Multiplicities over local rings, Lech's conjecture, and lim Ulrich sequence

Linquan Ma

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Overview

1. Hilbert–Samuel multiplicity
2. Lech's conjecture
3. Lim Ulrich sequence

Throughout, all rings are commutative, Noetherian, with multiplicative identity 1, and all modules are finitely generated.

Definition

Let $(R, \mathfrak{m})$ be a local ring of dimension d. Let I be an $\mathfrak{m}$-primary ideal of R and M an R-module. The Hilbert–Samuel multiplicity M with respect to I can be defined as:

$$e(I, M) = \lim_{n\to\infty} d! \cdot \frac{l_R(M/I^n M)}{n^d}.$$

We abbreviate our notation by setting $e(R) := e(\mathfrak{m}, R)$.

Here $l(M/I^n M)$ is the length function: if R is a k-algebra for $k = R/\mathfrak{m}$, then $l_R(M/I^n) = \dim_k(M/I^n M)$.

For $n \gg 0$, $l_R(M/I^nM)$ is actually a polynomial in n of degree d (the Hilbert–Samuel polynomial), thus $e(I, M)$ is the normalized leading coefficient of this polynomial. In general, $e(I, M)$ is a non-negative integer, and is positive if and only if $\dim M = d$.

It is clear from the definition that $e(I, M)$ only depends on the associated graded module $gr_I M$. It follows that $e(I, M) = e(I, \widehat{M})$ where M is the $\mathfrak{m}$-adic completion of M. In particular $e(R) = e(\widehat{R})$.

We collect some examples of multiplicities.

1. Let $R = k[x_1, \dots, x_d]$ and $\mathfrak{m} = (x_1, \dots, x_d)$. Then we have $l(R/\mathfrak{m}^n) = \binom{n+d-1}{d} \sim \frac{1}{d!} n^d$ so $e(R) = 1$. In fact $e(R) = 1$ for all regular local rings.
2. If R is a hypersurface (e.g., $R = k[x_0, \dots, x_d]/f$), then $e(R) = ord(f)$, which is the degree of the smallest order term of f.
3. Let $R = k[\underline{x}]/(f_1, \dots, f_c)$ such that $f_1, \dots, f_c$ are homogeneous of degree $d_1, \dots, d_c$ that forms a regular sequence. Then $e(R) = \prod_{i=1}^c d_i$. Caution: if f_i are not necessarily homogenous, then $e(R) \geq \prod_{i=1}^c ord(f_i)$, but $>$ frequently happens.

Example: $R = k[x, y, z]/(x^2 - y^3, xy - z^3)$. Then $R \cong k[t^5, t^6, t^9]$ and one checks that $e(R) = 5$ while $\prod ord(f_i) = 4$.

4. More generally, suppose $X \subseteq \mathbb{P}^n_k$ is a projective variety over k, with $C(X) \subseteq \mathbb{A}^{n+1}_k$ its affine cone $C(X) = \mathrm{Spec}(R)$ (i.e., R is the ring of functions on $C(X)$). Then $e(R) = \deg X$.
5. Let $R = k[x_1, \dots, x_d]$ and I an $\mathfrak{m}$-primary monomial ideal. Then $e(I, R) = d!\mathrm{vol}(\mathrm{conv}(I)^c)$, where $\mathrm{conv}(I)$ denotes the Newton polyhedral of I: the convex hull of all integer points of $\mathbb{R}^n_{\geq 0}$ that correspond to monomials inside I.
6. Suppose $I \subseteq R$ is generated by a system of parameters $\underline{x} = x_1, \dots, x_d$. Then the multiplicity is equal to the Euler characteristic:

$$e(I, M) = e(\underline{x}, M) = \chi(\underline{x}, M) := \sum_{i=0}^{d} (-1)^i l(H_i(\underline{x}, M)).$$

In particular, if M is a maximal Cohen–Macaulay R-module, then $e(\underline{x}, M) = l(M/(\underline{x})M)$.

