
O
G
D
O
O
O
O
D

Fellowship of the ring

23 April 2020

Quadratic Gorenstein Algebras and the Koszul property

Mike Stillman

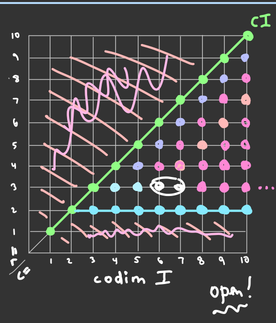
joint with:
Matt Mastroeni
Hal Schenck

arxiv:
1903.08265
1903.08273

- CI $\therefore$ koszul.
- CRV (koszul Caviglia.)
- Matsuda $\exists$ non-koszul

($R_{(r,c)} \otimes k[t]/t^2$ koszul $r+1, c+1$

$\exists$ non-koszuls MS (Jason McCullough $c=8, r=3$) Alex Seceleanu



The wonders of Betti diagrams

① $S = k[x,y,z]$.

$I = (x^2, y^2, z^2, xy - xz, yz - xz)$

$R = S/I$ codim $I = 3$ Gorenstein

Buchsbaum–Eisenbud. $I = Pf_4(5 \times 5$ skew matrix of linear forms$)$ M_2

free res:

$$0 \leftarrow R \leftarrow S \leftarrow S(-2)^5 \xleftarrow{M} S(-3)^5 \leftarrow S(-5) \leftarrow 0$$

Betti diagram:

	0	1	2	3 (j)
0	1	-	-	.
1	-	5	5	.
2	-	-	-	1

$i \rightarrow$; reg R $\rightarrow$ 2; projdim R $\rightarrow$ 3

$b_{ji} := \dim \mathrm{Tor}_j^S(R,k)_{i+j}$

- Hilbert series: $H_R(t) = \sum (\dim R_d) t^d$

$$= \frac{1 - 5t^2 + 5t^3 - t^5}{(1-t)^3}$$

$$= 1 + 3t + t^2$$

h vector $= (1,3,1)$.

- codim $I = 3$
- R is CM if projdim = codim - YES
- R is Gorenstein if R is CM and top betti # = 1 YES.

② Example: canonical curves

Let C = curve of genus g

Have $\varphi : C \hookrightarrow \mathbb{P}^{g-1}$ canonical embedding.

Let $I = I_C \subseteq S = k[x_1, \ldots, x_g]$

$R = S/I$.   homog coord ring.

codim $I = g - 2$.

e.g.: $g(C) = 7$ general curve.   codim = 5

i \ j	0	1	2	3	4	5
0	1					
1		10	16			
2				16	10	
3						1

CM
Gorenstein
reg $R = 3$

Gorenstein: $b_{ij} = b_{3-i, 5-j}$ duality.

h vector: $(1,5,5,1)$.

Both Quadratic (I generated), Gorenstein.

Abbreviate: $R = S/I$ is QG.

theorem (Petri) If C is not trigonal nor is it isomorphic to a plane quintic, then $R = S/I_C$ is QG.

Example Suppose $R = k[x_1, \ldots, x_c]/I$

codim $I = c$

R is QG. R is finite length.

Migliore–Nagel '13 study QG algebras

if reg $R = 4$ ($= r$)   ($R_r \neq 0$ dim 1, $R_{r+1} = 0$)

and $c = 5$ or 6: HF's?

and $R \neq$ CI.

- 1 5 8 5 1
- 1 6 10 6 1
- 1 6 11 6 1

} all occur

- 1 6 12 6 1 $\leftarrow$ occurs in char small.

koszul algebras (Priddy 1970)

Def $R = S/I$ is koszul if k has a linear R-free resolution.

$$0 \leftarrow k \leftarrow R \xleftarrow{d_1} R(-1)^{\#vars} \xleftarrow{d_2} R(-2)^{b_2} \xleftarrow{d_3} \cdots$$

properties

① if I has a quadratic GB then R is Koszul. (Fröberg, Eisenbud, Reeves, Totaro)

② no known algorithm to determine if R is Koszul.

③ (Finkelberg-Vishik '93, Polishchuk '95) $C \subseteq \mathbb{P}^{g-1}$ and $R = S/I_C$ is QG then R is Koszul.

Consider: R is QG $R = S/I$.

$\begin{cases} r = \text{reg } R \\ c = \text{cod } I \end{cases}$

Q: must R be Koszul?

known: ① $r \leq c$

② $r = c \iff I$ is a complete intersection $\Rightarrow$ (Tate '57) R is Koszul.

③ Conca–Rossi–Valla 2001

- $r = 2 \Rightarrow c \geq 2$, R is koszul.
- $r = 3 \Rightarrow$ if $c = 4,5$ Caviglia CRV

then R is Koszul.

