or 2 course release for running math circle**
- **start dreaming: math circle at every college and university**
- **comparison of math circles in US vs. football/debate teams?**
- **status of math circles in US society & educational system?**
- **popularity of math circles in US society?**

Does the US need Top-Tier Math Circles?

1.1. Math circles are weekly math programs that attract middle and high school students to mathematics by exposing them to intriguing and intellectually stimulating topics, rarely encountered in classrooms. Math circles vary in their organization, styles of sessions, and goals. But they all have one thing in common: *to inspire in students an understanding of and a lifelong love for mathematics.*

- inspire lifelong love for mathematics

1.2. Top-tier math circles *prepare our best young minds for their future roles as mathematics leaders.* Sessions are taught by accomplished mathematicians and explore advanced mathematical areas. They provide an educational opportunity for top pre-college mathematics students, not offered in any other setting in the U.S. education system. In addition to learning advanced mathematics topics, students are taught the technical writing skills needed to convey the solutions of complex problems.

- prepare our best young minds as future mathematics leaders

5.3. The gathering storm. There are a number of studies of the deteriorating situation in U.S. math and science education and its impact on the scientific and technological presence of the U.S. in the world. To describe just how critical the situation is, we refer below to three such reports.

- deteriorating math and science education in US

> *"The United States is losing its edge in innovation and is watching the erosion of its capacity to create new scientific and technological breakthroughs. Increased global competition, lackluster performance in mathematics and science education, and a lack of national focus on renewing its science and technology infrastructure have created a new economic and technological vulnerability as serious as any military or terrorist threat."*
>
> A Commitment to America's Future, 2005 [**13**]

- erosion of capacity for new scientific and technological breakthroughs
- vulnerability as serious as any military or terrorist threat

Recently the National Academy of Sciences has also called to our attention the need for the U.S. to raise its capabilities in mathematics, science and engineering, in a report entitled *"Rising Above the Gathering Storm: Energizing and Employing America for a Brighter Economic Future"* [**64**]. According to it:

- The U.S. has long depended on foreign-born and -trained mathematicians, engineers and scientists to help maintain its intellectual lead.
- The global competition for these talented individuals has greatly intensified in recent years and will continue to do so, as the rest of the world increases its technical capabilities and living standards.
- To remain competitive, the U.S. needs to devote considerably more effort and resources to foster excellence in mathematics, science and engineering.

The majority of talented individuals in these fields recruited by U.S. universities and technology companies are from China, Europe, India, and the former Soviet Union. A 2006 report on Science, Technology, Engineering, and Mathematics Education (STEM, [**68**]) brought forward related troubling trends and numbers:

- In 2004, China graduated approximately 500,000 engineers; India graduated 200,000 engineers, and the U.S. graduated 70,000 engineers. On the other hand, South Korea graduates as many engineers as the U.S. even though it has only one sixth of the U.S. population.
- More than half of all engineering doctorates awarded in the U.S. go to foreign-born students. In 2003, 25% of all college-educated workers and 40% of all doctorate holders were foreign-born. Over half of the doctorate holders in several fields who resided in the U.S. were foreign-born: computer science; electrical, civil, and mechanical engineering.
- From 1994 to 2004, there has been a steady increase in the percentage of U.S. patents granted with a foreign origin, including foreign-owned companies and foreign inventors. In one decade this number has increased from 18% in 1994 to 48% in 2004!

- raise capabilities in mathematics, science and engineering
- global competition for talented individuals had intensified
- more effort and resources in mathematics, science and engineering
- recruited from China, Europe, India and former Soviet Union
- China: 500,000
  India: 200,000
  US: 70,000
  S.Korea: 70,000
- foreign-born doctorate holders
- foreign-origin US patents

What do these foreign countries do differently from the U.S.? There are many differences and each country is unique. India and China value technical education as a path to prosperity; admission to technical schools there is based on rank in national exams. In the former Soviet Union and Eastern Europe, mathematically talented individuals are identified very early and are provided with the resources needed to reach their full potential.

5.4. Raising the ceiling. What can be done in the U.S.? Hung-Hsi Wu, Professor of Mathematics at UC Berkeley, has been involved in the education of U.S. mathematics teachers for the last decade. He was on the Task Group on Teachers in the National Mathematics Advisory Panel appointed by President Bush and is currently serving on the National Research Council Panel on the Study of Teacher Preparation Programs.

> *"A main purpose of both panels is to address the crisis in teacher quality among math teachers so as to insure the production of a large enough pool of mathematically literate students to fill our technological needs. However, to insure that we also produce* first rate scientists and mathematicians*, a different kind of approach would be necessary.*
>
> *This is where the* Math Circles *come in. It is programs like the Math Circles that can provide the needed guidance and stimulation for the cream of the crop of this pool. While the work done by the above-mentioned panels is designed to raise the floor to make our nation competitive in the global market, what the Math Circles do is to* raise the ceiling *in order to maintain our worldwide leadership position in science and technology.*
>
> *At a time of need in our nation's mathematics education, the work done in* top-tier math circles *such as the* Berkeley Math Circle *and the* San Jose Math Circle *is of vital importance."*

Hung-Hsi Wu
Professor of Mathematics
University of California at Berkeley

- value technical education as a path to prosperity
- early identify mathematically talented individuals and provide resources to reach full potential
- to produce first rate mathematicians requires different approach
- math circles raise the ceiling in order to maintain leadership in science and technology

"I wish to state in no uncertain terms how important programs for our talented young people are to the future of this country. The best place to develop the highest end mathematical talent is in groups where young people can feed off each others' excitement, guided by the best minds in the field. The model of top-tier math circles *has been honed over decades in other countries. An American version has been in place for a decade and has shown measurable and almost unbelievable results.*

Now is the time to make these programs a permanent feature of our educational landscape. The community is ready to assist in any way possible. Universities are happy to provide facilities. Professors are happy to volunteer their time. Parents are happy to spend countless hours.

And the reason we do this is that when you see these kids catch fire, it takes your breath away."

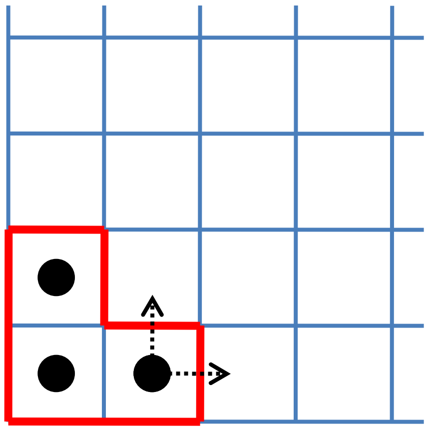
Ravi Vakil
Four-time Putnam Fellow
Professor of Mathematics
Stanford University

While it is unlikely that math circles will have a large impact on the value system of the American public, the top-tier math circles in the U.S. do play a significant role in meeting the challenges described above by *preparing our best young minds for their future role as mathematics, science, and technology leaders.* With your help, we can establish a dense network of math circles across the U.S.

