MONOTONICITY FORMULAS AND OBSTACLE TYPE PROBLEMS

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Abstract. Lecture notes for the mini-course given at MSRI, January 18–21, 2011, during the Introductory Workshop for the program on Free Boundary Problem, Theory and Applications. Based on the book A. Petrosyan, H. Shahgholian, N. Uraltseva, Regularity of free boundaries in obstacle type problems

Contents

Lecture 1

1. Obstacle type problems

1.1. The classical obstacle problem.

The Dirichlet Principle. A well-known variational principle of Dirichlet says that the solution of the boundary value problem

$$\Delta u = 0 \quad \text{in } D, \qquad u = g \quad \text{on } \partial D$$

can be found as the minimizer of the (Dirichlet) functional

$$J_0(u) = \int_D |\nabla u|^2 dx$$

among all u such that $u = g$ on ∂D. More precisely (and slightly more generally), if D is a bounded open set in $\mathbb{R}^n$, $g \in W^{1,2}(D)$ and $f \in L^\infty(D)$, then the minimizer of

$$J(u) = \int_D (|\nabla u|^2 + 2fu) dx \tag{1.1}$$

on the set

$$\mathfrak{K}_g = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D)\},$$

solves the Poisson equation

$$-\Delta u + f = 0 \quad \text{in } D, \qquad u = g \quad \text{on } \partial D$$

in the sense of distributions, i.e.

$$\int_D (\nabla u \nabla \eta + f\eta) dx = 0$$

for all test functions $\eta \in C_0^\infty(D)$ (and more generally for all $\eta \in W_0^{1,2}(D)$). One can think of the graph of u as a membrane attached to a thin wire (the graph of g over ∂D).

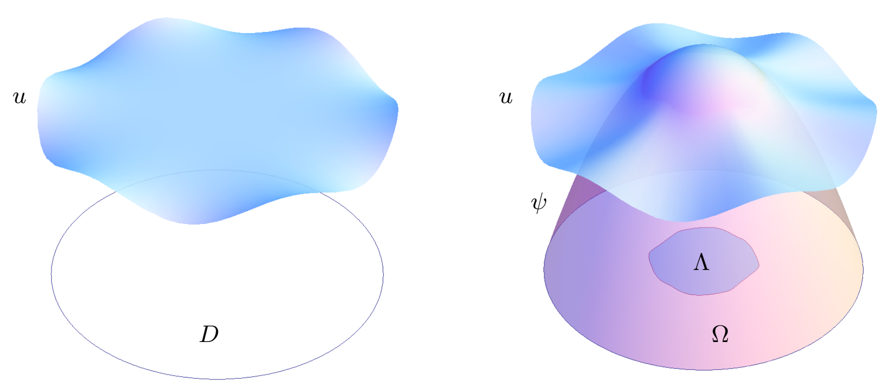

Figure 1.1. Free membrane and the solution of the obstacle problem

The Classical Obstacle Problem. Suppose now that we are given a certain function $\psi \in C^2(D)$, known as the *obstacle*, satisfying the compatibility condition $\psi \leq g$ on ∂D in the sense that $(\psi - g)_+ \in W_0^{1,2}(D)$. Consider then the problem of minimizing the functional (1.1), but now on the constrained set

$$\mathfrak{K}_{g,\psi} = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D), u \geq \psi \text{ a.e. in } D\}.$$

Since J is continuous and strictly convex on a convex subset $\mathfrak{K}_{g,\psi}$ of the Hilbert space $W^{1,2}(D)$, it has a unique minimizer on $\mathfrak{K}_{g,\psi}$.

As before, we may think of the graph of u as a membrane attached to a fixed wire, which is now forced to stay above the graph of ψ. A new feature in this problem is that the membrane can actually touch the obstacle, i.e. the set

$$\Lambda = \{u = \psi\},$$

known as the *coincidence set*, may be nonempty. We also denote

$$\Omega = D \setminus \Lambda.$$

The boundary

$$\Gamma = \partial\Lambda \cap D = \partial\Omega \cap D$$

is called the *free boundary*, as it is not known apriori.

To obtain the conditions satisfied by the minimizer, we note that using the so-called method of penalization (or regularization) one can show that the minimizer is not only in $W^{1,2}(D)$, but actually is in $W^{2,p}_{\mathrm{loc}}(D)$ for any $1 < p < \infty$ and consequently (by the Sobolev embedding theorem) are in $C^{1,\alpha}_{\mathrm{loc}}(D)$ for any $0 < \alpha < 1$ (see also Theorems 1.2 and 1.3). Then, it is straightforward to show that

$$-\Delta u + f = 0 \quad \text{in } \Omega = \{u > \psi\}, \qquad \Delta u = \Delta\psi \quad \text{a.e. on } \Lambda = \{u = \psi\}.$$

Besides,

$$-\Delta u + f \geq 0 \quad \text{in } D$$

in the sense of distributions, i.e.

$$\int_D (\nabla u \nabla \eta + f\eta) dx \geq 0$$

for any nonnegative $\eta \in W_0^{1,2}(D)$, which follows by passing to the limit $\varepsilon \to 0+$ in the inequality

$$\frac{J(u + \varepsilon\eta) - J(u)}{\varepsilon} \geq 0.$$

Combining the properties above, we obtain that the solution of the obstacle problem is a function $u \in W^{2,p}(D)$ for any $p < \infty$, which satisfies

$$-\Delta u + f \geq 0, \quad u \geq \psi, \quad (-\Delta u + f)(u - \psi) = 0 \quad \text{a.e. in } D \tag{1.2}$$

$$u - g \in W_0^{1,2}(D) \tag{1.3}$$

This is known as the *complementary problem* and uniquely characterizes the minimizers of J over $\mathfrak{K}_{g,\psi}$.

Reduction to the case of zero obstacle. Since the governing operator (Δ) is linear it is possible to reduce the problem to the case when the obstacle is 0. Indeed, if u is the solution of the obstacle problem as above, consider the difference $v = u - \psi$. It is straightforward to see that v is the minimizer of the functional

$$J_1(v) = \int_D (|\nabla v|^2 + 2 f_1 v) dx$$

on the set $\mathfrak{K}_{g_1,0}$, where

$$f_1 = f - \Delta\psi, \quad g_1 = g - \psi.$$

Moreover, v will satisfy

$$\Delta v = f_1 \chi_{\{v>0\}} \quad \text{in } D$$

in the sense of distributions.

Problem O. We will consider a simplified version of the problem above with $f_1 \equiv 1$

$$\begin{cases} \Delta u = \chi_{\{u>0\}} & \text{in } D \\ u \geq 0 & \text{in } D \end{cases} \tag{\textbf{O}}$$

The free boundary in this problem is $\Gamma = \partial\{u > 0\} \cap D$. Occasionally we will also use the notations $\Omega := \{u > 0\}$ and $\Lambda := D \setminus \Omega$.

1.2. **A problem from potential theory.** Let Ω be a bounded open set in $\mathbb{R}^n$ and f a certain bounded measurable function on Ω. Consider then the Newtonian potential of the distribution of mass $f\chi_\Omega$, i.e.

$$U(x) = \Phi_n * (f\chi_\Omega)(x) = \int_\Omega \Phi_n(x - y) f(y) dy$$

where Φ_n is the fundamental solution of the Laplacian in $\mathbb{R}^n$, i.e. $\Delta\Phi_n = \delta$ in the sense of distributions. It can be shown that the potential U is in $W^{2,p}_{\text{loc}}(\mathbb{R}^n)$ for any $p < \infty$ and satisfies

$$\Delta U = f\chi_\Omega \quad \text{in } \mathbb{R}^n$$

in the sense of distributions (or a.e., which is the same in this case). In particular, U is harmonic in $\mathbb{R}^n \setminus \overline{\Omega}$.

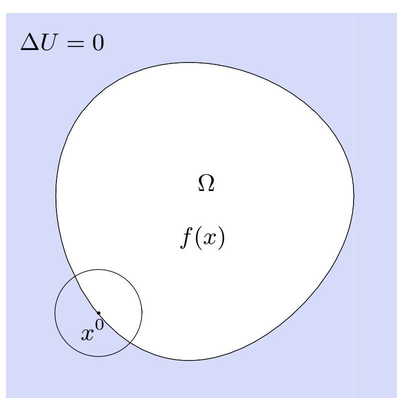

FIGURE 1.2. Harmonic continuation of Newtonian potentials

Let $x^0 \in \partial\Omega$ and suppose for some small $r > 0$ there is a harmonic function h in the ball $B_r(x^0)$ such that $h = U$ on $B_r \setminus \Omega$. We say in this case that h is a harmonic continuation of U into Ω at x^0. If such continuation exists, the difference $u = U - h$ satisfies

$$\Delta u = f\chi_\Omega \quad \text{in } B_r(x^0), \qquad u = |\nabla u| = 0 \quad \text{on } B_r(x^0) \setminus \Omega. \tag{1.4}$$

Using the Cauchy-Kowalevskaya theorem, it is straightforward to show that the harmonic continuation exists if $\partial\Omega$ and f are real-analytic in a neighborhood of x^0. We are interested in the converse question: if the solution of (1.4) exists for some $r > 0$, then what can be said about the regularity of $\partial\Omega$?

Problem I. The second model problem that we will study in this mini-course is going to be the one above with $f \equiv 1$

$$\begin{cases} \Delta u = \chi_\Omega & \text{in } D \\ u = |\nabla u| = 0 & \text{on } D \setminus \Omega \end{cases} \tag{\textbf{I}}$$

Note that we do not assume that $u \geq 0$. In fact, when $u \geq 0$ then Problem **I** is equivalent Problem **O**. For that reason we may refer to Problem **I** as the *obstacle problem without obstacle*. The free boundary in this problem is $\Gamma := \partial\Omega \cap D$. Note that without loss of generality we may assume that $\Omega = \{u \neq 0 \text{ or } |\nabla u| \neq 0\}$.

1.3. **Two-phase membrane problem.** Given a bounded open set D in $\mathbb{R}^n$, $g \in W^{1,2}(D)$ and nonnegative bounded measurable functions λ_+ and λ_- in D consider the problem of minimizing the functional

$$J(u) = \int_D (|\nabla u|^2 + 2\lambda_+(x)u^+ + 2\lambda_-(x)u^-)dx \tag{1.5}$$

over the set

$$\mathfrak{K}_g = \{u \in W^{1,2}(D) : u - g \in W^{1,2}_0(D)\}.$$

Here

$$u^+ = \max\{u, 0\}, \qquad u^- = \max\{-u, 0\}$$

The case $\lambda_- = 0$ and $g \geq 0$ the problem is equivalent to the obstacle problem with zero obstacle, see above.

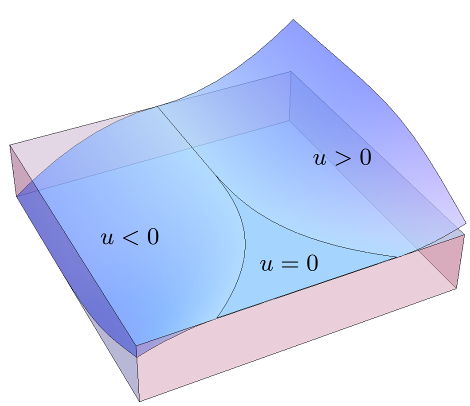

FIGURE 1.3. Two-phase membrane problem

Possible applications of this functional may come in several problems when the external force is a function of u itself, in this case the external force is

$$\lambda_+\chi_{\{u>0\}} - \lambda_-\chi_{\{u<0\}}.$$

As a specific example, imagine a membrane in $\mathbb{R}^{n+1}$ under the influence of an electric or a magnetic field of the form

$$F = -\lambda_+\chi_{\{x_{n+1}>0\}}e_{n+1} + \lambda_-\chi_{\{x_{n+1}<0\}}e_{n+1},$$

where e_{n+1} is $(n+1)$-th vector in the standard basis in $\mathbb{R}^{n+1}$. If we assume the membrane to be modeled by a graph in the x_{n+1}-direction and to be clamped in at the boundary, then the equilibrium state would correspond to the minimizer of our functional.

Another physical interpretation of this problem is the consideration of a thin membrane (film) which is fixed on the boundary of a given domain, and some part of the boundary data of this film is below the surface of a thick liquid (heavier than the film itself). Now the weight of the film produces a force downwards (call it λ_+) on that part of the film which is above the liquid surface. On the other side the part in the liquid is pushed upwards by a force λ_-, since the liquid is heavier than the film. Obviously the equilibrium state of the film is given by a minimization of the above mentioned functional.

One of the difficulties one confronts in this problem is that the interface $\{u = 0\}$ consists in general of two parts – one where the gradient of u is nonzero and one where the gradient of u vanishes. Close to points of the latter part we expect the gradient of u to have linear growth. However, because of the decomposition into two different types of growth, it is not possible to derive a growth estimate by classical techniques.