We next state some properties of multiplicities. Throughout $(R, \mathfrak{m})$ denotes a local ring of dimension d.

1. Suppose $\widehat{R}$ is unmixed. Then $e(R) = 1$ if and only if R is regular. Morally speaking, the larger the multiplicity, the worse the singularity of R (recall the hypersurface case).
2. $e(I, -)$ satisfies the additivity property: for every short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$, we have $e(I, M_2) = e(I, M_1) + e(I, M_3)$.
3. Let $J \subseteq I$. Suppose $I \subseteq \overline{J}$, then $e(I, M) = e(J, M)$ for all M. If $\widehat{R}$ is equidimensional, then $e(I, R) = e(J, R)$ implies $I \subseteq \overline{J}$.

Some consequences of these properties.

1. Additivity implies that $e(I, -)$ only sees the components of M of dimension d. In particular, if R is a domain, then $e(I, M) = \mathrm{rank}(M) \cdot e(I, R)$.
2. If $R/\mathfrak{m}$ is infinite, then every $\mathfrak{m}$-primary ideal I is integral over an ideal generated by a system of parameters $(x_1, \ldots, x_d)$, called a minimal reduction of I. Therefore $e(I, M) = e(\underline{x}, M) = \chi(\underline{x}, M)$.

Example: $R = k[x, y, z]/(x) \cap (y^2, z)$, then we have $e(R) = e(R/(x)) = e(k[y, z]) = 1$. Note that R is not regular (the condition $\widehat{R}$ is unmixed is necessary when characterizing regularity).

We want to study the behaviour of $e(R)$ under flat maps. There are two types of flat maps.

1. A localization map $R \to S = R_P$.
2. A flat local map $(R, \mathfrak{m}) \to (S, \mathfrak{n})$.

For localization, the singularities of R_P should be no worse than the singularities of R. For example: if R is regular, Cohen–Macaulay, rational singularity, strongly F-regular... then so is R_P. Therefore the natural expectation is $e(R) \geq e(R_P)$.

Conjecture (localization formula)

Let $(R, \mathfrak{m})$ be a local ring and $P \in Spec(R)$ such that $ht(P) + \dim(R/P) = \dim(R)$. Then $e(R) \geq e(R_P)$.

The hypothesis on height and dimension is needed: consider $R = k[x, y, z]/(x) \cap (y^2, z)$ and $P = (y, z)$, then $e(R) = 1$ while $e(R_P) = 2$.

Theorem (Nagata 60')

The localization formula holds when R is excellent.

This is a highly non-trivial result. For instance, it gives a proof that localizations of regular local rings are regular (need a small argument to reduce to the complete case) without using homological methods.

For flat local map $R \to S$ (e.g., $S = R[[x]]$ or S is finite-free over R), the singularities of R should be no worse than the singularities of S. For example, if S is regular, Cohen–Macaulay, rational singularity, strongly F-regular... then so is R. Therefore the natural expectation is $e(R) \leq e(S)$. This is a remarkable conjecture of Lech which dates back to 1960, and is still wide open!

Conjecture (Lech's conjecture)

Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local extension. Then $e(R) \leq e(S)$.

We can also put stronger conditions on the map.

1. If $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is flat with $S/\mathfrak{m}S$ regular (e.g., $R \to S$ is formally smooth), then it is easy to show that $e(R) = e(S)$.
2. If $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is flat with $S/\mathfrak{m}S$ a complete intersection (e.g., $R \to S$ is lci map), then Lech proved that $e(R) \leq e(S)$. This can be also proved using Cohen-factorization of Avramov–Foxby–B. Herzog.

Lech's conjecture was known in the following cases:

1. (Lech 60') $\dim R \leq 2$.
2. (Backelin–J. Herzog–Ulrich 90') R is a strict complete intersection: $gr_{\mathfrak{m}}R$ is a complete intersection.
3. (Ma 16) $\dim R = 3$ and R has equal characteristic.
4. (B. Herzog 80' 90') under some technical conditions on $S/\mathfrak{m}S$.