For the bored die-hards:

Build a math circle session around each problem.

1. Freedom for the Clones



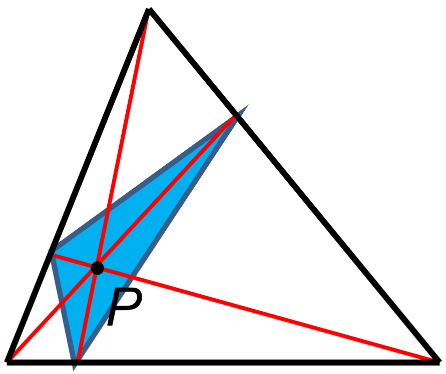
Question:
Can you free all clones from the prison?

2. Infinite Raffle

For every natural number n the total number of tickets on which divisors of n are written is exactly n.

Show:
Every such n appears on at least 1 ticket.

3. The Golden Centroid



Show:
P is the centroid of the big triangle iff P is the centroid of the small triangle.

Questions to think about:

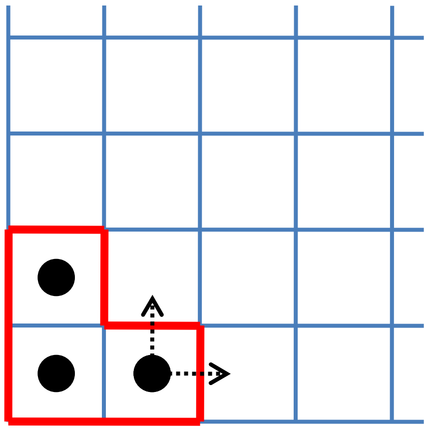
- **Math area(s)?**
  - **Math theory?**
    - **Problem Solving Techniques?**
      - **Other related problems?**
        - **Generalizations? Pushing on?**

- **How many different math circles sessions do you envision per problem? On what did you base your decision?**

For the bored die-hards:

Build a math circle session around each problem.

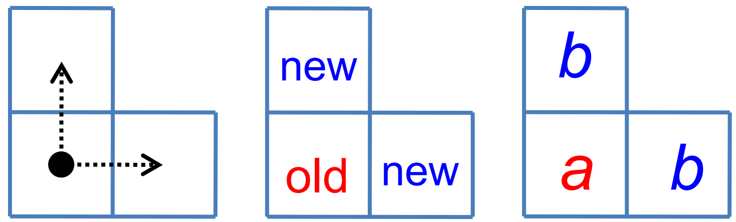
1. Freedom for the Clones



Question:
Can you free all clones from the prison?

Idea: Find an *invariant* feature of the game.

Catch 22: No *integer* invariant can be readily found here.



$a = b + b = 2b$

old sum = *new sum*

1/32						
1/16	1/32					
1/8	1/16	1/32				
1/4	1/8	1/16	1/32			
1/2	1/4	1/8	1/16	1/32		
1	1/2	1/4	1/8	1/16	1/32	

$a = 2b$

old = 2 x *new*

new sum
= **2**

sum after
every move
= **2**

final sum
= **2**

1/32	1/64	1/128	1/256	1/512	1/1024	
1/16	1/32	1/64	1/128	1/256	1/512	
1/8	1/16	1/32	1/64	1/128	1/256	
1/4	1/8	1/16	1/32	1/64	1/128	
1/2	1/4	1/8	1/16	1/32	1/64	
1	**1/2**	1/4	1/8	1/16	1/32	

old sum
= 1+ ½ + ½
= **2**

Idea: The *sum of all numbers* of the board remains unchanged after any move. This is our *invariant*!

Detour: Forget about the barbed fence, the inside and outside of the prison, and mess around with the whole table.

1/32	1/64	1/128	1/256	1/512	1/1024	
1/16	1/32	1/64	1/128	1/256	1/512	
1/8	1/16	1/32	1/64	1/128	1/256	
1/4	1/8	1/16	1/32	1/64	1/128	
1/2	1/4	1/8	1/16	1/32	1/64	
1	1/2	1/4	1/8	1/16	1/32	

Question: How to calculate the sum of the numbers in the whole table?

Question: How to calculate the sum of the numbers in the first row?

1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + … = **2**

1/32	1/64	1/128	1/256	1/512	1/1024		→ **1/16**
1/16	1/32	1/64	1/128	1/256	1/512		→ **1/8**
1/8	1/16	1/32	1/64	1/128	1/256		→ **1/4**
1/4	1/8	1/16	1/32	1/64	1/128		→ **1/2**
1/2	1/4	1/8	1/16	1/32	1/64		→ **1**
1	1/2	1/4	1/8	1/16	1/32		→ **2**

Total Sum = 4

$$2 + 1 + 1/2 + 1/4 + 1/8 + 1/16 + \ldots = \frac{2}{1-\frac{1}{2}} = \frac{2}{\frac{1}{2}} = 4$$