A good reference for this problem is Shahgholian-Uraltseva-Weiss [SUW07].

Problem II. Our next model problem is going to be the two-phase obstacle problem with $\lambda_\pm \equiv const$

$$\Delta u = \lambda_+\chi_{\{u>0\}} - \lambda_-\chi_{\{u<0\}} \quad \text{in } D. \tag{\textbf{II}}$$

We will denote $\Omega_+ = \{u > 0\}$ and $\Omega_- = \{u < 0\}$ and call the the *positive* and *negative phases*, respectively. The boundaries of these apriori unknown regions will form the free boundary in this problem. Namely, we denote $\Gamma_\pm = \partial\Omega_\pm \cap D$ and $\Gamma = \Gamma_+ \cup \Gamma_-$.

1.4. General framework of obstacle type problems.

Some of our results will be proved in a more general framework of what we call *obstacle-type problems.*

Let D be an open subset in $\mathbb{R}^n$ and suppose that we are given a function $u \in L^\infty_{\text{loc}}(D)$ such that

$$\Delta u = g \quad \text{in } D \tag{1.6}$$

in the sense of distributions for some function $g \in L^\infty(D)$. This means that

$$\int_D u\Delta\eta\, dx = \int_D g\eta\, dx$$

for all test functions $\eta \in C_0^\infty(D)$. Assume further that there exists an open subset $G \subset D$ such that

$$g(x) = f(u(x))\chi_G \quad \text{in } D, \tag{1.7}$$

$$|\nabla u| = 0 \quad \text{on } D \setminus G, \tag{1.8}$$

where f is a monotone nondecreasing function

$$f(s) \le f(t) \quad \text{if } s \le t. \tag{1.9}$$

Depending on the problem, the free boundary is going to be $\partial G \cap D$ and/or the set where f has a discontinuity.

The Problems **O**, **I**, and **II** fit into the framework (1.6)–(1.9) in the following way.

- Problem **O**: $G = \{u > 0\}$, $g = \chi_G$.
- Problem **I**: $G = \Omega$, $g = \chi_G$.
- Problem **II**: $G = D$, $g = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}}$.

One of the results that we are going to prove in this course is the following version of the theorem of Shahgholian [Sha03] (for the classical obstacle problem the result goes back to Frehse [Fre72]).

Theorem 1.1 ($C^{1,1}$-regularity)**.** *Let $u \in L^\infty(D)$ satisfy* (1.6)–(1.9)*. Then $u \in C^{1,1}_{\mathrm{loc}}(D)$ and*

$$\|u\|_{C^{1,1}(K)} \le C \left(\|u\|_{L^\infty(D)} + \|g\|_{L^\infty(D)} \right),$$

for any $K \Subset D$, where $C = C(n, \operatorname{dist}(K, \partial D))$.

The proof of this theorem will require the use of the so-called Alt-Caffarelli-Friedman monotonicity formula [ACF84], that we will discuss in the next lecture. However, the proof of this result for Problem **O** is rather elementary and can be found in the appendix to this lecture.

1.5. **$W^{2,p}$-Regularity of solutions.** Even though the optimal $C^{1,1}$ regularity will require some efforts, it turns out that the $W^{2,p}$ regularity for any $1 < p < \infty$ is immediate. In fact, one has to use only the equation (1.6).

Theorem 1.2. *Let $u \in L^p(D)$, $g \in L^p(D)$, $1 < p < \infty$, satisfy $\Delta u = g$ in D in the sense of distributions. Then $u \in W^{2,p}_{\mathrm{loc}}(D)$ and*

$$\|u\|_{W^{2,p}(K)} \le C \left(\|u\|_{L^p(D)} + \|g\|_{L^p(D)} \right)$$

for any $K \Subset D$ with $C = C(p, n, K, D)$. □

Thus, for solutions of (1.6) we obtain

$$u \in W^{2,p}_{\mathrm{loc}}(D), \quad \text{for all} \quad 1 < p < \infty, \tag{1.10}$$

Consequently, we also have

$$u \in C^{1,\alpha}_{\mathrm{loc}}(D), \quad \text{for all} \quad 0 < \alpha < 1, \tag{1.11}$$

by the Sobolev embedding $W^{2,p} \hookrightarrow C^{1,\alpha}$ with $\alpha = 1 - n/p$ for $p > n$. It is worth noting that for $u \in W^{2,p}_{\mathrm{loc}}(D)$ the validity of (1.6) a.e. in D is equivalent to the one in the sense of distributions.

An easy counterexample shows that in general we cannot have $p = \infty$ in (1.10) and $\alpha = 1$ in (1.11). Instead we have the following

Theorem 1.3. *Let $u \in L^\infty(D)$, $g \in L^\infty(D)$ satisfy $\Delta u = g$ in the sense of distributions. Then $u \in W^{2,p}_{\mathrm{loc}}(D) \cap C^{1,\alpha}_{\mathrm{loc}}(D)$ for all $1 < p < \infty$, $0 < \alpha < 1$ and*

$$|\nabla u(x) - \nabla u(y)| \le C|x-y| \log \frac{1}{|x-y|}, \quad \text{for } x, y \in K \Subset D,\ |x-y| \le 1/e$$

and $C = C(n, K, D)\left(\|u\|_{L^\infty(D)} + \|g\|_{L^\infty(D)}\right)$. □

Essentially, we will show that the logarithmic term in this theorem can be dropped if one assumes the additional structure as in (1.6)–(1.9).

1.6. **The Thin Obstacle (Signorini) problem.** Let D be a domain in $\mathbb{R}^n$ and $\mathcal{M}$ a smooth $(n-1)$-dimensional manifold in $\mathbb{R}^n$ that divides D into two parts: D_+ and D_-. For given functions $\phi : \mathcal{M} \to \mathbb{R}$ and $g : \partial D \to \mathbb{R}$ satisfying $g > \phi$ on $\mathcal{M} \cap \partial D$, consider the problem of minimizing the Dirichlet integral

$$J(u) = \int_D |\nabla u|^2 dx$$

on the closed convex set

$$\mathfrak{K} = \{u \in W^{1,2}(D) : u = g \text{ on } \partial D,\ u \ge \phi \text{ on } \mathcal{M} \cap D\}.$$

This problem is known as the *thin obstacle problem*, with ϕ known as the *thin obstacle*. Them main difference from the *classical obstacle problem* is that u is constrained to stay above the obstacle ϕ only on $\mathcal{M}$ and not on the entire domain D.

The thin obstacle problem arises in a variety of situations of interest for the applied sciences. It presents itself in elasticity, when an elastic body is at rest, partially laying on a surface $\mathcal{M}$. It also arises in financial mathematics in situations in which the random variation of an underlying asset changes discontinuously (in the form of the obstacle problem for the half Laplacian). It models the flow of a saline concentration through a semipermeable membrane when the flow occurs in a preferred direction.

When $\mathcal{M}$ and ϕ are smooth, it has been proved by Caffarelli [Caf79] (and in more general case by Uraltseva [Ura85]) that the minimizer u in the thin obstacle problem is of class $C^{1,\alpha}_{\mathrm{loc}}(D_\pm \cup \mathcal{M})$. Since we can make free perturbations away from $\mathcal{M}$, it is easy to see that u satisfies

$$\Delta u = 0 \quad \text{in } D \setminus \mathcal{M} = D_+ \cup D_-,$$

but in general u does not need to be harmonic across $\mathcal{M}$. Instead, on $\mathcal{M}$, one has the following *complementary conditions*

$$u - \phi \ge 0, \quad \partial_{\nu^+} u + \partial_{\nu^-} u \ge 0, \quad (u-\phi)(\partial_{\nu^+} u + \partial_{\nu^-} u) = 0,$$

where $\nu^\pm$ are the outer unit normals to $D_\pm$ on $\mathcal{M}$. One of the main objects of study in this problem is the so-called *coincidence set*

$$\Lambda(u) := \{x \in \mathcal{M} : u(x) = \phi(x)\}$$

and its boundary (in the relative topology on $\mathcal{M}$)

$$\Gamma(u) := \partial_{\mathcal{M}} \Lambda(u),$$

known as the *free boundary*.

A similar problem is obtained when $\mathcal{M}$ is a part of ∂D and one minimizes $J(u)$ over the convex set

$$\mathfrak{K} = \{u \in W^{1,2}(D) : u = g \text{ on } \partial D \setminus \mathcal{M},\ u \geq \phi \text{ on } \mathcal{M}\}.$$

In this case u is harmonic in D and satisfies the so-called *Signorini boundary conditions*

$$u - \phi \geq 0, \quad \partial_\nu u \geq 0, \quad (u-\phi)\partial_\nu u = 0$$

on $\mathcal{M}$, where ν is the outer unit normal on ∂D. This problem is known as the *boundary thin obstacle problem* or the *Signorini problem*. Note that in the case when $\mathcal{M}$ is a plane and D and g are symmetric with respect to $\mathcal{M}$, then the thin obstacle problem in D is equivalent to the boundary obstacle problem in D_+.

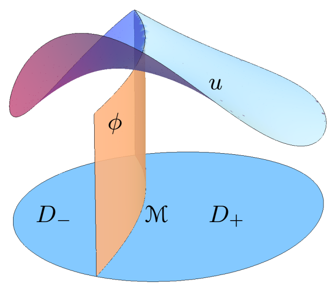

FIGURE 1.4. The thin obstacle (Signorini) problem

Problem S. We will make the following simplifying assumptions in considering the problem. We will assume that $\mathcal{M} = \{x_n = 0\}$, $\phi \equiv 0$, and D is symmetric with respect to $\mathcal{M}$, the solution u is even in x_n variable and satisfies

$$(\mathbf{S}) \qquad \begin{aligned} \Delta u = 0 \quad &\text{in } D_+ \cup D_- \\ u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u\, \partial_{x_n} u = 0 \quad &\text{on } D' = \mathcal{M} \cap D \end{aligned}$$

The free boundary in the problem is the set $\Gamma = \partial\{u(\cdot, 0) > 0\}$.

One of the results that we are going to prove for Problem **S** is the following optimal regularity result, that was first proved by Athanasopoulos-Caffarelli [AC04].

Theorem 1.4 (Optimal regularity in Signorini problem)**.** *Let $u \in W^{1,2}(D)$ be a solution of Problem* **S**. *Then $u \in C^{1,1/2}_{\text{loc}}(D_\pm \cup D')$, moreover*

$$\|u\|_{C^{1,1/2}(K_\pm \cup K')} \leq C \|u\|_{L^2(D)},$$

for any $K \Subset D$, symmetric with respect to $\mathcal{M}$, where $C = C(n, \operatorname{dist}(K, \partial D))$.

Note that this is indeed the best regularity possible for the solution of the Signorini problem, since the function

$$u(x) = \operatorname{Re}(x_1 + i|x_n|)^{3/2}$$

is a solution of the Signorini problem. The proof of the theorem above will require the application of Almgren's frequency formula [Alm00], combined with Alt-Caffarelli-Friedman monotonicity formula [ACF84].

As a starting point we will take the following regularity result due to Caffarelli [Caf79] and Ural'tseva [Ura85] (in a more general setting).

Theorem 1.5. *Let $u \in W^{1,2}(D)$ be a solution of Problem* **S**. *Then there exists $\alpha > 0$ such that $u \in C^{1,\alpha}_{\mathrm{loc}}(D_\pm \cup D')$, moreover*

$$\|u\|_{C^{1,\alpha}(K_\pm \cup K')} \le C\|u\|_{L^2(D)},$$

for any $K \Subset D$, symmetric with respect to $\mathcal{M}$, where $C = C(n, \operatorname{dist}(K, \partial D))$.

Appendix.

Proof of Theorem 1.1 for Problem **O**. We start with the following result on the growth of u away from the free boundary $\partial\{u > 0\}$.

Lemma 1.6 (Quadratic growth)**.** *Let u be a solution of Problem* **O**, *$\Omega = \{u > 0\}$, $x^0 \in \partial\Omega$, and $B_{2R}(x^0) \subset D$. Then*

$$\sup_{B_R(x^0)} u \le C_n R^2.$$

Proof. Split u into the sum $u_1 + u_2$ in $B_{2R}(x^0)$, where

$$\Delta u_1 = \Delta u, \quad \Delta u_2 = 0 \quad \text{in } B_{2R}(x^0)$$
$$u_1 = 0, \quad u_2 = u \quad \text{on } \partial B_{2R}(x^0).$$

We then estimate the functions u_1 and u_2 separately.