In general, for a flat local $R \to S$ with $d = \dim(R)$, we know that:

1. (Lech 60') $e(R) \leq d! \cdot e(S)$
2. (Ma 16) $e(R) \leq \max\{1, d!/2^d\}e(S)$ if R has equal characteristic.

Are the localization formula and Lech's conjecture related? A strange and surprising result:

Theorem (Larfeldt–Lech 80')

The conjecture on localization formula and Lech's conjecture are equivalent.

On the other hand, the fact that localization formula is known for excellent rings has nothing to do with Lech's conjecture: in fact, it is easy to see that Lech's conjecture immediately reduces to the case that both R and S are complete (thus excellent).

Methods:

1. Lech, B. Herzog: delicate analysis on Hilbert–Samuel polynomials $H^1(t) = l(R/\mathfrak{m}^t)$, and its iterates $H^i(t) = \sum_{j=0}^{t} H^{i-1}(j)$
2. Backelin–J. Herzog–Ulrich: constructing Ulrich modules over R.
3. Ma: positive characteristic methods and reduction mod p.

We will focus on Ulrich modules.

Recall that if M is maximal Cohen–Macaulay, then for every system of parameters $\underline{x} = x_1, \ldots, x_d$ of R, we have $e(\underline{x}, M) = l(M/(\underline{x})M)$. In particular, if $\underline{x}$ is a minimal reduction of $\mathfrak{m}$, then

$$e(\mathfrak{m}, M) = l(M/(\underline{x})M) \geq l(M/\mathfrak{m}M) = \nu_R(M).$$

Definition

Let $(R, \mathfrak{m})$ be a local ring. M is called a Ulrich module over R if M is a maximal Cohen–Macaulay module and $e(\mathfrak{m}, M) = \nu_R(M)$.

An important observation of Hochster–Huneke and Hanes:

Theorem

Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local extension. Suppose R/P admits a Ulrich module for each minimal prime P of R such that $\dim(R/P) = \dim(R)$. Then $e(R) \leq e(S)$.

Sketch of proof.

One first complete and use the localization formula to reduce to the case that $\dim(R) = \dim(S)$. By the additivity property, it is enough to show that $e(R/P) \leq e(S/PS)$ for each minimal prime P of R such that $\dim(R/P) = \dim(R)$. Let U be a Ulrich module over R/P, then

$$e(R/P) = \frac{e(\mathfrak{m}, U)}{\operatorname{rank} U} = \frac{\nu_R(U)}{\operatorname{rank} U} = \frac{\nu_S(U \otimes S/PS)}{\operatorname{rank} U} \leq \frac{e(\mathfrak{n}, U \otimes S/PS)}{\operatorname{rank} U} = e(S/PS)$$

where the only $\leq$ is because $U \otimes S/PS$ is maximal Cohen–Macaulay over S/PS since S/PS is flat over R/P with $\dim(S/PS) = \dim(R/P)$. □

Unfortunately, the existence of Ulrich sequence is known in very limited cases:

1. (easy) $\dim(R) \leq 1$.
2. (Brennan–J. Herzog–Ulrich 80') R is standard graded Cohen–Macaulay domain such that $\dim(R) = 2$ with $R/\mathfrak{m}$ infinite.
3. (Backelin–J. Herzog–Ulrich 90') R is a strict complete intersection.
4. (Eisenbud–Schreyer–Weyman 03) R is a Veronese subring of a polynomial ring.
5. (Bruns–Römer–Wiebe 04) R is a generic determinantal ring.
6. If $X \subseteq \mathbb{P}^n_k$ is a projective variety, then graded Ulrich modules over the affine cone $C(X)$ corresponds to Ulrich sheafs on X (=Ulrich bundles if X is smooth).