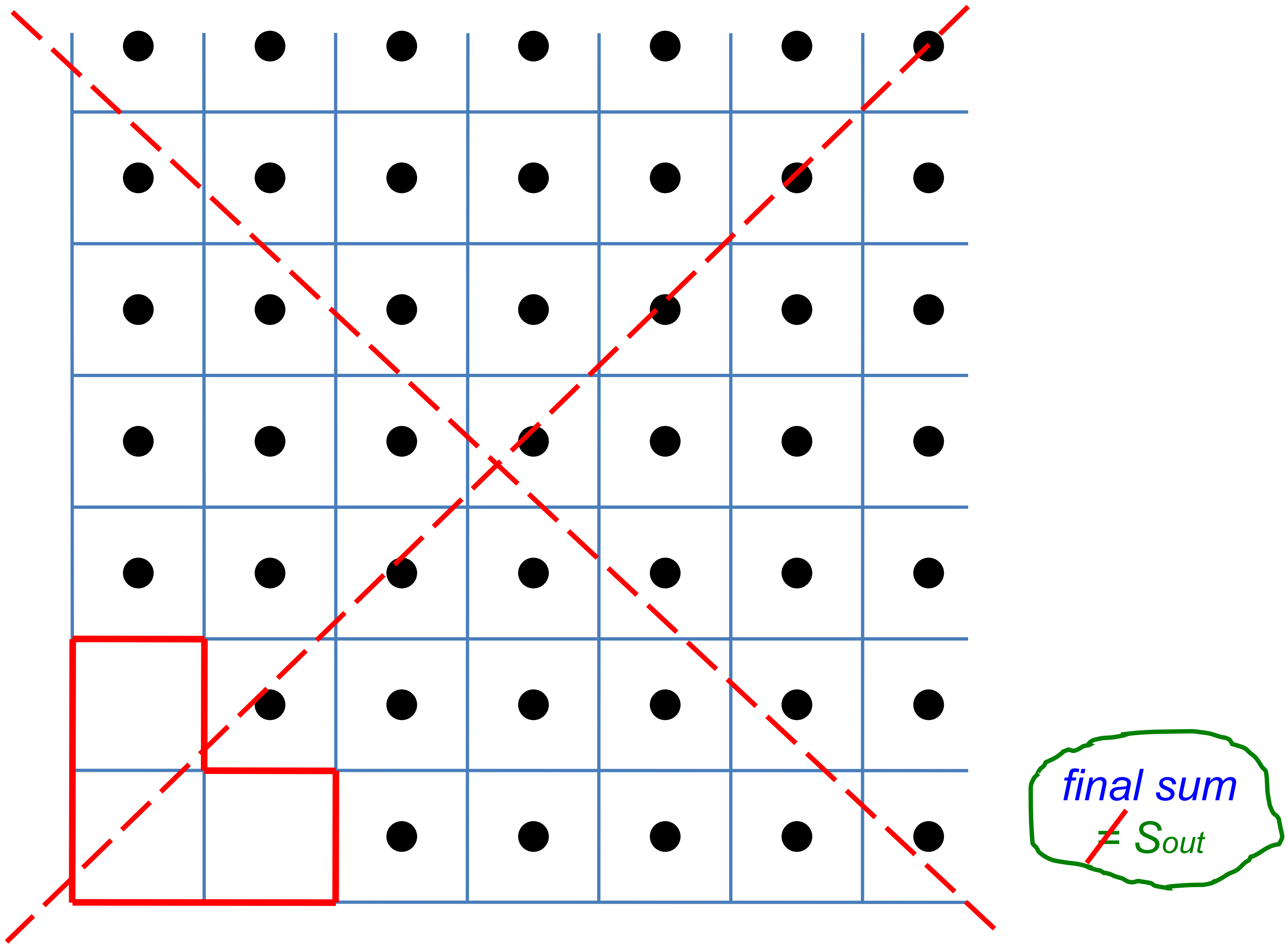
S_{in} = sum in prison = **2**

final sum = **2**

1/32	1/64	1/128	1/256	1/512	1/1024	
1/16	1/32	1/64	1/128	1/256	1/512	
1/8	1/16	1/32	1/64	1/128	1/256	
1/4	1/8	1/16	1/32	1/64	1/128	
1/2	1/4	1/8	1/16	1/32	1/64	
1	**1/2**	1/4	1/8	1/16	1/32	

final sum = S_{out}

S_{out} = sum outside prison = total sum – inside sum = 4 – 2 = **2**

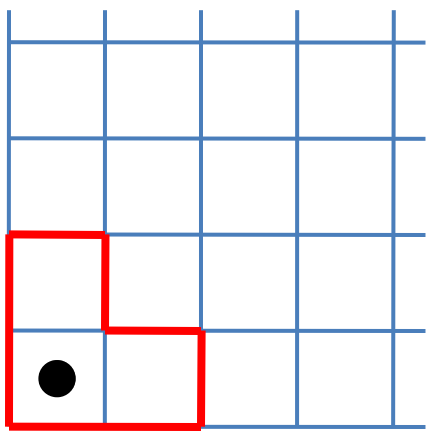


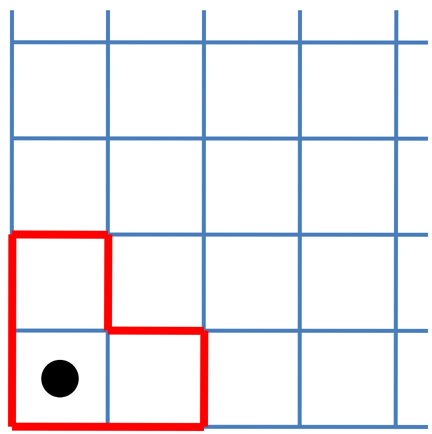
Finate game!!

Playing beyond the given game:

change initial clones configuration

change initial barbed fence

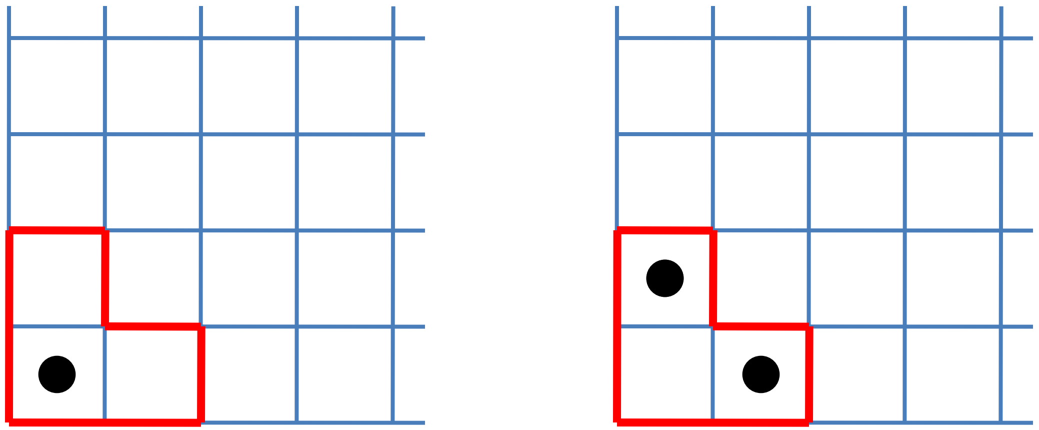




Playing beyond the given game:

change initial clones configuration

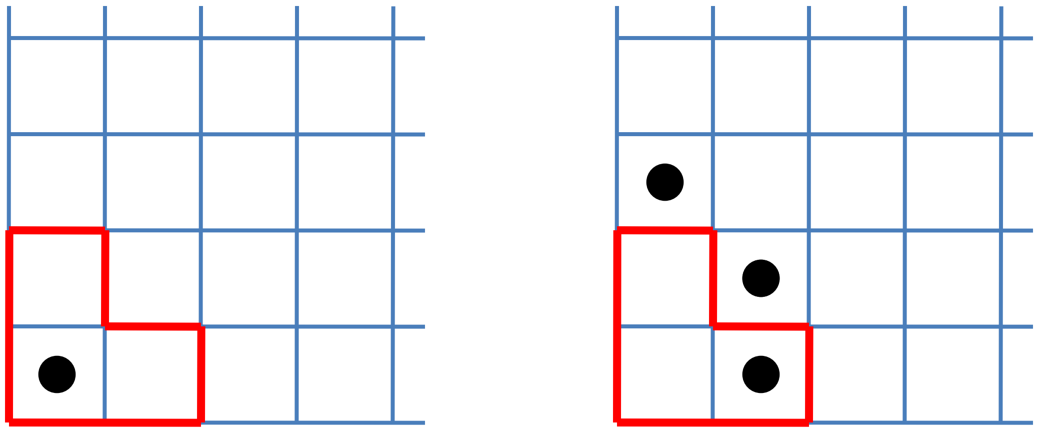
change initial barbed fence



Playing beyond the given game:

change initial clones configuration

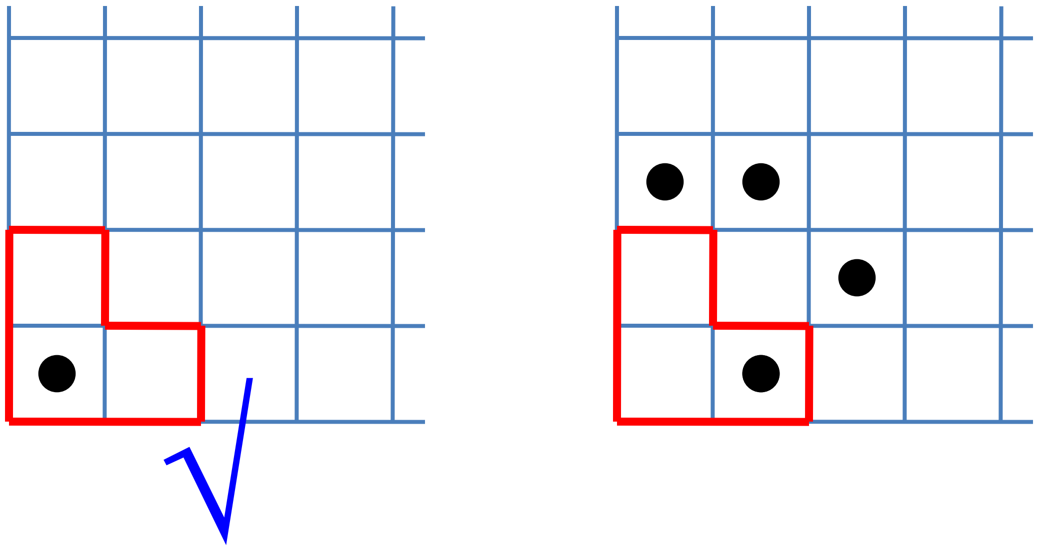
change initial barbed fence



Playing beyond the given game:

change initial clones configuration

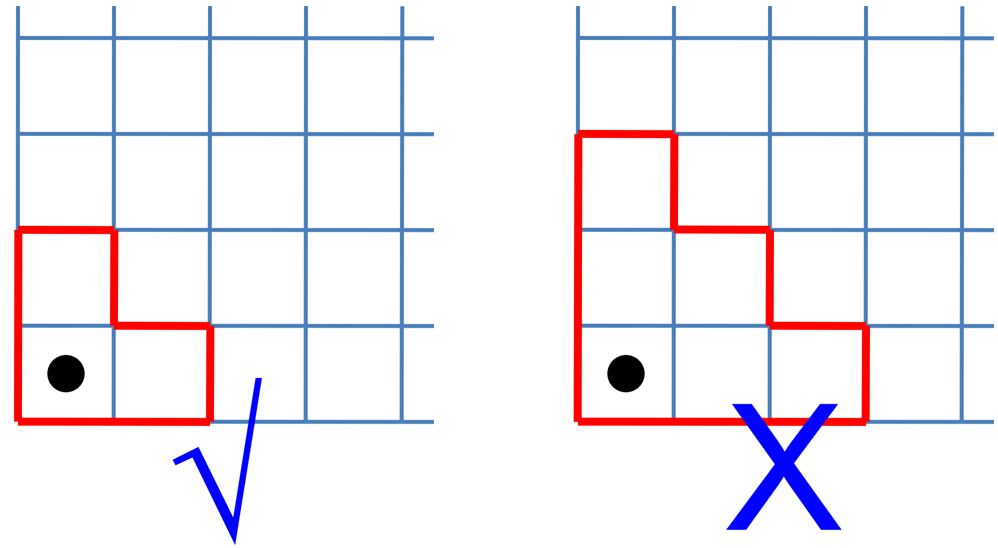
change initial barbed fence



Playing beyond the given game:

change initial clones configuration

change initial barbed fence



Ultimate question:

- Starting with one clone in position (1,1), for which barbed fences can the clones escape and for which they cannot?

- Equivalently, which are the **"minimal" inescapable barbed fences**?

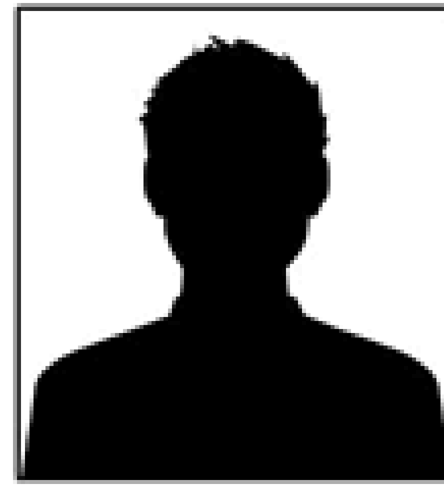
Pebbling a Chessboard

Fan Chung, Ron Graham, John Morrison, and Andrew Odlyzko

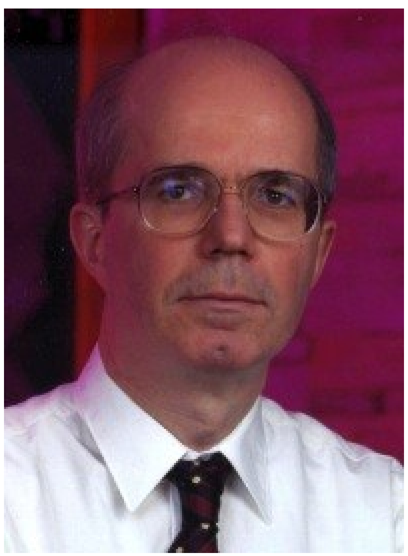
1. INTRODUCTION. The following puzzle has attracted some attention recently. We first learned of it through Martin Gardner [**6**]. A version of it appeared in Omni magazine in 1993 [**11**]. However, it was proposed over 10 years ago by Kontsevich [**9**], and a partial analysis of it was published shortly thereafter by Khodulev [**8**]. We begin with an infinite "chessboard" B covering the first quadrant. The cells of the board are labelled by integer coordinates (i, j) with $i, j \geq 0$. Initially, a single "pebble" is located in cell $(0,0)$ (the lower left corner; see Figure 1). The first step or "move" consists of replacing this pebble by two pebbles, located at cells $(1,0)$ and $(0,1)$, respectively. In general, a move will consist of removing some pebble, say in cell (i, j), and placing *two* pebbles on the board, in positions $(i + 1, j)$ and $(i, j + 1)$, *provided each of these positions is not already occupied*.