1) To estimate u_1, we consider the auxiliary function

$$\phi(x) = \frac{1}{2n}(4R^2 - |x - x^0|^2),$$

which is the solution of

$$\Delta\phi = -1 \quad \text{in } B_{2R}(x^0), \qquad \phi = 0 \quad \text{on } \partial B_{2R}(x^0).$$

Then we have

$$-\phi(x) \le u_1(x) \le 0, \quad x \in B_{2R}(x^0).$$

This follows from the comparison principle, since

$$0 \le \Delta u_1 \le 1 \quad \text{in} \quad B_{2R}(x^0),$$

and that both u_1 and ϕ vanish on $\partial B_{2R}(x^0)$. In particular, this implies that

$$|u_1(x)| \le \frac{2}{n} R^2, \quad x \in B_{2R}(x^0).$$

2) To estimate u_2, observe that $u_2 \ge 0$ in $B_{2R}(x^0)$, since $u_2 = u \ge 0$ on $\partial B_{2R}(x^0)$. Also note that $u_1(x^0) + u_2(x^0) = u(x^0) = 0$ and the estimate of u_1 gives

$$u_2(x^0) \le C_n R^2.$$

Applying now the Harnack inequality, we obtain

$$u_2(x) \le C(n) u_2(x^0) \le C_n R^2, \quad x \in B_R(x^0).$$

Finally, combining the estimates for u_1 and u_2, we obtain the desired estimate for u. □

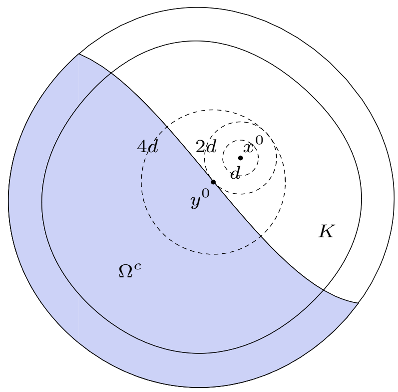

FIGURE 1.5. Case 1) $d < \delta/5$

We can now proceed to the proof of Theorem 1.1.

Let $K \Subset D$, $x^0 \in K$, and let $\delta = \frac{1}{2}\operatorname{dist}(K, \partial D)$ and $d = \frac{1}{2}\operatorname{dist}(x^0, \Omega^c)$. Then we have two possibilities.

1) $d < \delta/5$. In this case, let $y_0 \in \partial B_{2d}(x^0) \cap \partial\Omega$. Then $B_{8d}(y_0) \subset B_{10d}(x^0) \subset\subset D$. Applying Lemma 1.6, we have

$$\|u\|_{L^\infty(B_{4d}(y_0))} \le C_n d^2.$$

Now note that $B_{2d}(x^0) \subset B_{4d}(y_0)$ and $\Delta u = 1$ in $B_{2d}(x^0)$. By the interior elliptic estimate we then have

$$\|D^2 u\|_{L^\infty(B_d(x^0))} \le C_n.$$

2) $d \ge \delta/5$. In this case, the interior derivative estimate for u in $B_d(x^0)$ gives

$$\|D^2 u\|_{L^\infty(B_d(x^0))} \le C(n)\left(\frac{\|u\|_{L^\infty(D)}}{\delta^2} + 1\right).$$

Combining cases 1) and 2) above, we obtain

$$\|u\|_{C^{1,1}(K)} \le C(n)\left(\frac{\|u\|_{L^\infty(D)}}{\delta^2} + 1\right).$$

Lecture 2

2. The optimal regularity in obstacle type problems

2.1. ACF monotonicity formula.

2.1.1. *Harmonic Functions.* For a continuous $u \in W^{1,2}(B_1)$ define the following quantity:

$$J(r,u) = \frac{1}{r^2}\int_{B_r} \frac{|\nabla u|^2 dx}{|x|^{n-2}}, \quad 0<r<1.$$

It is relatively straightforward to show that $r \mapsto J(r,u)$ is monotone if u is a harmonic function. Namely, if we represent u as a locally uniformly convergent series

$$u(x) = \sum_{k=0}^{\infty} f_k(x),$$

where $f_k(x)$ are k-th order homogeneous harmonic polynomials, and use the orthogonality of homogeneous harmonic polynomials of different order, we will have

$$\begin{aligned} J(r,u) &= \frac{1}{r^2}\int_0^r \int_{\partial B_1} |\nabla u(\rho\theta)|^2 \rho \, d\theta d\rho = \\ &= \frac{1}{r^2}\int_0^r \int_{\partial B_1} \rho \sum_{k=1}^{\infty} |\nabla f_k(\rho\theta)|^2 d\theta d\rho \\ &= \frac{1}{r^2}\int_0^r \int_{\partial B_1} \sum_{k=1}^{\infty} \rho^{2k-1} |\nabla f_k(\theta)|^2 d\theta d\rho \\ &= \sum_{k=1}^{\infty} a_k r^{2(k-1)}, \end{aligned}$$

with

$$a_k = \frac{1}{2k}\int_{\partial B_1} |\nabla f_k(\theta)|^2 d\theta \geq 0.$$

This implies that $r \mapsto J(r,u)$ is monotone increasing.

We next illustrate how to use this monotonicity formula to obtain interior gradient estimates for harmonic functions.

a) Letting $r \to 0+$, we obtain

$$J(0+,u) \leq J(1/2,u).$$

On the other hand, since u is C^1 (actually real analytic) at the origin, it is easy to see that $J(0+,u) = c_n|\nabla u(0)|^2$, for $c_n > 0$, which implies that

$$c_n|\nabla u(0)|^2 \leq J(1/2,u).$$

b) Next, we claim that $J(1/2,u)$ is controllable by the L^2-norm of u over B_1. Indeed, consider the function $|x|^{2-n}$ in $B_{1/2}$ and extend it to a function V on B_1 in a smooth nonnegative way, so that $V \equiv 0$ near ∂B_1. Let also $\delta > 0$ be a small number and $\hat{V} = \hat{V}_\delta = \min\{V, \delta^{2-n}\}$.

Then, using the equality $|\nabla u|^2 = \Delta(u^2/2)$, we have

$$\begin{aligned}\int_{B_{1/2}\setminus B_\delta} \frac{|\nabla u|^2}{|x|^{n-2}} dx &\le \int_{B_1} \Delta\left(\frac{u^2}{2}\right)\hat{V} dx\\ &= -\int_{\partial B_\delta} \frac{u^2}{2}(n-2)\delta^{1-n} dH^{n-1} + \int_{B_1\setminus B_\delta}\left(\frac{u^2}{2}\right)\Delta V\\ &\le \int_{B_1\setminus B_{1/2}}\left(\frac{u^2}{2}\right)\Delta V dx,\end{aligned}$$

which implies as $\delta \to 0$

$$J(1/2, u) \le C_n \|u\|^2_{L^2(B_1)}.$$

Combining the estimates in a) and b) above we arrive at

$$|\nabla u(0)| \le C_n \|u\|_{L^2(B_1)}$$

Obviously, this is not the best way to establish the inequality above. This method is rather a prelude to the application of the monotonicity formula of Alt-Caffarelli-Friedman [ACF84] for a pair of nonnegative subharmonic functions with "disjoint" supports.

2.1.2. *ACF monotonicity formula.*

Theorem 2.1 (Alt-Caffarelli-Friedman (ACF) monotonicity formula)**.** *Let $u_\pm$ be a pair of continuous functions in B_1 such that*

$$u_\pm \ge 0, \quad \Delta u_\pm \ge 0, \quad u_+ \cdot u_- = 0 \quad \text{in } B_1.$$

Then the functional

$$\begin{aligned} r \mapsto \Phi(r) = \Phi(r, u_+, u_-) &:= J(r, u_+)J(r, u_-)\\ &= \frac{1}{r^4}\int_{B_r}\frac{|\nabla u_+|^2 dx}{|x|^{n-2}}\int_{B_r}\frac{|\nabla u_-|^2 dx}{|x|^{n-2}}\end{aligned}$$

is finite and nondecreasing for $0 < r < 1$.

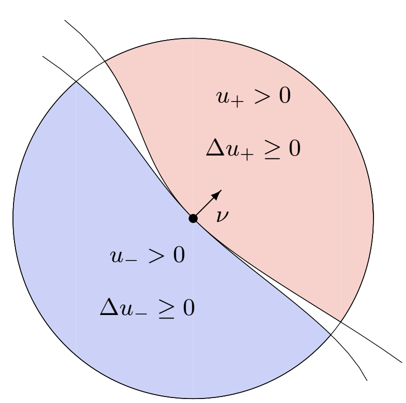

FIGURE 2.1. ACF monotonicity formula

We then have the following series of remarks.

a) Each of the terms $J(r,u_\pm)$ can be understood as a weighted average of $|\nabla u_\pm|^2$. For instance, if $u_\pm=\alpha_\pm x_1^\pm$, then

$$J(r,u_\pm)\equiv c_n\alpha_\pm^2,\quad \Phi(r,u_+,u_-)\equiv c_n^2\alpha_+^2\alpha_-^2$$

b) Let $\mathcal{C}$ be a cone with vertex at the origin, i.e. given a subset $\Sigma_0\subset\partial B_1$,

$$\mathcal{C}=\{r\,\theta : r>0,\ \theta\in\Sigma_0\}.$$

Consider a homogeneous harmonic function in $\mathcal{C}$ of the form

$$h(r\,\theta)=r^\alpha f(\theta),\qquad \alpha>0,$$

vanishing on $\partial\mathcal{C}$. We have

$$\begin{aligned}\Delta h&=\partial_{rr}h+\frac{n-1}{r}\,\partial_r h+\frac{1}{r^2}\Delta_\theta h\\&=r^{\alpha-2}[(\alpha(\alpha-1)+(n-1)\alpha)f(\theta)+\Delta_\theta f(\theta)].\end{aligned}$$

Thus, we have that h is harmonic in $\mathcal{C}$ iff f is an eigenfunction for the spherical Laplacian Δ_θ in Σ_0:

$$-\Delta_\theta f(\theta)=\lambda f(\theta)\quad\text{in }\Sigma_0,$$

where

$$\lambda=\alpha(n-2+\alpha).$$

Thus, if we take two disjoint open set $\Sigma_\pm$ on the unit sphere, find there first eigenvalues $\lambda_\pm$ and the corresponding eigenfunctions $f_\pm$, then the homogeneous harmonic functions

$$u_\pm=r^{\alpha_\pm}f_\pm(\theta),\quad\text{in }\mathcal{C}_\pm=\{r\,\theta : r>0,\ \theta\in\Sigma_\pm\}$$

where $\alpha_\pm>0$ are found from the identity

$$\lambda_\pm=\alpha_\pm(n-2+\alpha_\pm).$$

Observe that $u_\pm$ extended by zero in the complements of $\mathcal{C}_\pm$ are subharmonic in $\mathbb{R}^n$. Further, it is easy to calculate that

$$\Phi(r,u_+,u_-)=Cr^{2(\alpha_++\alpha_--2)}$$

for $C>0$ and therefore the monotonicity formula will follow in this case once we know

$$\alpha_++\alpha_-\geq 2.$$

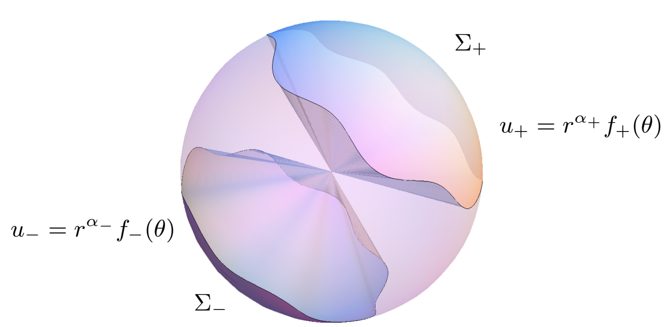

FIGURE 2.2. Two homogeneous harmonic functions $u_\pm$ in cones generated by spherical regions $\Sigma_\pm$

This inequality has been established first by Friedland-Hayman [FH76]. What is interesting is that it actually implies the monotonicity formula for all $u_\pm$ as in Theorem 2.1, not necessarily homogeneous, see the appendix to this lecture. We refer to the book of Caffarelli-Salsa [CS05, Chapter 12] for a detailed proof of the Friedland-Hayman inequality.

If u is a nonnegative subharmonic function, then $J(r,u)$ can be controlled in terms of L^2-norm of u, arguing as in the case of harmonic function and using that $|\nabla u|^2 \leq \Delta(u^2/2)$. More precisely, one has that

$$J(1/2,u) \leq C_n \|u\|^2_{L^2(B_1)}.$$

Combining this with the ACF monotonicity formula, we obtain the following estimate.

Theorem 2.2 (ACF estimate)**.** *Let $u_\pm$ be as in Theorem 2.1. Then*

$$\Phi(r,u_+,u_-) \leq C_n \|u_+\|^2_{L^2(B_1)} \|u_-\|^2_{L^2(B_1)}$$

for $0 < r \leq 1/2$.