Even more unfortunately, very recently, F. Yhee constructed complete local domain of dimension 2 that does not admit any Ulrich module. So in general Ulrich modules do not always exist! (but the question is still open for Cohen–Macaulay rings)

We will introduce a much weaker variant of Ulrich module, which we call **lim Ulrich sequence**, to prove Lech's conjecture for a large class or rings:

Theorem (Ma 20)

Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local extension. Suppose R is standard graded over a perfect field k (localized or completed at the homogenous maximal ideal). Then $e(R) \leq e(S)$.

By Artin approximation and a standard reduction mod $p > 0$ technique (not obvious!), in order to prove this theorem we can assume k has characteristic $p > 0$.

We need the notion of lim Cohen–Macaulay sequence, developed by Bhatt–Hochster–Ma.

Definition

Let $(R, \mathfrak{m})$ be a local ring of dimension d. A sequence of modules $\{M_n\}_n$ of dimension d is called lim Cohen–Macaulay if there exists a system of parameters $\underline{x} = x_1, \ldots, x_d$ of R such that

$$\lim_{n\to\infty} \frac{l(H_i(\underline{x}, M_n))}{\nu_R(M_n)} = 0 \qquad (*)$$

for all $i > 0$.

It is true, but not obvious, that this is independent of the choice of the system of parameters. In other words, if $(*)$ holds for one system of parameters, then it holds for all system of parameters. Moreover, if R is a domain, then we can use $\operatorname{rank}(M_n)$ instead of $\nu_R(M_n)$ in the denominator.

Obviously, if M is a maximal Cohen–Macaulay module over R, then the constant sequence $M_n = M$ is a lim Cohen–Macaulay sequence.

If $(R, \mathfrak{m})$ is an F-finite local ring of characteristic $p > 0$, then $M_n = F_*^n R \cong R^{1/p^n}$ is a lim Cohen–Macaulay sequence: this is a reformulation of a result of Dutta and Roberts (also follows from tight closure theory).

Bhatt–Hochster–Ma proved that if every complete local domain of mixed characteristic (with perfect residue field) admits a lim Cohen–Macaulay sequence, then Serre's positivity conjecture holds. This greatly extends the earlier observation of Hochster that the existence of finitely generated maximal Cohen–Macaulay modules implies Serre's conjecture.

We also need a weaker variant of lim Cohen–Macaulay sequence.

Definition

Let $(R, \mathfrak{m})$ be a local ring of dimension d. A sequence of modules $\{M_n\}_n$ of dimension d is called weakly lim Cohen–Macaulay if there exists a system of parameters $\underline{x} = x_1, \ldots, x_d$ of R such that

$$\lim_{n\to\infty} \frac{\chi_1(\underline{x}, M_n)}{\nu_R(M_n)} = 0.$$

Here $\chi_1(\underline{x}, M) := \sum_{i=1}^{d} (-1)^{i-1} l(H_i(\underline{x}, M))$. Again, it is true, but not obvious, that this is independent of the choice of the system of parameters, and if R is a domain, then we can use $\mathrm{rank}(M_n)$ instead of $\nu_R(M_n)$ in the denominator. It is clear that any lim Cohen–Macaulay sequence is weakly lim Cohen–Macaulay, but they are not equivalent.

A consequence of these facts is the following:

Lemma

Suppose $R/\mathfrak{m}$ is infinite, if $\{M_n\}_n$ is a weakly lim Cohen–Macaulay sequence, then

$$\lim_{n\to\infty} \frac{e(\mathfrak{m}, M_n)}{\nu_R(M_n)} \geq 1.$$

Sketch of proof.