Question: if $r = 3$, $c \geq 6$ is R koszul?

Matsuda 2017

- found a toric ring $R = S/I$ (I binomial prime), QG

with cod $= 7$
reg $= 4$

BUT not koszul!

asks: can one find such example for $c < 7$, $r = 4$?

Inverse systems

Let $S = k[x_1, \ldots, x_c]$
$D = k[y_1, \ldots, y_c]$

think of monomials in D as fractions

$$y_1^{\beta_1} \cdots y_c^{\beta_c} = y^\beta \longleftrightarrow \frac{1}{x^\beta}$$

S acts on D:

$$x^\alpha \cdot y^\beta = \begin{cases} y^{\beta - \alpha} & \text{if } \beta - \alpha \geq 0 \\ 0 & \text{else.} \end{cases}$$

Let $f \in D$ be homog. of degree r

define $f^\perp := \{ g \in S : gf = 0 \} = \mathrm{ann}\, f \subseteq S$ ideal. ($= I$)

(M2 package InverseSystems)

example $f = xy + xz + yz$

$f^\perp = (x^2, y^2, z^2, xy - xz, xy - yz)$

same as our 1st example.

prop If $f \in D_r$, $I = f^\perp \subset S$, $R = S/I$

then ⓐ R is finite length

ⓑ R is Gorenstein

ⓒ reg $R = r$ ($=$ socdeg R) $R_r \neq 0$, $R_{r+1} = 0$.

theorem Every Gorenstein, finite length, reg $R = r$ can R arises as $R = S/f^\perp$ for some $f \in D_r$.

(Eis. 21.2).   (Kanev–Iarrobino).

Difficulties:

- hard to control HF of $f^\perp$
- hard to get quadratic generation.

CRV '01:

if $f \in D_3$ defines a smooth hypersurface in $\mathbb{P}^{c-1}$, then $R = S/f^\perp$ is koszul.

Case $R = S/I$ $r = 4$, codim $= c$ $c = 7$ Matsuda.

if R is QG, must R be koszul?

theorem Let $S = k[x_1, \ldots, x_c]$ $c \geq 7$

Let $f = \sum_{i \in \mathbb{Z}/c\mathbb{Z}} y_i y_{i+1} y_{i+2}^2 \in D_4$.

then $R = S/f^\perp$ satisfies:

ⓐ R is quadratic (of course Gorenstein)

ⓑ HF is $(1, c, 2c, c, 1)$ $(1, c, ?, c, 1)$

ⓒ R is not koszul. $(1,6,12,6,1)$

Also $\exists$ such an example for $c = 6$.

Case $R = S/I$:   codim $= c$, $r = c - 1$

e.g.: $r = 4$, $c = 5$ $(1, 5, h_2, 5, 1)$.

Migliore–Nagel '13:

if R is QG and $r = c - 1$

then the HF of R is uniquely determined.

in particular: I is gen by $c+2$ quadrics (5 linear syzygies) theorem (MSS) Given $R = S/I$ QG, $r = c-1$

then

$$I = \mathrm{Pfaff}_4(M) + (Q_6, \dots, Q_{c+2})$$

where

- M is 5×5 skew symm linear matrix
  $\mathrm{Pf}_4(M)$ is Goren codim 3.
- $(Q_6, \dots, Q_{c+2})$ reg. seq. mod $\mathrm{Pf}_4(M)$.

$\therefore R = S/I$ is koszul.

Open questions

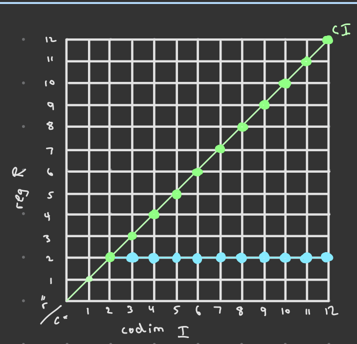
- R QG, $r=3$, $c=6$ or $c=7$ $\overset{?}{\Longrightarrow}$ R koszul ?
- consider $r=4$ case, $c \leq 6$

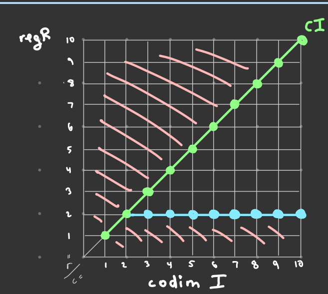
  MN '13:

1	5	8	5	1	koszul
1	6	10	6	1	does ∃ non-koszul.
1	6	11	6	1	
1	6	12	6	1	∃ non koszul.

- Consider $\mathrm{Gor}(T)$ (T = hilb fcn)

  identify:
  - locus or comps. of QG ideals
  - locus where koszul-ness fails at step n.


CI
reg R
codim I


CI
regR
codim I