In some applications, this weaker form of the monotonicity formula turns out to be sufficient. However, in other applications, one needs to use Theorem 2.1 at its full strength, moreover, one needs to have the information for the case of $\Phi(r)$ being a constant in some interval.

Theorem 2.3 (Case of equality in ACF monotonicity formula)**.** *Let $u_\pm$ be as in Theorem 2.1 and suppose that $\Phi(r_1) = \Phi(r_2)$ for some $0 < r_1 < r_2 < 1$. Then one of the following holds:*

(i) *either $u_+ = 0$ in B_{r_2} or $u_- = 0$ in B_{r_2},*
(ii) *for every $r_1 < r < r_2$, $\operatorname{supp} u_\pm \cap \partial B_r$ are complementary half-spheres and $u_+ \Delta u_+ = u_- \Delta u_- = 0$ on B_{r_2} in the sense of measures.*

This follows directly from analyzing the proof of the ACF Monotonicity Formula, in particular, from analyzing the case of inequality in Friedland-Hayman inequality: $\alpha_+ + \alpha_- = 2$ if and only if $\Sigma_\pm$ are complementary half-spherical caps. For more details we refer to the paper by Caffarelli-Karp-Shahgholian [CKS00] where this theorem has first appeared.

2.2. **Optimal regularity in obstacle type problems.** In this section, following the idea of Shahgholian [Sha03], we use the ACF estimate to prove Theorem 1.1.

One of the important ingredients in the proof is the following fundamental lemma. Basically, it says that the positive and negative parts of the directional derivatives of solutions to obstacle type-problems satisfy to the assumptions in the ACF monotonicity formula.

Lemma 2.4. *Let $u \in C^1(D)$ satisfy* (1.6)–(1.9)*. Then for any unit vector e,*

$$\Delta(\partial_e u)^\pm \geq 0 \quad \text{in } D.$$

Proof. Fix a direction e and let $v = \partial_e u$. Let also

$$V := \{v > 0\}.$$

Note that $V \subset G$ because of the assumption (1.8). Then, formally, for $x \in V$,

$$\Delta(v^+) = \partial_e \Delta u(x) = f'(u)\partial_e u = f'(u)v \geq 0.$$

To justify this computation, observe that $\Delta(v^+) \geq 0$ in D is equivalent to the inequality

$$-\int_D \nabla(v^+)\nabla\eta \, dx \geq 0 \tag{2.1}$$

for any nonnegative $\eta \in C_0^\infty(D)$. Suppose first that $\operatorname{supp}\eta \subset \{v > \delta\}$ with $\delta > 0$. Then writing the equation

$$-\int_D \nabla u \nabla\eta \, dx = \int_D f\eta \, dx$$

with $\eta = \eta(x)$ and $\eta = \eta(x-he)$, we obtain an equation for the incremental quotient

$$v_{(h)}(x) := \frac{u(x+he) - u(x)}{h}.$$

Namely, we obtain

$$-\int_D \nabla v_{(h)}\nabla\eta \, dx = \frac{1}{h}\int_D [f(u(x+he)) - f(u(x))]\eta \, dx \geq 0 \tag{2.2}$$

for small $h > 0$, where we have used that $u(x+he) > u(x)$ on $\operatorname{supp}\eta \subset \{v > \delta\}$ for small enough h, combined with the monotonicity of f. Letting in (2.2) $h \to 0$ and then $\delta \to 0$ we arrive at

$$-\int_D \nabla v \nabla\eta \, dx \geq 0$$

for arbitrary $\eta \geq 0$ with $\operatorname{supp}\eta \Subset \{v > 0\}$.

Thus, we proved that $\Delta v \geq 0$ in the open set $V = \{v > 0\}$ in the sense of distributions. Then it is a simple exercise to show that (2.1) holds for any nonnegative $\eta \in C_0^\infty(D)$.

To prove the same inequality for v^-, we simply reverse the direction e.

□

Proof of Theorem 1.1. Without loss of generality, we may assume that $D = B_1$ and $K = B_{1/2}$. Further observe the function u is twice differentiable at every Lebesgue point $x^0 \in B_1$ of the Hessian of u, since $u \in W^{2,p}_{\mathrm{loc}}(B_1)$ with $p > n$, see e.g. Evans [Eva98, Theorem 5.8.5]. Then fix such a point $x^0 \in B_{1/2}$ and define

$$v(x) = \partial_e u(x)$$

for a unit vector e orthogonal to $\nabla u(x^0)$ (if $\nabla u(x^0) = 0$, take arbitrary unit e). Again, without loss of generality we may assume $x^0 = 0$. Our aim is to obtain a uniform estimate for $\partial_{x_j e} u(0) = \partial_{x_j} v(0)$, $j = 1, \dots, n$. By construction, $v(0) = 0$ and v is differentiable at 0. Hence, we have the Taylor expansion

$$v(x) = \xi \cdot x + o(|x|), \quad \xi = \nabla v(0).$$

Now, if $\xi = 0$ then $\partial_{x_j} v(0) = 0$ for all $j = 1, \dots, n$ and there is nothing to estimate. If $\xi \neq 0$, consider the cone

$$\mathcal{C} = \{x \in \mathbb{R}^n : \xi \cdot x \geq |\xi||x|/2\},$$

which has a property that

$$\mathcal{C} \cap B_r \subset \{v > 0\}, \quad -\mathcal{C} \cap B_r \subset \{v < 0\}$$

for sufficiently small $r>0$. Consider also the rescalings

$$v_r(x)=\frac{v(rx)}{r},\quad x\in B_1.$$

Note that $v_r(x)\to v_0(x):=\xi\cdot x$ uniformly in B_1 and $\nabla v_r\to\nabla v_0$ in $L^p(B_1)$, $p>n$. The latter follows from the equality

$$\int_{B_1}|\nabla v_r(x)-\xi|^p dx=\frac{1}{r^n}\int_{B_r}|\nabla v(x)-\nabla v(0)|^p dx,$$

where the right-hand side goes to zero as $r\to 0$, since $x^0=0$ is a Lebesgue point for ∇v. Then we have

$$\begin{aligned}
c_n|\xi|^4&=\int_{\mathcal{C}\cap B_1}\frac{|\nabla v_0(x)|^2dx}{|x|^{n-2}}\int_{-\mathcal{C}\cap B_1}\frac{|\nabla v_0(x)|^2dx}{|x|^{n-2}}\\
&=\lim_{r\to 0}\int_{\mathcal{C}\cap B_1}\frac{|\nabla v_r(x)|^2dx}{|x|^{n-2}}\int_{-\mathcal{C}\cap B_1}\frac{|\nabla v_r(x)|^2dx}{|x|^{n-2}}\\
&=\lim_{r\to 0}\frac{1}{r^4}\int_{\mathcal{C}\cap B_r}\frac{|\nabla v(x)|^2dx}{|x|^{n-2}}\int_{-\mathcal{C}\cap B_r}\frac{|\nabla v(x)|^2dx}{|x|^{n-2}}\\
&\le\lim_{r\to 0}\Phi(r,v^+,v^-),
\end{aligned}$$

where Φ is as in ACF Monotonicity Formula (Theorem 2.1). Then applying the ACF estimate (Theorem 2.2, slightly scaled) we obtain

$$c_n|\xi|^4\le\liminf_{r\to 0}\Phi(r,v^+,v^-)\le C_n\|\nabla u\|^4_{L^\infty(B_{3/4})}$$

Hence, we obtain that $|\xi|\le C_n\|\nabla u\|_{L^\infty(B_{3/4})}$, which implies that

$$|\nabla\partial_e u(x^0)|\le C_n(\|u\|_{L^\infty(B_1)}+\|g\|_{L^\infty(B_1)})=N.$$

This doesn't give the desired estimate on $|D^2u|$ yet, since e is subject to the condition $e\cdot\nabla u(x^0)=0$, unless $\nabla u(x^0)=0$. If $\nabla u(x^0)\neq 0$, we may choose the coordinate system so that $\nabla u(x^0)$ is parallel to e_1. Then, taking $e=e_2,\dots,e_n$ in the estimate above, we obtain

$$|\partial_{x_ix_j}u(x^0)|\le N,\quad i=2,\dots,n,\quad j=1,2,\dots,n$$

To obtain the estimate in the missing direction e_1, we use the equation $\Delta u=g$:

$$\begin{aligned}
|\partial_{x_1x_1}u(x^0)|&\le|\Delta u(x^0)|+|\partial_{x_2x_2}u(x^0)|+\dots+|\partial_{x_nx_n}u(x^0)|\\
&\le\|g\|_{L^\infty(D)}+(n-1)N\le C_nN.
\end{aligned}$$

This completes the proof of the theorem. □

Appendix.

Reduction of ACF monotonicity formula to Friedland-Hayman inequality. Here we follow Caffarelli [Caf98].

We start with a remark that the functional J scales linearly, in the sense that if

$$u_\lambda(x)=\frac{1}{\lambda}u(\lambda x),$$

then

$$J(r/\lambda,u_\lambda)=J(r,u).$$

In particular, this implies that we can assume $u_\pm$ to be defined in B_R for a certain $R>1$. Then it will suffice to show that $\Phi'(r)\ge 0$ only for $r=1$.

It will be convenient to introduce

$$I(r,u)=\int_{B_r}\frac{|\nabla u|^2}{|x|^{n-2}}dx$$

Thus, $J(r,u)=\frac{1}{r^2}I(r,u)$ and $\Phi(r,u_+,u_-)=\frac{1}{r^4}I(r,u_+)I(r,u_-)$. Note that we may assume that $I(1,u_\pm)<\infty$. Besides, since $I_\pm$ are absolutely continuous functions of r, without loss of generality we may assume that $r=1$ is a Lebesgue point for their respective integrands. For simplicity, we will denote $I_\pm=I(1,u_\pm)$ and $I'_\pm=I'(1,u_\pm)$. Then we have

$$\Phi'(1)=I'_+I_-+I_+I'_--4I_+I_-.$$

Thus, we want to show

$$\frac{I'_+}{I_+}+\frac{I'_-}{I_-}\geq 4.$$

We now want to rewrite this as an inequality on the unit sphere. To this end, for $u=u_\pm$, let $\Sigma=\{u>0\}\cap\partial B_1$. Let u_m be mollifications of u. Observe that $\Delta(1/|x|^{n-2})$ is a nonpositive measure. Then we have

$$\begin{aligned}I(1,u_m)&=\int_{B_1}\frac{|\nabla u_m|^2}{|x|^{n-2}}dx\leq\int_{B_1}\frac{\Delta\left(\frac{u_m^2}{2}\right)}{|x|^{n-2}}dx\\&=\int_{\partial B_1}\left(u_m\,\partial_r u_m+\frac{n-2}{2}u_m^2\right)d\theta.\end{aligned}$$

Letting $m\to\infty$ we obtain

$$I(1,u)\leq\int_\Sigma\left(u\,\partial_r u+\frac{n-2}{2}u^2\right)d\theta.$$

On the other hand

$$I'(1,u)=\int_\Sigma|\nabla u|^2d\theta.$$

Thus,

$$\frac{I'(1,u)}{I(1,u)}\geq\frac{\displaystyle\int_\Sigma[(\partial_r u)^2+|\nabla_\theta u|^2]d\theta}{\displaystyle\int_\Sigma[u\,\partial_r u+\frac{n-2}{2}u^2]d\theta}$$

Note at this point that

$$\frac{\displaystyle\int_\Sigma|\nabla_\theta u|^2}{\displaystyle\int_\Sigma u^2}\geq\lambda,$$

where $\lambda=\lambda(\Sigma)$ is the first eigenfunction of the spherical Laplacian Δ_θ in Σ, so we want to split $u\,\partial u_r$ in an optimal fashion, to spread its control between $\int(\partial_r u)^2$ and $\int|\nabla_\theta u|^2$, i.e.,

$$\int_\Sigma u\,\partial_r u\leq\frac{1}{2}\left[A\int_\Sigma u^2+\frac{1}{A}\int_\Sigma(\partial_r u)^2\right].$$

This will leave us with

$$\frac{I'(1,u)}{I(1,u)} \geq 2\,\frac{\displaystyle\int_\Sigma (\partial_r u)^2 + |\nabla_\theta u|^2}{\displaystyle\frac{1}{A}\int_\Sigma (\partial_r u)^2 + (A+n-2)\int_\Sigma u^2}.$$

To perfectly balance both terms, we want

$$\frac{1}{A} = \frac{A+n-2}{\lambda}, \quad \text{or} \quad A[A+n-2] = \lambda$$

This choice will give us

$$\frac{I'(1,u)}{I(1,u)} \geq 2A.$$

But now observe that A is precisely the homogeneity of the homogeneous harmonic function, constructed from the first eigenfunction of the spherical Laplacian in Σ. So, if $\Sigma_\pm = \{u_\pm > 0\} \cap \partial B_1$, then these are disjoint open sets on ∂B_1 and if $A_\pm$ are the corresponding homogeneities, then we have

$$\frac{I'_+}{I_+} + \frac{I'_-}{I_-} - 4 \geq 2(A_+ + A_- - 2)$$

and therefore the required inequality will follow from the Friedland-Hayman inequality

$$A_+ + A_- - 2 \geq 0.$$

Lecture 3

3. The free boundary in obstacle type problems

3.1. **Normalized solutions, rescalings, and blowups.**

3.1.1. *Local and global solutions.* Our analysis of the free boundary is based on the study of so-called blowups. Since the regularity of the free boundary is a local question, we may restrict ourselves to the solutions defined in balls centered at free boundary points. We may further translate these points to the origin.