Fan Chang

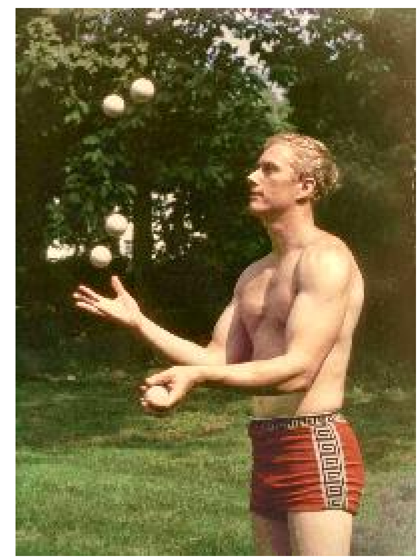
Maxim Kontsevich



John Morrison



Andrew Odlyzko



Ron Graham

Definition: Let *M(k)* be the family of "minimal" inescapable barbed fences enclosing exactly *k* prison cells, and let *f(k)* be the number of such "minimal" inescapable barbed fences in *M(k).*

Result 1: Characterize all such minimal inescapable barbed fences in *M(k)*.

Result 2: Give a polynomial algorithm for recognizing such "minimal" such inescapable barbed fences.

Result 3: Determine the asymptotic growth of *f(k)*:

$$f(k) \sim c\,\gamma^{K-1} \quad \text{as } k \to \infty$$

where c and γ are roots of certain degree 3 (cubic) polynomials.

The paper invokes
some advanced concepts:

Ramanujan-type
continued franctions

Algebraic
numbers

Recursive
relations

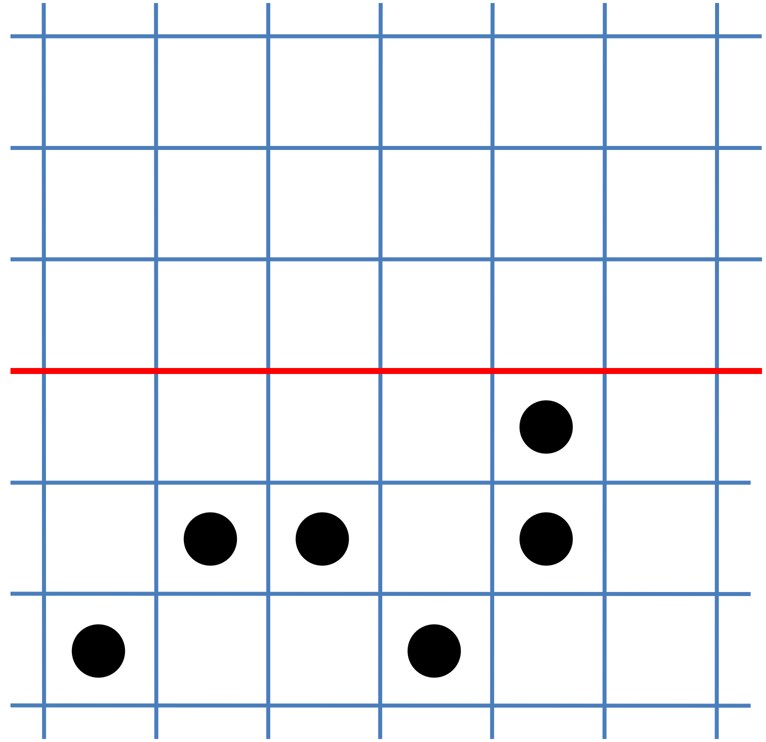
Partial
derivatives

Generating
functions

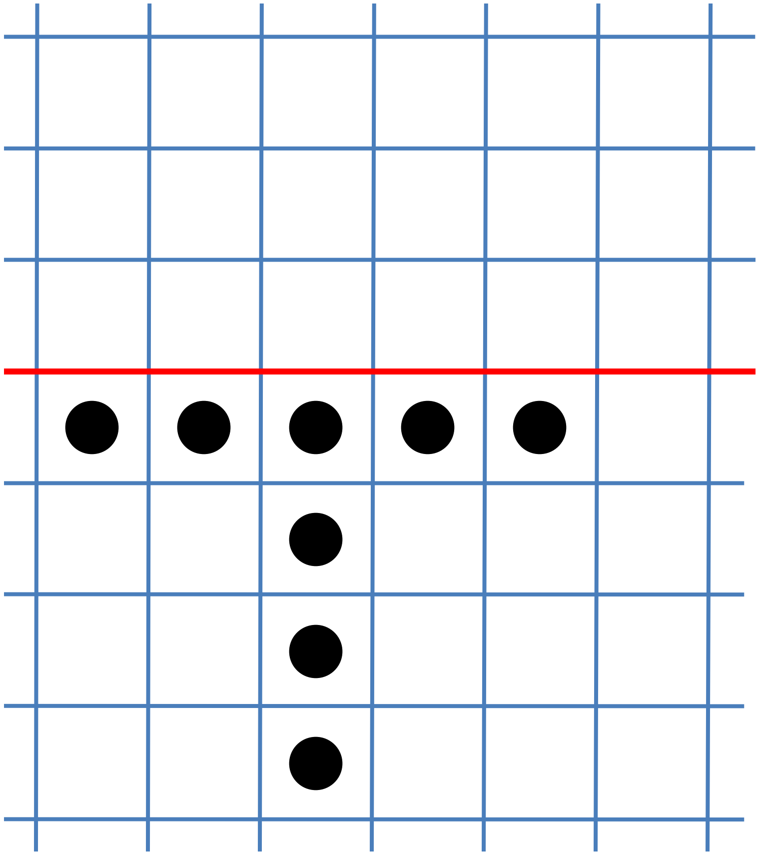
Functional
analysis

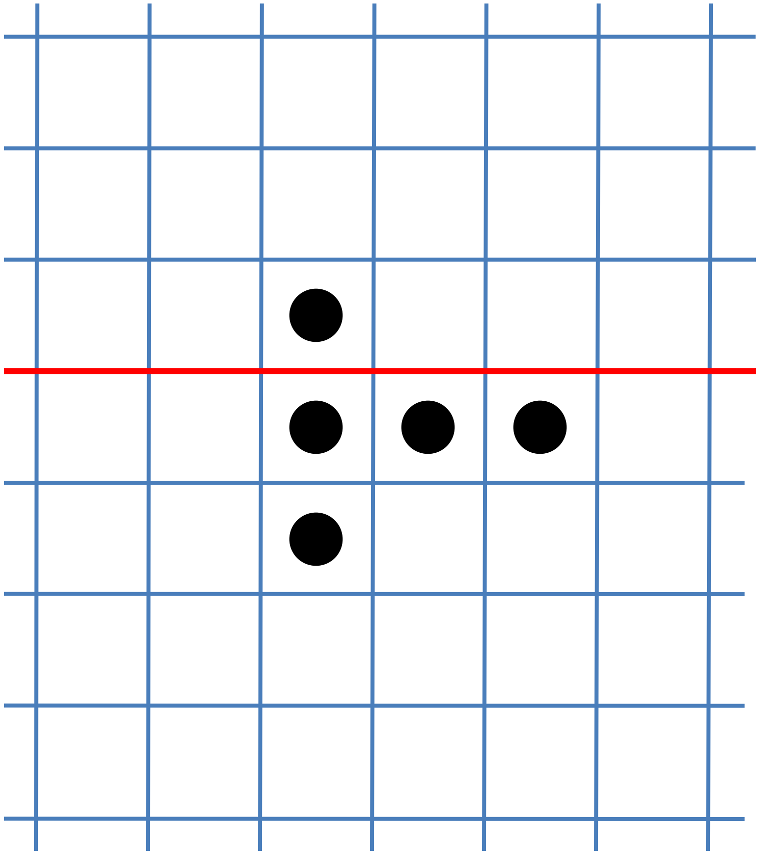
John Conway

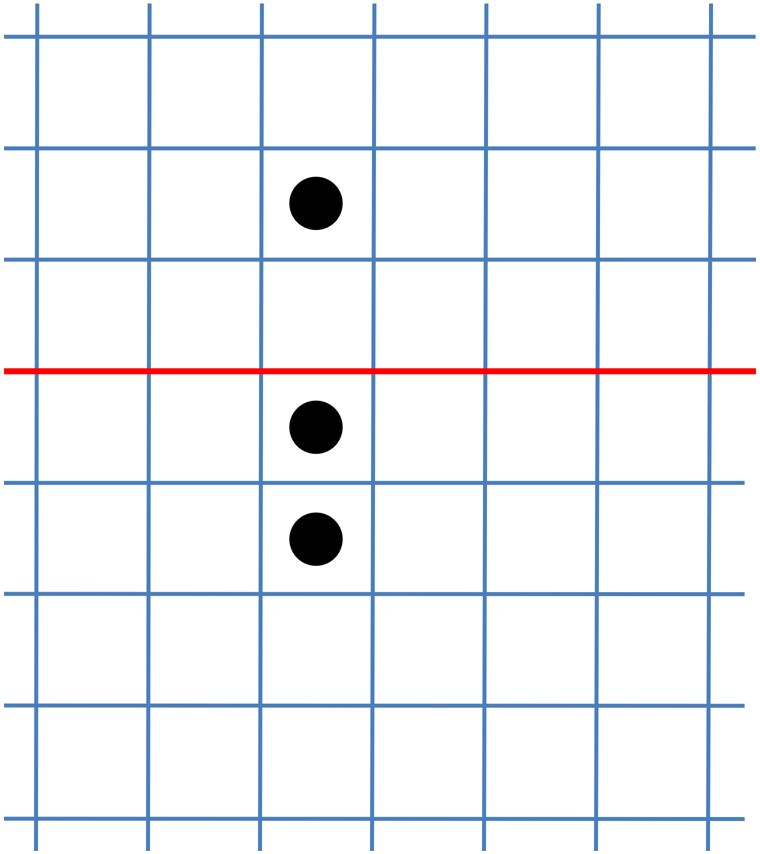
For the Die-Hards: Conway Checkers

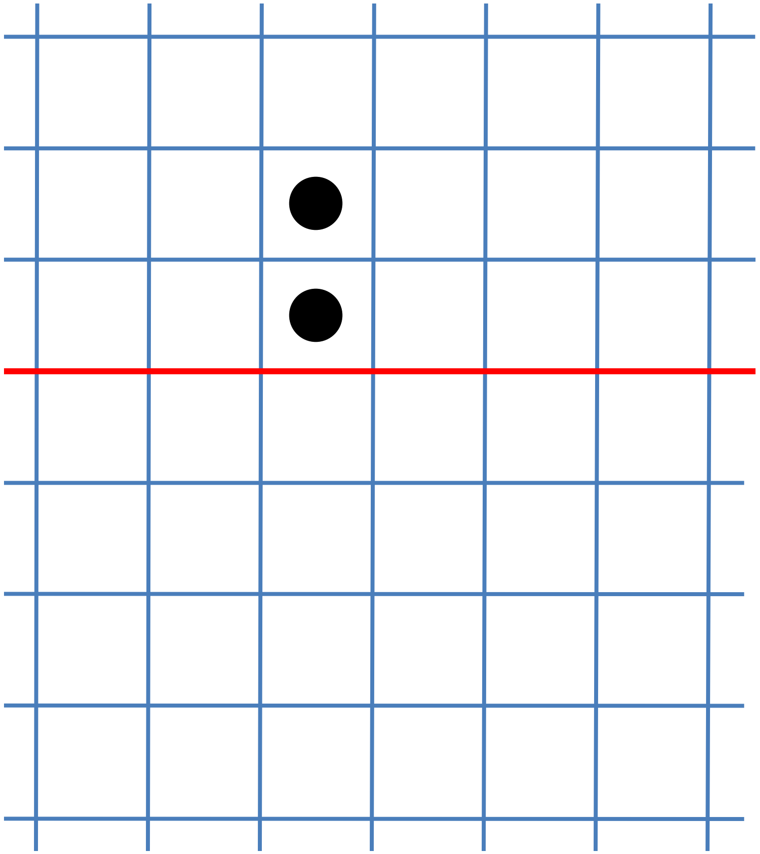


Question: What is the highest row above the designated line that can be reached?