Let $\underline{x}$ be a minimal reduction of $\mathfrak{m}$. Note that $e(\mathfrak{m}, M_n) = \chi(\underline{x}, M_n)$ and the weakly lim Cohen–Macaulay assumption applied to $\underline{x}$ implies that $\chi_1(\underline{x}, M_n) = o(\nu_R(M_n))$. So asymptotically, the limit tends to

$$\lim_{n\to\infty} \frac{l(M_n/(\underline{x})M_n)}{\nu_R(M_n)} \geq 1.$$

□

Now we define (weakly) lim Ulrich sequence

Definition

A sequence of modules $\{U_n\}_n$ is called (weakly) lim Ulrich if:

1. $\{U_n\}_n$ is (weakly) lim Cohen–Macaulay;
2. $\lim_{n\to\infty} \frac{e(\mathfrak{m}, U_n)}{\nu_R(U_n)} = 1$.

Obviously, if U is a Ulrich module over R, then the constant sequence $U_n = U$ is a lim Ulrich sequence. Thus the following is a generalization of the observation of Hochster–Huneke and Hanes.

Theorem

Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ *be a flat local extension. Suppose* R/P *admits a weakly lim Ulrich sequence for each minimal prime* P *of* R *such that* $\dim(R/P) = \dim(R)$*. Then* $e(R) \le e(S)$*.*

Sketch of proof.

The proof almost follows the same line! First, by the same argument we can assume $\dim(R) = \dim(S)$ and we can work with $R/P \to S/PS$ for each minimal prime P such that $\dim(R/P) = \dim R$. But then

$$e(R/P) = \lim_{n\to\infty} \frac{e(\mathfrak{m}, U_n)}{\mathrm{rank}(U_n)} = \lim_{n\to\infty} \frac{\nu_R(U_n)}{\mathrm{rank}(U_n)}$$

$$= \lim_{n\to\infty} \frac{\nu_S(U_n \otimes S/PS)}{\mathrm{rank}(U_n)} \le \lim_{n\to\infty} \frac{e(\mathfrak{n}, U_n \otimes S/PS)}{\mathrm{rank}(U_n)} = e(S/PS).$$

Here the second $=$ uses that U_n is weakly lim Ulrich, and the only $\le$ uses the fact that $U_n \otimes S/PS$ is weakly lim Cohen–Macaulay over S/PS (as the map is flat local of the same dimension) and the lemma. □

Putting all these together, in order to prove our main result on Lech's conjecture, it remains to prove the following.

Theorem (Ma 20)

Suppose $(R, \mathfrak{m})$ is a standard graded domain over an infinite F-finite field of characteristic $p > 0$. Then R admits a weakly lim Ulrich sequence.

We sketch the construction when R **is Cohen–Macaulay and** k **is perfect**. The proof is substantially less technical in this case (while revealing the idea). Moreover, in this case, the construction actually yields lim Ulrich sequence.

There are 4 steps.

Step 1: Let $T_n = k[x_1, y_1]\#k[x_2, y_2]\# \cdots \#k[x_n, y_n]$ be the Segre product of n copies of $k[x, y]$ (which is the affine cone of the product of n projective lines). Let $q = p^e$: one should think that q being very large compared to n. Set

$$W_q^n := k[x_1, y_1]\#k[x_2, y_2](q)\# \cdots \#k[x_n, y_n]((n-1)q).$$

Next pick $z_1, \ldots, z_{n+1}$ general degree one forms of T_n such that

$$A_n := k[z_1, \ldots, z_{n+1}] \to T_n$$

is a Noether normalization. We view W_q^n as graded modules over A_n.

Step 2: Let $\underline{z} = z_1, \ldots, z_d$ be general degree one forms of R. Then

$$A := k[z_1, \ldots, z_d] \to R$$

is a Noether normalization. We identify A with A_{d-1} in Step 1 and consider $R \otimes_A W_q^{d-1}$ as a graded module over R. We set

$$U_e := F_*^e \left((R \otimes_A W_q^{d-1})_{-1 \mod q} \right)$$

and we claim that $\{U_e\}_e$ is a lim Ulrich sequence over R.