Definition 3.1 (Local solutions). For given R, $M > 0$, let $P_R(M)$ be the class of $C^{1,1}$ solutions u of Problems **O**, **I**, or **II** in B_R such that

- $\|D^2u\|_{L^\infty(B_R)} \le M$,
- $0 \in \Gamma$ and additionally $|\nabla u(0)| = 0$ in the case of problem **II**.

Note that in the case when $|\nabla u| > 0$ at a free boundary point in Problem **II**, by the implicit function theorem the free boundary is a $C^{1,\alpha}$ surface near that point and therefore we concentrate on the behavior near the free boundaries where the gradient vanishes.

Taking formally $R = \infty$ in the above definition, we obtain solutions in the entire space $\mathbb{R}^n$, which grow quadratically at infinity. Slightly abusing the terminology, we call them *global solutions*.

Definition 3.2 (Global solutions). For given $M > 0$ let $P_\infty(M)$ be the class of $C^{1,1}_{\mathrm{loc}}$ solutions u of Problems **O**, **I**, or **II** in $\mathbb{R}^n$, such that

- $\|D^2u\|_{L^\infty(\mathbb{R}^n)} \le M$,
- $0 \in \Gamma$ and additionally $|\nabla u(0)| = 0$ in the case of problem **II**.

3.1.2. *Rescalings and blowups.* The following scaling and translation properties are enjoyed by the solutions in the above classes. If $u \in P_R(M)$ and $\lambda > 0$, then the *rescaling* of u at 0

$$u_\lambda(x) = \frac{u(\lambda x)}{\lambda^2}, \quad x \in B_{R/\lambda} \tag{3.1}$$

will be from class $P_{R/\lambda}(M)$. Using this simple observation, we will often state the results for $P_1(M)$ as the corresponding statements for classes $P_R(M)$ can be easily recovered.

Observe that for $u \in P_R(M)$ the rescalings u_λ satisfy the estimate $|D^2u_\lambda(x)| \le M$ in $B_{R/\lambda}$ for all $\lambda > 0$. Therefore, we also have

$$|\nabla u_\lambda(x)| \le M|x|, \quad x \in B_{R/\lambda}$$
$$|u_\lambda(x)| \le \frac{1}{2}M|x|^2, \quad x \in B_{R/\lambda}.$$

Hence, we can find a sequence $\lambda = \lambda_j \to 0$ such that

$$u_{\lambda_j} \to u_0 \quad \text{in } C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n) \text{ for any } 0 < \alpha < 1$$

where $u_0 \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$. Such u_0 is called a *blowup of u at the origin* and Proposition 3.4 below says that u_0 is a global solution; more precisely, $u_0 \in P_\infty(M)$.

Remark 3.3. An important remark is that it is apriori not clear if u_0 is unique, as different sequences $\lambda_j \to 0$ may lead to different limits u_0.

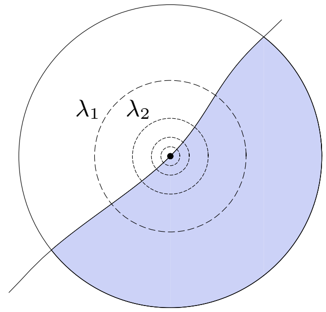

FIGURE 3.1. Blowup with a fixed center

Proposition 3.4 (Blowups)**.** *Let $u \in P_R(M)$ and $u_{\lambda_j} \to u_0$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ for some sequence $\lambda_j \to 0$. Then $u_0 \in P_\infty(M)$.*

The proposition contains a hidden statement that $0 \in \Gamma$, which in particular implies that u_0 is not identically zero. The proof of that fact is contained in the following important lemma (the proof of which is in the appendix).

Lemma 3.5 (Nondegeneracy)**.** (i) *Let $u \in P_R(M)$ be a solution of Problem* **O** *or* **I**. *Then*

$$\sup_{B_r} u \geq c_n r^2, \quad 0 < r < R.$$

(ii) *Let $u \in P_R(M)$ be a solution of Problem* **II**. *If $0 \in \Gamma_+$ then*

$$\sup_{B_r} u \geq c_n \lambda_+ r^2, \quad 0 < r < R.$$

and if $0 \in \Gamma_-$ then

$$\inf_{B_r} u \leq -c_n \lambda_- r^2, \quad 0 < r < R.$$

3.2. **Weiss's type monotonicity formulas.** We next introduce a useful tool in the study of our problems: the so-called Weiss's monotonicity formula [Wei99]. We are going to define this functional for model problems **O**, **I**, and **II**.

Definition 3.6. For a solution $u \in P_R(M)$ and $0 < r < R$ define *Weiss's Energy Functional* as follows:

- In Problem **O** and **I**

$$W(r,u) := \frac{1}{r^{n+2}} \int_{B_r} (|\nabla u|^2 + 2u)\, dx - \frac{2}{r^{n+3}} \int_{\partial B_r} u^2 dH^{n-1}. \tag{3.2}$$

- In Problem **II**

$$W(r,u) := \frac{1}{r^{n+2}} \int_{B_r} (|\nabla u|^2 + 2\lambda_+ u^+ + 2\lambda_- u^-)\, dx - \frac{2}{r^{n+3}} \int_{\partial B_r} u^2 dH^{n-1}. \tag{3.3}$$

The functional W has the following scaling property

$$W(rs,u)=W(s,u_r) \tag{3.4}$$

for any $0<r<R$, $0<s<R/r$, where $u_r=u_{x^0,r}$ is the rescaling as in (4.5). This observation leads to a simple formula for $\frac{d}{dr}W$ and ultimately to the proof of the Monotonicity Formula.

To simplify the notation, we introduce the following operators

$$\partial' v := x\cdot\nabla v(x)-2v(x)=\frac{d}{d\lambda}\Big|_{\lambda=1}\frac{v(\lambda x)}{\lambda^2}$$

Theorem 3.7 (Weiss's monotonicity formula)**.** *Let $u\in P_R(M)$. Then $r\mapsto W(r,u)$ is a nondecreasing absolutely continuous function for $0<r<R$ and*

$$\frac{d}{dr}W(r,u)=\frac{2}{r^{n+4}}\int_{\partial B_r}|\partial' u|^2 dH^{n-1} \tag{3.5}$$

for a.e. $0<r<R$.

Moreover, the identity $W(r,u)\equiv const$ for $r_1<r<r_2$ implies the homogeneity of u with respect to the origin, i.e.

$$x\cdot\nabla u(x)-2u(x)\equiv 0,\qquad in\ B_{r_2}\setminus B_{r_1}. \tag{3.6}$$

Proof. We only prove the differentiation formula (3.5), as the rest of the theorem is its simple corollary.
Problems **O**, **I**. Using the scaling property (3.4) we will have

$$\begin{aligned}\frac{d}{dr}W(r,u)&=\frac{d}{dr}W(1,u_r)\\&=\int_{B_1}\frac{d}{dr}(|\nabla u_r|^2+2u_r)\,dx-2\int_{\partial B_1}\frac{d}{dr}(u_r^2)dH^{n-1}.\end{aligned}$$

If we now use that

$$\begin{aligned}\frac{d}{dr}(\nabla u_r)&=\nabla\frac{du_r}{dr}\\ \frac{du_r}{dr}&=\frac{\partial' u_r}{r},\end{aligned}$$

then integrating by parts, we will obtain

$$\frac{d}{dr}W(r,u)=\frac{2}{r}\int_{B_1}(-\Delta u_r+1)\partial' u_r\,dx+\frac{2}{r}\int_{\partial B_1}(\partial_\nu u_r-2u_r)\partial' u_r\,dH^{n-1},$$

where $\partial_\nu u_r$ is the outer normal derivative of u_r on ∂B_1. Finally, noting that $(-\Delta u_r+1)\partial' u_r=0$ and that $\partial_\nu u_r=x\cdot\nabla u_r$ on ∂B_1, we obtain

$$\frac{d}{dr}W(r,u)=\frac{2}{r}\int_{\partial B_1}|\partial' u_r|^2 dH^{n-1},$$

which implies (3.5) after scaling.

Problem **II**. In this case, the proof is almost identical to the one for Problems **O** and **I** and requires only minor adjustments. □

Corollary 3.8 (Homogeneity of blowups with fixed center)**.** *Let $u\in P_R(M)$ and $u_0(x)=\lim_{j\to\infty}u_{\lambda_j}$ be a blowup of u at the origin. Then u_0 is homogeneous of degree two, i.e.*

$$u_0(\lambda x)=\lambda^2 u_0(x),\quad x\in\mathbb{R}^n,\ \lambda>0.$$

Proof. We may assume that the convergence $u_{x^0,\lambda_j} \to u_0$ is in $C^{1,\alpha}_{\rm loc}(\mathbb{R}^n)$. Then, we have

$$W(r,u_0) = \lim_{j\to\infty} W(r,u_{\lambda_j}) = \lim_{j\to\infty} W(\lambda_j r, u) = W(0+,u)$$

for any $r>0$, which means that $W(r,u_0)$ is constant. Hence, using the second part of Theorem 3.7, we obtain that u_0 is homogeneous of degree 2 in $\mathbb{R}^n$. □

3.3. **Homogeneous global solutions.** Even though the information on blowup that we obtained is significant, it is still far from being complete. We will need to invoke the ACF monotonicity formula (to be more precise, the case of equality) in order to complete the classification of blowups.

Theorem 3.9 (Homogeneous solutions of obstacle type problems)**.** *Let $u_0 \in P_\infty(M)$ be a homogeneous of degree two global solution. Then u_0 is either a monotone function of one independent variable or a homogeneous quadratic polynomial.*

Being a monotone function of one independent variable here means that there exists a unit vector e and a monotone function $\phi:\mathbb{R}\to\mathbb{R}$ such that $u_0(x)=\phi(x\cdot e)$ for all $x\in\mathbb{R}^n$.

We start with identifying the obstacle type problem solved by u_0.

Proof of Theorem 3.9. Let e be a direction in $\mathbb{R}^n$ and consider the pair of functions $(\partial_e u_0)^\pm$, positive and negative parts of the directional derivative of u_0. Recall that by Lemma 2.4 $(\partial_e u_0)^\pm$ are subharmonic and therefore satisfy the assumptions of the ACF monotonicity formula. Thus, if we define

$$\begin{aligned}\phi_e(r,u_0) &:= \Phi(r,(\partial_e u_0)^+,(\partial_e u_0)^-)\\ &= \frac{1}{r^4}\int_{B_r}\frac{|\nabla(\partial_e u_0)^+|^2\,dx}{|x|^{n-2}}\int_{B_r}\frac{|\nabla(\partial_e u_0)^-|^2\,dx}{|x|^{n-2}}.\end{aligned}$$

then $\phi_e(r,u_0)$ must be monotone nondecreasing in r.

On the other hand, if u_0 is homogeneous of degree two, then the simple change of variables shows that

$$\phi_e(r,u_0)\equiv \mathit{const}.$$

Thus, we are in the case of equality in the ACF monotonicity formula, and therefore we have the following two alternatives (see Theorem 2.3):

(i) one of the functions $(\partial_e u_0)^+$ and $(\partial_e u_0)^-$ vanishes identically in $\mathbb{R}^n$, or
(ii) the supports of $(\partial_e u_0)^+$ and $(\partial_e u_0)^-$ intersected with ∂B_r are complementary hemispheres for any $r>0$ and $(\partial_e u_0)^+\Delta(\partial_e u_0)^+=0$ in $\mathbb{R}^n$ and $(\partial_e u_0)^-\Delta(\partial_e u_0)^-=0$ in $\mathbb{R}^n$ in the sense of measures.

To finish the proof we need to consider several possibilities.