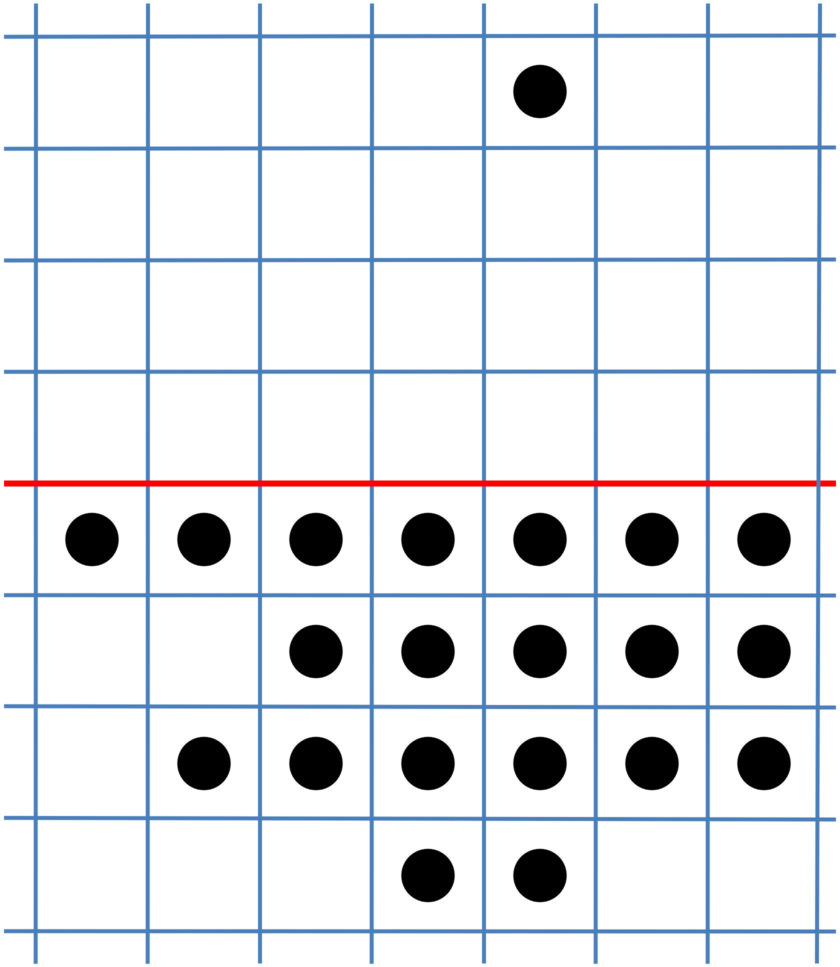




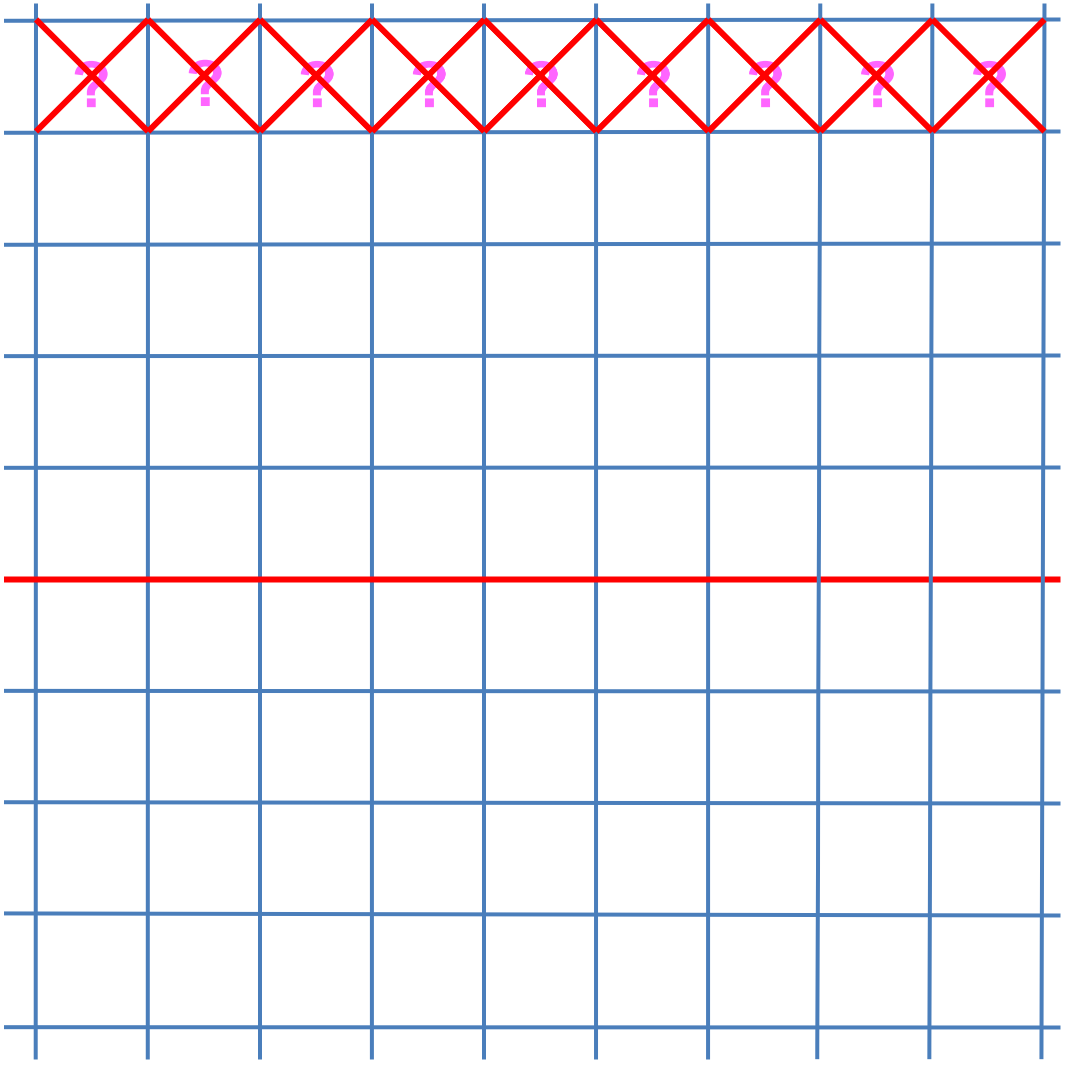




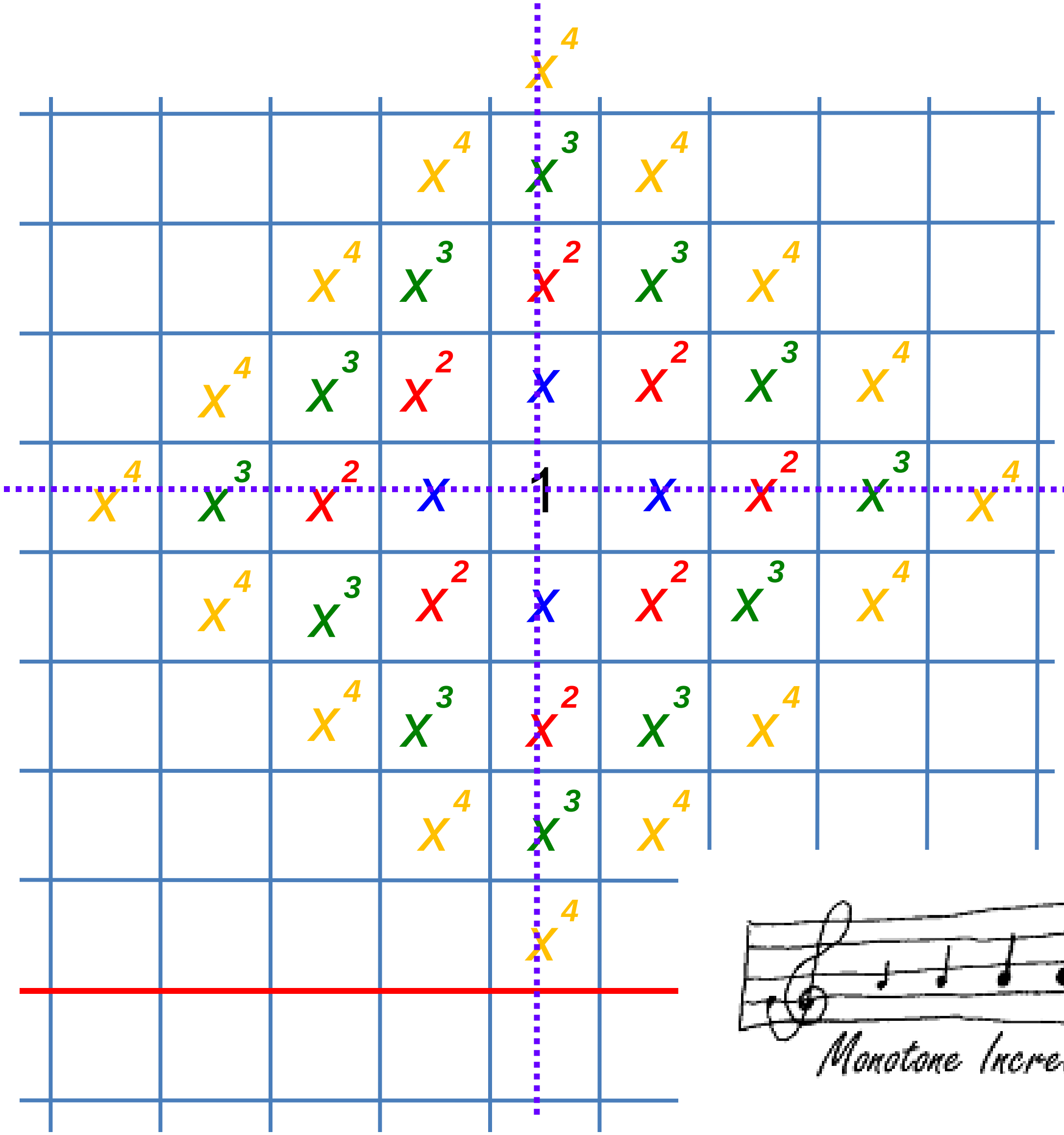
Possible to reach 3rd row

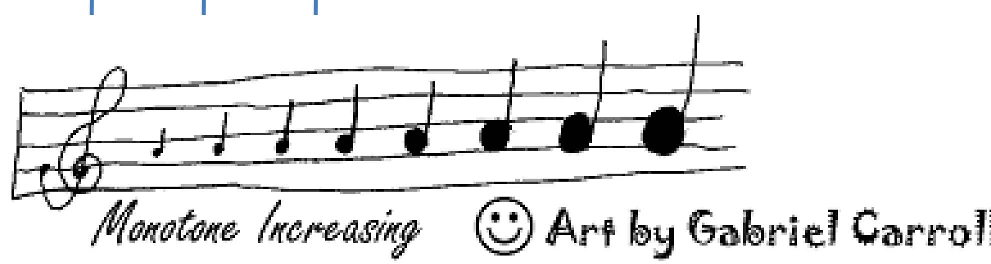


Possible to reach 4th row



Impossible to reach 5th row! Why?





Desired Sum: at least 1!

Conclusion: Impossible to reach 5th row b/c game is finite!

For the bored die-hards:

Build a math circle session around each problem.

Answer:

Every such n appears on exactly $\varphi(n)$ tickets.

$\varphi(n)$ = Euler function

$\varphi(n)$ counts how many integers between 1 and n are *relatively prime* with n.

$\varphi(n) \geq 1$ for all n

2. Infinite Raffle

3	3	1	6	6

For every natural number n the total number of tickets on which divisors of n are written is exactly n.

Show:
Every such n appears on at least 1 ticket.

- **arithmetic functions**
- **multiplicative functions**
- **Dirichlet product (convolution of functions)**
- **Mobius inversion formula**

$\varphi(2009) = \varphi\,(7^2.41) = (7^2 - 7)(41\text{-}1) = 1680$ tickets!

For the bored die-hards:

Build a math circle session around each problem.

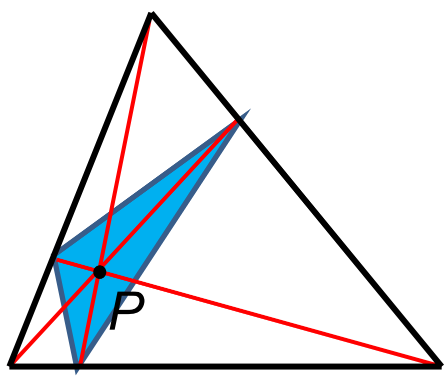
1. Classic geometry approach

- medians, centroids;
  midsegment and parallelogram theorems
- Menelaus's and Ceva's Theorems

2. Projective geometry approach

- Desargue's Theorem
- Yi Sun's brilliancy solution at BAMO '06
- projective geometry

3. The Golden Centroid



Show:
P is the centroid of the big triangle iff P is the centroid of the small triangle.

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