Here for a $\mathbb{Z}$-graded R-module M, $M_{-a \mod q} := \oplus_{j \in \mathbb{Z}} M_{-a+qj}$. In general, $M_{-a \mod q}$ is not necessarily an R-module. But U_e's are R-modules because of the e-th Frobenius pushforward (thus we need characteristic $p > 0$ here!).

Step 3: We verify that $\{U_e\}_e$ is lim Cohen–Macaulay. Since R is Cohen–Macaulay, we know

$$R \cong A(-a_1) \oplus \cdots \oplus A(-a_s)$$

for some $a_i \geq 0$, as graded A-modules. Therefore

$$U_e \cong \oplus_{i=1}^{s} F_*^e \left((W_q^{d-1})_{-1-a_i \mod q} \right).$$

Hence it is enough to show $F_*^e \left((W_q^{d-1})_{-1-a_i \mod q} \right)$ is lim Cohen–Macaulay over A.

By a local cohomology criterion for lim Cohen–Macaulay sequence (Bhatt–Hochster–Ma), it is enough to control the lengths of the lower local cohomology modules of $F^e_*\left((W_q^{d-1})_{-1-a_i \mod q}\right)$ (in this case they have finite length) by their ranks.

By direct computations (details omitted):

$$\mathrm{rank}_A F^e_*\left((W_q^{d-1})_{-1-a_i \mod q}\right) = (d-1)! q^{d-1}$$

$$l\left(H^i_m(F^e_*((W_q^{d-1})_{-1-a_i \mod q}))\right) \sim o(q^{d-1})$$

This shows $\{U_e\}_e$ is lim Cohen–Macaulay. Note: this step doesn't work if R is not Cohen–Macaulay, because then the lower local cohomology modules of U_e do not necessarily have finite length.

Step 4: Finally we check the lim Ulrich condition. Since $\underline{z} = z_1, \ldots, z_d$ is a minimal reduction of $\mathfrak{m}$, we have

$$e(\mathfrak{m}, U_e) = e(\underline{z}, U_e) = \mathrm{rank}_A U_e = s(d-1)! q^{d-1}.$$

On the other hand,

$$\nu_R(U_e) \geq \dim_k F^e_*(R \otimes_A W_q^{d-1})_{q-1} = \dim_k (R \otimes_A W_q^{d-1})_{q-1}$$

because elements in the maximal ideal of R act by their q-th powers, so they cannot hit the degree $\leq q-1$ pieces of $R \otimes_A W_q^{d-1}$ since $R \otimes_A W_q^{d-1}$ only lives in non-negative degrees.

But what is $\dim_k(R\otimes_A W_q^{d-1})_{q-1}$? This is $\dim_k \oplus_{i=1}^{s}(W_q^{d-1})_{q-1-a_i}$ and we can compute directly! Recall that

$$W_q^{d-1} := k[x_1,y_1]\#k[x_2,y_2](q)\#\cdots\#k[x_{d-1},y_{d-1}]((d-2)q).$$

By the definition of Segre product, we have ($q=p^e$ is large compared with each a_i)

$$\dim_k(W_q^{d-1})_{q-1-a_i} = (q-a_i)(2q-a_i)\cdots((d-1)q-a_i) \sim (d-1)!q^{d-1}.$$

Therefore $\dim_k(R\otimes_A W_q^{d-1})_{q-1} \sim s(d-1)!q^{d-1} = e(\mathfrak{m}, U_e)$. Hence

$$\lim_{n\to\infty}\frac{e(\mathfrak{m},U_n)}{\nu_R(U_n)} \leq \lim_{n\to\infty}\frac{e(\mathfrak{m},U_n)}{\dim_k(R\otimes_A W_q^{d-1})_{q-1}} = 1.$$

But then the limit $=1$ since it is always ≥ 1 as we already proved $\{U_e\}_e$ is lim Cohen–Macaulay.

Thank you!