1) Suppose $\{|\nabla u_0|=0\}$ has a positive Lebesgue measure. Then the supports of $(\partial_e u_0)^\pm$ cannot be complementary spheres for some $r>0$. This means the situation (ii) above is impossible and therefore we are in situation (i). This means that $\partial_e u_0$ has a sign for any unit vector e. This is equivalent to u_0 being a monotone function of one variable.

2) Suppose now that $\{|\nabla u_0|=0\}$ has a Lebesgue measure zero. In the case of Problems **O** and **I** this means $\Delta u_0=1$ a.e. in $\mathbb{R}^n$ and therefore by the Liouville

theorem u_0 is a quadratic polynomial. Therefore, it is left to consider the case of Problem **II**, i.e. when u_0 satisfies

$$\Delta u_0 = \lambda_+ \chi_{\{u_0>0\}} - \lambda_- \chi_{\{u_0<0\}} \quad \text{a.e. in } \mathbb{R}^n,$$

where $\lambda_\pm > 0$.

2a) Suppose now that there exists $y^0 \in \Gamma = \Gamma_+ \cup \Gamma_-$ with $|\nabla u_0(y^0)| > 0$. (Recall that $\Gamma_\pm = \partial\{\pm u_0 > 0\}$.) Then necessarily $y^0 \in \Gamma_+ \cap \Gamma_-$. Let $\nu^0 = \nabla u_0(y^0)/|\nabla u_0(y^0)|$ be the direction of the gradient of u_0 at y^0. Then there exists a small neighborhood $B_\rho(y^0)$ where $\partial_{\nu^0} u_0 > 0$ and such that $\{u_0 = 0\} \cap B_\rho(y^0)$ is a $C^{1,\alpha}$-surface. If $e \cdot \nu^0 \neq 0$ then $\partial_e u_0(y^0) \neq 0$ and for sufficiently small δ we obtain

$$|\Delta \partial_e u_0|(B_\delta(y^0)) = (\lambda_+ + \lambda_-) \int_{\{u_0=0\}\cap B_\delta(y^0)} |e \cdot \nu| \, dH^{n-1} > 0,$$

where $\nu = \nabla u_0/|\nabla u_0|$ is the normal to the surface $u_0 = 0$. Thus, the alternative (ii) cannot hold for directions e non-orthogonal to ν^0. Consequently, (i) holds for all such directions and, by continuity, for all directions. As before, this implies that u_0 is a monotone function of one variable.

2b) Finally, consider the case when $|\nabla u_0|$ vanishes on $\Gamma = \Gamma_+ \cup \Gamma_-$. Then it is easy to see that $u_0^\pm$ are global solutions of the classical obstacle problem

$$\Delta u_0^\pm = \lambda_\pm \chi_{\{u_0^\pm>0\}} \quad \text{in } \mathbb{R}^n$$

We then invoke the following very well known result for the solution of the classical obstacle problem.

Lemma 3.10 (Global solutions of Problem **O**). *Let $u \in P_\infty(M)$ be a global solution of Problem* **O**. *Then u is a convex function.*

Proof. See the appendix to this lecture. □

Thus, both u_0^+ and u_0^- are convex functions. This is possible only if the sets $\{u_0^\pm > 0\}$ are disjoint halfspaces separated by parallel hyperplanes. The only global solutions of the classical obstacle problem vanishing on a halfspace have the form $v(x) = C((x \cdot e - c)^+)^2$, where e is the outward unit normal to the boundary of the halfspace and $c \in \mathbb{R}$. Thus, putting $u_0^\pm$ together, we obtain that u_0 is indeed a monotone increasing function of $x \cdot e$. □

3.4. **Classification of free boundary points.** With the use of Theorem 3.9 we can give a full description of blowups in our model problems **O**, **I**, and **II**.

Theorem 3.11 (Classification of blowups). *Let u_0 be a blowup at the origin of a solution of Problem* **O**, **I**, *or* **II**. *Then u_0 has of one of the following forms.*

- In Problems **O**, **I**:

- Polynomial solution $u_0(x) = \frac{1}{2}(x \cdot Ax)$, $x \in \mathbb{R}^n$. *where A is an $n \times n$ symmetric matrix with* $\operatorname{Tr} A = 1$. *In Problem* **O** *we additionally have that A is a nonnegative matrix.*
- Halfplane solutions $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$, *$x \in \mathbb{R}^n$, where e is a unit vector.*

-In Problem **II**:

- Polynomial solutions (positive or negative) $u_0(x) = \frac{\lambda_+}{2}(x \cdot Ax)$ *or* $u_0(x) = -\frac{\lambda_-}{2}(x \cdot Ax)$, *$x \in \mathbb{R}^n$, where A is an $n \times n$ nonnegative symmetric matrix with* $\operatorname{Tr} A = 1$.

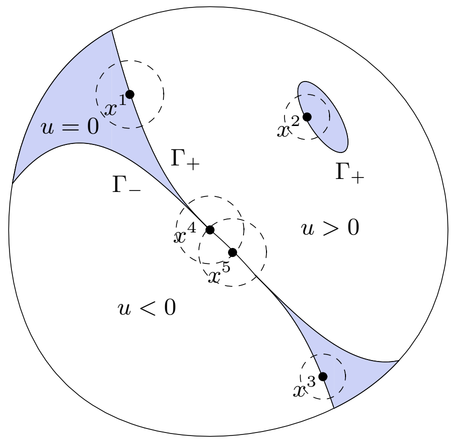

FIGURE 3.2. Problem **II**: Free boundary $\Gamma = \Gamma_+ \cup \Gamma_-$; x^1, x^2, x^3 one-phase points; x^4 branching two-phase point; x^5 non-branching two-phase point.

- Halfplane solutions (positive or negative) $u_0(x) = \frac{\lambda_+}{2}(x \cdot e)_+^2$ *or* $u_0(x) = -\frac{\lambda_-}{2}(x \cdot e)_-^2$, $x \in \mathbb{R}^n$, *for a unit vector* e.
- Two-plane solution $u_0(x) = \frac{\lambda_+}{2}(x \cdot e)_+^2 - \frac{\lambda_-}{2}(x \cdot e)_-^2$, $x \in \mathbb{R}^n$, *for a unit vector* e.

Proof. The proof consists in identifying the quadratic polynomial and one-dimensional solutions of Problems **O**, **I**, and **II**. This is left to the reader as an easy exercise. □

Theorem 3.12 (Unique type of the blowup)**.** *Let* $u \in P_R(M)$*. Then all possible blowups of* u *at the origin have the same type, i.e., they fall into the same category as described in Theorem 3.11.*

Before giving a proof of this theorem, we would like to mention that Theorems 3.11 and 3.12 above lead to the following classification of free boundary points. Even though we have defined the blowups only at the origin, it is of course straightforward to define the blowups at any free boundary point $x^0 \in \Gamma$ (with and additional assumption $|\nabla u(x^0)| = 0$ in Problem **II**). Indeed, we consider the rescalings

$$u_{x^0,\lambda}(x) = \frac{u(x^0 + \lambda x)}{\lambda^2}$$

and let $\lambda \to 0$. Clearly, the results above are applicable for the blowups at any such point x^0 (just translate x^0 to the origin!).

Definition 3.13 (Classiffication of free boundary points).

– In Problems **O**, **I** for $x^0 \in \Gamma$ we will use the following terminology:
  - x^0 is a *regular point*, if one and therefore every blowup of u at x^0 is a halfplane solution
  - x^0 is a *singular point*, if one and therefore every blowups of u at x^0 is polynomial.

– In Problem **II**, for $x^0 \in \Gamma$ we say
  - x^0 is a *two-phase point*, if $x^0 \in \Gamma_+ \cap \Gamma_-$

- x^0 is an *one-phase point*, otherwise.

Thus, the function u does not change sign in a neighborhood of an one-phase point. Equivalently, x^0 is a two-phase point if either $|\nabla u(x^0)| \neq 0$ or $|\nabla u(x^0)| = 0$ and one (and consequently all) blowups at x^0 are two-plane solutions. Further, similarly to Problems **O**, **I** we distinguish *regular* and *singular* one-phase points, depending on their blowups.

Proof of Theorem 3.12. To prove the theorem Problems **O** and **I**, we consider the limit as $r \to 0$ in Weiss's monotonicity formula leads to a useful notion of balanced energy. To do so, let us introduce the notation

$$W(r, u, x^0) = W(r, u(\cdot - x^0)),$$

which we call Weiss's energy functional centered at x^0.

Definition 3.14 (Balanced energy). Let u be a solution of Problem **O** or **I**. Then for any $x^0 \in \Gamma$ the limit

$$\omega(x^0) := W(0+, u, x^0) = \lim_{r \to 0} W(r, u, x^0), \tag{3.7}$$

which exists by Theorem 3.7, is called the *balanced energy* of u at x^0.

As, we saw in the proof of Corollary 3.8,

$$\omega(x^0) = W(r, u_0) \equiv W(1, u_0)$$

for any blowup u_0 of u at x^0. Thus, the balanced energy at a point coincides with the Weiss energy of any of blowups with fixed center x^0.

Uniqueness of the type of the blowup will follow from the following lemma.

Lemma 3.15 (Weiss's energy of homogeneous solutions)**.** *Let u_0 be a homogeneous global solution of Problem* **O** *or* **I**. *Then*

$$W(r, u_0) \equiv W(1, u_0) = \begin{cases} \alpha_n/2 & \text{if } u_0 \text{ is a halfplane solution,} \\ \alpha_n & \text{if } u_0 \text{ is a polynomial solution,} \end{cases}$$

where α_n is a dimensional constant.

Proof. Integrating by parts in the expression for $W(r, u_0)$ and using that $\Delta u_0 = 1$ in $\Omega(u_0)$, we obtain that at

$$\begin{aligned} W(r, u_0) \equiv W(1, u_0) &= \int_{B_1} \left(|\nabla u_0|^2 + 2u_0\right) dx - 2 \int_{\partial B_1} u_0^2 \, dH^{n-1} \\ &= \int_{B_1} (-\Delta u_0 + 2) u_0 \, dx - \int_{\partial B_1} \partial' u_0 u_0 \, dH^{n-1} = \int_{B_1} u_0 \, dx. \end{aligned}$$

Thus, if we denote by

$$\alpha_n = \frac{1}{2} \int_{B_1} x_1^2 \, dx = \frac{1}{2n} \int_{B_1} |x|^2 dx = \frac{H^{n-1}(\partial B_1)}{2n(n+2)},$$

then for polynomial solutions $u_0(x) = \frac{1}{2}(x \cdot Ax)$ we can compute that

$$W(r, \tfrac{1}{2}(x \cdot Ax)) = \frac{1}{2} \int_{B_1} x \cdot Ax \, dx = \alpha_n \operatorname{Tr} A = \alpha_n.$$

On the other hand, for halfplane solutions $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$ we will have

$$W(r, \tfrac{1}{2}(x \cdot e)_+^2) = \frac{1}{2} \int_{B_1} (x \cdot e)_+^2 dx = \frac{\alpha_n}{2}.$$

□

This completes the proof of Theorem 3.12 in the case of Problems **O** and **I**. It also completes the proof in the case of Problem **II** if 0 is a one-phase point. In the case when 0 is a two-phase point, the nondegeneracy lemma (Lemma 3.5) implies that 0 is also a two-phase free boundary point for any blowup u_0, which is possible only if u_0 is two-plane solution (as defined in Theorem 3.11). □

Appendix.

Proof of Lemma 3.5. We will consider only the case of Problems **O** and **I**, the proof for Problem **II** being analogous.

In fact, we will prove a slightly more general statement: if u is a solution of Problem **I** in a domain D, $x^0 \in \overline{\Omega} \cap D$, then

$$\sup_{B_r(x^0)} u \geq u(x^0) + c_n r^2,$$

provided $B_{2r}(x^0) \Subset D$.

Before giving the proof, we would like to illustrate one of its main ideas by proving a similar nondegeneracy statement for solutions of $\Delta u = 1$.

Lemma 3.16. *Let u satisfy $\Delta u = 1$ in the ball B_R. Then*

$$\sup_{\partial B_r} u \geq u(0) + \frac{r^2}{2n}, \quad 0 < r < R$$

Proof. Consider the auxiliary function

$$w(x) = u(x) - \frac{|x|^2}{2n}, \quad x \in B_R.$$

Then w is harmonic in B_R. Therefore by the maximum principle we obtain that

$$w(0) \leq \sup_{\partial B_r} w = \left(\sup_{\partial B_r} u \right) - \frac{r^2}{2n},$$

which implies the required inequality. □

1) Assume first that $x^0 \in \Omega$ and moreover $u(x^0) > 0$. Consider then the auxiliary function

$$w(x) = u(x) - u(x^0) - \frac{|x - x^0|^2}{2n}, \tag{3.8}$$

similar to the one in the proof the previous lemma. We have $\Delta w = 0$ in $B_r(x^0) \cap \Omega$. Since $w(x^0) = 0$, by the maximum principle we have that

$$\sup_{\partial(B_r(x^0) \cap \Omega)} w \geq 0.$$

Besides, $w(x) = -u(x^0) - |x - x^0|^2/(2n) < 0$ on $\partial\Omega$. Therefore, we must have

$$\sup_{\partial B_r(x^0) \cap \Omega} w \geq 0.$$

The latter is equivalent to

$$\sup_{\partial B_r(x^0) \cap \Omega} u \geq u(x^0) + \frac{r^2}{2n}$$

and the lemma is proved in this case.

2) Suppose now $x^0 \in \Omega(u)$ and $u(x^0) \le 0$. If $B_{r/2}(x^0)$ contains a point x^1 such that $u(x^1) > 0$, then

$$\sup_{B_r(x^0)} u \ge \sup_{B_{r/2}(x_1)} u \ge u(x^1) + \frac{(r/2)^2}{2n} \ge u(x^0) + \frac{r^2}{8n}$$

which implies the lemma in this case.

If it happens that $u \le 0$ in $B_{r/2}(x^0)$, from subharmonicity of u and the strong maximum principle we will have that either $u = 0$ identically in $B_{r/2}(x^0)$, or $u < 0$ in $B_{r/2}(x^0)$. The former case is impossible, as $x^0 \in \Omega(u)$, and the latter case implies that $B_{r/2}(x^0) \subset \Omega(u)$ and therefore $\Delta u = 1$ in $B_{r/2}(x^0)$. Then Lemma 3.16 finishes the proof in this case and we obtain

$$\sup_{B_r(x^0)} u \ge \sup_{B_{r/2}(x^0)} u \ge u(x^0) + \frac{r^2}{8n}.$$

3) Finally, for $x^0 \in \overline{\Omega(u)}$, we take a sequence $x^j \in \Omega(u)$ such that $x^j \to x^0$ as $j \to \infty$ and pass to the limit in the corresponding nondegeneracy statement at x^j.

This completes the proof of Lemma 3.5 □

Proof of Lemma 3.10. Fix any direction e. Without loss of generality suppose that $e = e_n = (0, \cdots, 0, 1)$. Assume, on the contrary, that

$$-m := \inf_{\Omega} \partial_{x_n x_n} u < 0,$$

and let $x^j \in \Omega(u)$ be a minimizing sequence for the value $-m$, i.e.

$$\lim_{j \to \infty} \partial_{x_n x_n} u(x^j) = -m.$$

Let $d_j = \operatorname{dist}(x^j, \Gamma)$ and consider the rescalings

$$u_j(x) = u_{x^j, d_j}(x) = \frac{u(x^j + d_j x)}{d_j^2}.$$

Observe that $B_1 \subset \Omega(u_j)$ and the free boundary $\Gamma(u_j)$ contains at leas one point on ∂B_1. Since also $|D^2 u_j|$ are uniformly bounded by M we have the uniform estimates

$$|u_j(x)| \le \frac{M}{2}(|x| + 1)^2$$

and therefore we can extract a subsequence converging in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ to a global solution u_0 of Problem **O**. Moreover, similarly to u_j, observe that $B_1 \subset \Omega(u_0) = \{u_0 > 0\}$, and ∂B_1 contains at least one free boundary point.

Next observe, since all functions u_j satisfy $\Delta u_j = 1$ in B_1, the convergence to u_0 can be assumed to be at least in $C^2_{\mathrm{loc}}(B_1)$. Hence, the limit function u_0 satisfies

$$\Delta u_0 = 1, \qquad \partial_{x_n x_n} u_0 \ge -m \quad \text{in } B_1, \qquad \partial_{x_n x_n} u_0(0) = -m.$$

Since $\partial_{x_n x_n} u_0$ is harmonic in B_1, the minimum principle implies that $\partial_{x_n x_n} u_0 \equiv -m$ in B_1. In fact we have even more, $\partial_{x_n x_n} u_0 = -m$ in the connected component of $\Omega(u_0)$ which contains B_1. Hence we obtain the representations

$$\partial_n u_0(x) = g_1(x') - m x_n, \quad x' = (x_1, \ldots, x_{n-1}) \tag{3.9}$$

and

$$u_0(x) = g_2(x') + g_1(x') x_n - \frac{m}{2} x_n^2, \tag{3.10}$$

in B_1. Now let us choose a point $(x',0) \in B_1$ and start moving in the direction e_n. Observe that as long as we stay in $\Omega(u_0)$, we still have $\partial_{x_n x_n} u = -m$ and therefore still have the representations (3.9)–(3.10). However, sooner or later we will reach $\partial\Omega(u_0)$, otherwise if x_n becomes too large, (3.10) will imply $u_0 < 0$, contrary to our assumption. Since $u_0 = |\nabla u_0| = 0$ on $\partial\Omega(u_0)$, from (3.9) we obtain that the first value $\xi(x')$ of x_n for which we arrive at $\partial\Omega(u_0)$ is given by

$$\xi(x') = \frac{g_1(x')}{m}.$$

Hence from (3.10) we deduce that

$$g_2(x') = -\frac{g_1(x')^2}{2m}.$$

Now, the representation (3.10) takes the form

$$u_0(x) = -\frac{m}{2}(x_n - \xi(x'))^2,$$

which is not possible since $u_0 \geq 0$. This concludes the proof. □

Lecture 4

4. The Signorini Problem

In this lecture we are going to prove the optimal regularity in Problem **S** (Theorem 1.4). One of the main ingredients is Almgren's frequency formula [Alm00].

4.1. Almgren's frequency formula.

Harmonic functions. For a harmonic function u in the ball B_R consider the following quantity

$$N(r) = N(r,u) = \frac{r\int_{B_r}|\nabla u|^2}{\int_{\partial B_r} u^2}, \quad 0<r<R,$$

which we will call *Almgren's frequency* (the justification for the name will be given a bit later). A theorem of Almgren then says that $N(r)$ is monotone nondecreasing. To see that, let us introduce

$$H(r) = \int_{\partial B_r} u^2, \quad 0<r<R.$$

Using the spherical coordinates it is then straightforward to see that

$$H'(r) = \frac{n-1}{r}H(r) + 2\int_{\partial B_r}(\partial_\nu u)u.$$

Then applying the divergence theorem on the last term an using that $\Delta(u^2) = 2|\nabla u|^2$ in B_R, we arrive at

$$H'(r) = \frac{n-1}{r}H(r) + 2\int_{B_r}|\nabla u|^2.$$

Hence, we can write that

$$\frac{rH'(r)}{H(r)} = n-1+2N(r).$$

Thus, the monotonicity of $N(r)$ is equivalent to the monotonicity of

$$\Phi(r) = r\frac{d}{dr}\log H(r),$$

which in turn is equivalent to showing the log-convexity of $H(r)$ in $\log r$ (which is a fancy name for the convexity of $\log H(e^t)$ in t).

Now, writing u in the form of

$$u(x) = \sum_{k=0}^\infty f_k(x),$$

where f_k are homogeneous harmonic polynomials of degree k, we will have that

$$H(r) = \sum_{k=0}^\infty \int_{\partial B_1} f_k^2(\theta) r^{n-1+2k} d\theta = r^{n-1}\sum_{k=0}^\infty a_k r^{2k},$$

where $a_k = \int_{\partial B_1} f_k(\theta)^2 d\theta \ge 0$. Then by the Cauchy-Schwarz inequality we obtain that

$$H(\sqrt{r_1 r_2}) \le \sqrt{H(r_1)}\sqrt{H(r_2)}, \quad 0<r_i<R,\ i=1,2,$$

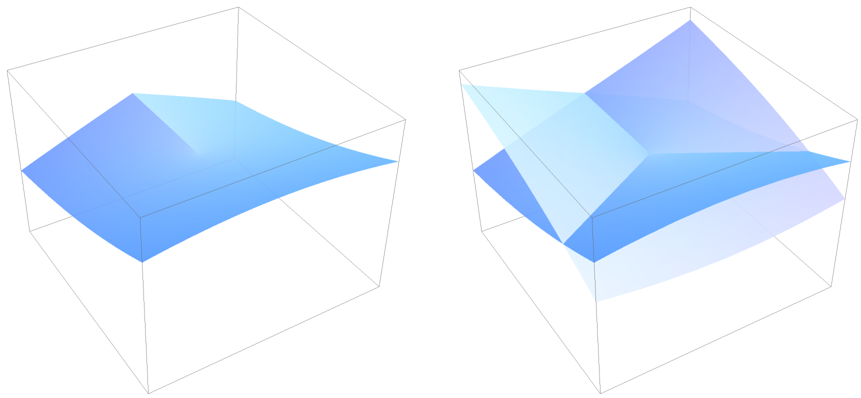

FIGURE 4.1. Solution of the Signorini problem $\mathrm{Re}(x_1 + i|x_2|)^{3/2}$ and multi-valued harmonic function $\mathrm{Re}(x_1 + ix_2)^{3/2}$

which is the required convexity for H. This implies that $N(r)$ is indeed monotone. Moreover, the case of equality in the Cauchy-Schwarz implies that if $N(r_1) = N(r_2)$, $r_1 \neq r_2$ then necessarily all a_k but one must be zero. This means that

$$u(x) = f_k(x)$$

for some nonnegative integer k, i.e. u is a homogeneous harmonic polynomial. Moreover, in that case $H(r) = a_k r^{n-1+2k}$ and therefore $N(r) \equiv k$. Now the name *frequency* comes from the fact that in dimension two, in polar coordinates

$$f_k = c_k r^k \cos(k\theta + \phi)$$

and $N(r)$ coincides with the frequency of $\cos(k\theta + \phi)$.

Signorini problem. It turns out that Almgren's frequency formula is valid also for the solutions of Problem **S** (as was first observed by Athanasopoulos-Caffarelli-Salsa [ACS08]). This should come at no surprise, since the solutions of Problem **S** extended from D_+ to D_-, by using both even and odd reflections, form a two-valued harmonic function and Almgren's formula was proved originally for multivalued harmonic functions (see Fig. 4.1).

Theorem 4.1 (Almgren's frequency formula)**.** *Let u be a nonzero solution of Problem* **S** *in B_R, then the frequency of u*

$$r \mapsto N(r) = N(r, u) = \frac{r \int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2}$$

is nondecreasing for $0 < r < R$. Moreover, $N(r, u) \equiv \kappa$ for $0 < r < R$ if and only if u is homogeneous of degree κ in B_R, i.e.

$$x \cdot \nabla u - \kappa u = 0 \quad \text{in } B_R.$$

Proof. Consider the quantities

$$H(r) = \int_{\partial B_r} u^2, \quad D(r) = \int_{B_r} |\nabla u|^2. \tag{4.1}$$

Denoting by $u_\nu = \partial_\nu u$, where ν is the outer unit normal on ∂B_r, we have

$$H'(r) = \frac{n-1}{r}\,H(r) + 2\int_{\partial B_r} u u_\nu. \tag{4.2}$$

On the other hand, using that $\Delta(u^2/2) = u\Delta u + |\nabla u|^2 = |\nabla u|^2$ and integrating by parts, we obtain

$$\int_{\partial B_r} U_\nu = \int_{B_r} |\nabla u|^2 = D(r). \tag{4.3}$$

Further, to compute $D'(r)$ we use Rellich's formula

$$\int_{\partial B_r} |\nabla u|^2 = \frac{n-2}{r}\int_{B_r} |\nabla u|^2 + 2\int_{\partial B_r} u_\nu^2 - \frac{2}{r}\int_{B_r} (x\cdot\nabla u)\Delta u.$$

Notice that in view of the fact $(x\cdot\nabla u)u_{x_n} = 0$ on B_1' the last integral in the right-hand side vanishes. Hence,

$$D'(r) = \frac{n-2}{r}\,D(r) + 2\int_{\partial B_r} u_\nu^2. \tag{4.4}$$

Thus, we have

$$\begin{aligned}\frac{N'(r)}{N(r)} &= \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)}\\ &= \frac{1}{r} + \frac{n-2}{r} - \frac{n-1}{r} + 2\left\{\frac{\int_{\partial B_r} u_\nu^2}{\int_{\partial B_r} u u_\nu} - \frac{\int_{\partial B_r} u u_\nu}{\int_{\partial B_r} u^2}\right\} \geq 0.\end{aligned}$$

The last inequality is obtained form the Cauchy-Schwarz inequality and implies the monotonicity statement in the theorem. Analyzing the case of equality in Cauchy-Schwarz, we obtain the second part of the theorem. □

4.2. **Rescalings and blowups.** To study the behavior of u near the free boundary we want to use the method of rescalings and blowups. And here we see an important difference of Problem **S** for Problems **O**, **I**, **II**: whereas in the latter problems one has to scale quadratically to preserve the equation, in Problem **S**, any scaling of the type

$$u_r(x) = c(r)u(rx)$$

where $c(r) > 0$ is an arbitrary constant, will preserve the structure of the problem. There is, however, a special choice of $c(r)$ that plays well with Almgren's frequency function. Namely, if we define

$$u_r(x) = \frac{u(rx)}{\left(\frac{1}{r^{n-1}}\int_{\partial B_r} u^2\right)^{1/2}}, \tag{4.5}$$

then we will have an identity

$$N(\rho, u_r) = N(r\rho, u), \quad r > 0,\ \rho < R/r. \tag{4.6}$$

Note that we will also have the following normalization property

$$\|u_r\|_{L^2(\partial B_1)} = 1. \tag{4.7}$$

We then want to study the *blowups* of u at the origin (assuming $0 \in \Gamma$), which are the limits of the rescalings u_r over subsequences $r = r_j \to 0+$. And here again we remark that blowups might be different over different subsequences $r = r_j \to 0+$.

The existence of blowups is justified as follows. The scaling property of the frequency and Theorem 4.1 imply that

$$\int_{B_1} |\nabla u_r|^2 = N(1, u_r) = N(r, u) \leq N(R, u).$$

for $r < R$. Combining with (4.7), we will have a uniform bound for the family $\{u_r\}$ in $W^{1,2}(B_1)$. Now, this implies that there exists a nonzero function $u_0 \in W^{1,2}(B_1)$, such that for a subsequence $r = r_j \to 0+$

$$\begin{aligned} u_{r_j} &\to u_0 \quad \text{in } W^{1,2}(B_1) \\ u_{r_j} &\to u_0 \quad \text{in } L^2(\partial B_1) \\ u_{r_j} &\to u_0 \quad \text{in } C^1_{\mathrm{loc}}(B_1^\pm \cup B_1'). \end{aligned} \tag{4.8}$$

It is easy to see the weak convergence in $W^{1,2}(B_1)$ and the strong convergence in $L^2(\partial B_1)$. The third convergence (and consequently the strong convergence in $W^{1,2}$) follows from uniform $C^{1,\alpha}_{\mathrm{loc}}$ estimates on u_r in $B_1^\pm \cup B_1'$ in terms of $W^{1,2}$-norm of u_r in B_1.

Proposition 4.2 (Homogeneity of blowups). *Let u be a solution of Problem* **S** *in B_R and let u_0 be a blowup of u as described above. Then u_0 is a nonzero global solution of Problem* **S**, *homogeneous of degree $\kappa = N(0+, u)$.*

Proof. The fact that u_0 solves Problem **S** follows from the above mentioned $C^{1,\alpha}_{\mathrm{loc}}$ estimates on u_r in $B_1^\pm \cup B_1'$. For the blowup u_0 over a sequence $r_j \to 0+$ we have

$$N(r, u_0) = \lim_{r_j \to 0+} N(r, u_{r_j}) = \lim_{r_j \to 0+} N(r r_j, u) = N(0+, u)$$

for any $0 < r < 1$. This implies that $N(r, u_0)$ is a constant. In view of the last part of Theorem 4.1 we conclude that u_0 is a homogeneous function. The fact that $u_0 \not\equiv 0$ follows from the convergence $u_{r_j} \to u_0$ in $L^2(\partial B_1)$ and that equality $\int_{\partial B_1} u_{r_j}^2 = 1$, implying that $\int_{\partial B_1} u_0^2 = 1$. □

We emphasize here that although the blowups at the origin might not be unique, as a consequence of Proposition 4.2 they all have the same homogeneity.

4.3. **Homogeneous global solutions.** So an important questions we ask is for what values of κ we have homogeneous global solutions of Problem **S** of degree κ. Because of the $C^{1,\alpha}$ regularity in $\mathbb{R}^n_\pm$, we immediately have that $\kappa \geq 1+\alpha$. However, the general question of possible values of κ is still open, except in dimension $n = 2$. Using the polar coordinates in dimension $n = 2$ one can show that the only homogeneous global solutions are:

$$u_\kappa = C_\kappa \operatorname{Re}(x_1 + i|x_2|)^\kappa, \quad \kappa = 2m - 1/2,\ 2m,\ m \in \mathbb{N}.$$

and

$$v_\kappa = C_\kappa \operatorname{Im}(x_1 + i|x_2|)^\kappa, \quad \kappa = 2m + 1,\ m \in \mathbb{N}.$$

However, it can be shown that the solutions v_κ can never appear as blowups. Therefore the range of possible values of κ for blowups in dimension $n = 2$ is

$$\kappa = 2m - 1/2,\ 2m,\ m \in \mathbb{N}.$$

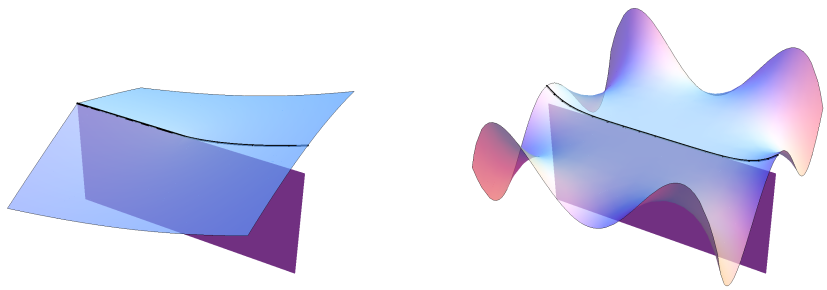

FIGURE 4.2. Graphs of $\operatorname{Re}(x_1 + i\,|x_2|)^{3/2}$ and $\operatorname{Re}(x_1 + i\,|x_2|)^6$

In higher dimensions, very few results are known in this direction, however, the next theorem establishes the minimal possible value of κ and leads to the optimal regularity of solutions of Problem **S**.

Theorem 4.3. *Let u be a homogeneous of degree κ solution of Problem* **S** *with $1 < \kappa < 2$. Then $\kappa = 3/2$ and*

$$u(x) = C_n \operatorname{Re}(x_1 + i|x_n|)^{3/2},$$

after a possible rotation in $\mathbb{R}^{n-1}$.

Proof. For a direction $e \in \partial B_1'$ consider two functions

$$v_e^\pm = \max\{\pm\partial_e u, 0\}.$$

Then they satisfy the following conditions

$$\Delta v_e^\pm \geq 0, \quad v_e^\pm \geq 0, \quad v_e^+ \cdot v_e^- = 0 \quad \text{in } \mathbb{R}^n.$$

Hence we can apply ACF monotonicity formula to the pair $v_e^\pm$. Namely, the functional

$$\phi_e(r) = \frac{1}{r^4}\int_{B_r}\frac{|\nabla v_e^+|^2}{|x|^{n-2}}\int_{B_r}\frac{|\nabla v_e^-|^2}{|x|^{n-2}},$$

is monotone nondecreasing in r. On the other hand, from the homogeneity of u, it is easy to see that

$$\phi_e(r) = r^{4(\kappa-2)}\phi_e(1), \quad r > 0.$$

Since $\kappa < 2$, $\phi(r)$ can be monotone increasing if and only if $\phi(1) = 0$ and consequently $\phi(r) = 0$ for all $r > 0$.

From here it follows that one of the functions $v_e^\pm$ is identically zero, which is equivalent to $\partial_e u$ being ether nonnegative or nonpositive on the entire $\mathbb{R}^n$. Since this is true for any tangential direction $e \in \partial B_1'$, it follows then that u depends only on one tangential direction, and is monotone in that direction. Therefore, without loss of generality we may assume that $n = 2$. However, we already know all possible homogeneous solutions in dimension $n = 2$, and the only one with $1 < \kappa < 2$ is

$$u_{3/2}(x) = C_{3/2} \operatorname{Re}(x_1 + i|x_2|)^{3/2}.$$

□

Corollary 4.4 (Minimal homogeneity). *Let u be a solution of Problem* **S** *in B_R and $0 \in \Gamma$. Then*

$$N(0+, u) \geq 2 - \frac{1}{2}.$$

Moreover, either

$$N(0+, u) = 2 - \frac{1}{2} \quad \text{or} \quad N(0+, u) \geq 2. \quad \square$$

The minimal homogeneity allows to establish the following maximal growth of the solution near free boundary points.

Lemma 4.5 (Growth estimate). *Let u be a solution of Problem* **S** *in B_1, $0 \in \Gamma$, and $N(0+, u) \geq \kappa$. Then*

$$\sup_{B_r} |u| \leq C_0\, r^\kappa, \quad 0 < r < 1/2.$$

with $C_0 = C(n, \kappa, \|u\|_{L^2(B_1)})$.

Proof. Form the monotonicity of N we will have $N(r) \geq \kappa$ for all r. This is equivalent to having

$$r\frac{H'(r)}{H(r)} \geq n - 1 + 2\kappa.$$

Dividing by r and integrating from r to $3/4$ we will obtain

$$\log \frac{H(3/4)}{H(r)} \geq (n - 1 + 2\kappa) \log \frac{3/4}{r},$$

and consequently

$$H(r) \leq C_0 r^{n-1+2\kappa}, \quad 0 < r < 3/4.$$

Next, observing that $u_\pm = \max\{\pm u, 0\}$ are subharmonic, we will obtain

$$\begin{aligned} \sup_{B_r} u_\pm &\leq C_n \Big(\frac{1}{r^{n-1}} \int_{\partial B_{(3/2)r}} u_\pm^2 \Big)^{1/2} \\ &\leq C_n \Big(\frac{H((3/2)r)}{r^{n-1}} \Big)^{1/2} \leq C_0 r^\kappa \end{aligned}$$

which implies the desired estimate. □

The proof of Theorem 1.4 now follows from the growth estimate above with $\kappa = 3/2$ and the interior elliptic estimates.

Proof of Theorem1.4. Without loss of generality we will assume that $D = B_1$, $K = B_{1/2}$.

For any $x \in B^+_{1/2}$ let

$$d(x) = \operatorname{dist}(x, \Gamma).$$

Note that $B_{d(x)}(x) \cap \{x_n = 0\}$ is fully contained in either $\{u(\cdot, 0) = 0\}$ or $\{u(\cdot, 0) > 0\}$ and therefore either the odd or even reflection of u to B_1^- is going to be harmonic in $B_{d(x)}(x)$. We will denote that extension by $\tilde{u}$.

Now take two points $x^1, x^2 \in B^+_{1/2}$ with $|x^1 - x^2| \leq 1/8$. We want to show that

$$|\nabla u(x^1) - \nabla u(x^2)| \leq C|x^1 - x^2|^{1/2} \tag{4.9}$$

with C depending on L^2 norm of u in B_1.

1) Assume first that $d(x^1) \geq 1/4$ (or $d(x^2) \geq 1/4$). Then $\tilde{u}$ is harmonic in $B_{1/4}(x^1)$ and therefore (4.9) follows from the interior estimates for harmonic functions.

2) Suppose now $d(x^2) \leq d(x^1) \leq 1/4$ and $|x^1 - x^2| \geq d(x^1)/2$. We then have that $\tilde{u}$ is harmonic in $B_{d(x^i)}(x^i)$, $i = 1, 2$, and from Lemma 4.5 we have that

$$|\tilde{u}| \leq C_0 d(x^i)^{3/2} \quad \text{in } B_{d(x^i)}(x^i).$$

By the interior gradient estimates we then have

$$|\nabla u(x^i)| = |\nabla \tilde{u}(x^i)| \leq C_0 d(x^i)^{1/2}.$$

Hence, in this case

$$|\nabla u(x^1) - \nabla u(x^2)| \leq |\nabla u(x^1)| + |\nabla u(x^2)| \leq C_0 d(x^1)^{1/2} \leq C_0 |x^1 - x^2|^{1/2}.$$

3) Finally, suppose $d(x^2) \leq d(x^1) \leq 1/4$ and $|x^1 - x^2| \leq d(x^1)/2$. Then from harmonicity of $\tilde{u}$ in $B_{d(x^1)}(x^1)$ and the estimate $|\tilde{u}| \leq C_0 d(x_1)^{3/2}$, combined with the interior derivative estimates, we will have

$$|D^2 \tilde{u}| \leq C_0 d(x_1)^{-1/2} \quad \text{in } B_{d(x^1)/2}(x^1)$$

and therefore

$$|\nabla u(x^1) - \nabla u(x^2)| = |\nabla \tilde{u}(x^1) - \nabla \tilde{u}(x^2)| \leq C_0 d(x_1)^{-1/2} |x^1 - x^2| \leq C_0 |x^1 - x^2|^{1/2}.$$

This completes the proof of the theorem. □

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