Wheel Signatures on the Neural Ideal

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Motivation

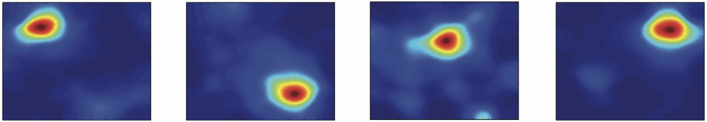

Figure: Place fields for four neurons from Curto 2017

- Neurons fire when reacting to environmental stimuli. This activity has patterns which give rise to receptive fields.

- These fields are represented by a collection of sets, $\mathcal{U} = \{U_i\}_{i=1}^n$, contained within a stimulus space.

- It is interesting to explore if these collections are open convex, closed convex, or neither.

Neural Codes

Definition

A **code** on n neurons is a subset $\mathcal{C} \subseteq 2^{[n]}$, where $[n] = \{1, ..., n\}$. Elements of a code $\mathcal{C}$ are **codewords**, and a **maximal codeword** is a codeword in $\mathcal{C}$ that is maximal with respect to inclusion.

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Example:

$$\mathcal{C} = \{\mathbf{1234}, \mathbf{123}, \mathbf{234}, \mathbf{345}, 12, 13, 24, 34, 35, 45, 1, 2, 3, 4, 5, \emptyset\}$$

- 24 is a codeword
- 1234 is a maximal codeword

Realization of a Code

Definition

A **realization of a code** $\mathcal{C} \in \mathbb{R}^d$ is a collection $\mathcal{U} = \{U_i\}_{i=1}^n$ of open subsets of a **stimulus space** $X \subseteq \mathbb{R}^d$ such that $c \in \mathcal{C}$ if and only if

$$\left(\bigcap_{i \in c} U_i\right) \setminus \left(\bigcup_{j \in [n] \setminus c} U_j\right) \neq \emptyset.$$

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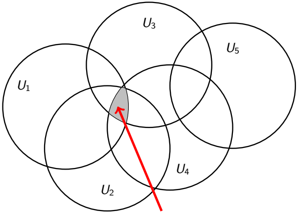

codeword $\{1, 2, 3, 4, 0\} = \{\mathbf{1234}\}$

Simplicial Complex

Definition

The simplicial complex of a code $\mathcal{C}$ on n neurons is:

$$\Delta(\mathcal{C}) := \{\sigma \in 2^{[n]} \mid \sigma \subseteq c \text{ for some } c \in \mathcal{C}\}$$

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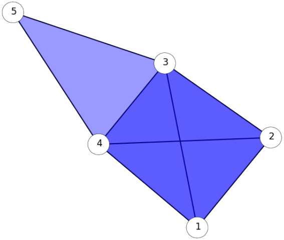

Figure: The simplicial complex $\Delta(\mathcal{C})$

Trunks

Definition

Given a code $\mathcal{C}$ on n neurons and subset $\sigma \subseteq [n]$, the **trunk** of σ in $\mathcal{C}$, denoted by $\mathrm{Tk}_{\mathcal{C}}(\sigma)$, is the set of all codewords containing σ; denoted as

$$\mathrm{Tk}_{\mathcal{C}}(\sigma) := \{c \in \mathcal{C} \mid \sigma \subseteq c\}$$

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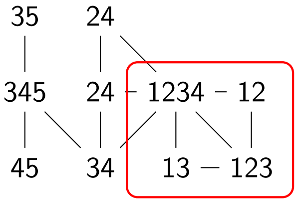


Figure: The trunk of 1

Note: This containment graph doesn't include 1, 2, 3, 4, or 5

Neural Ideal

Definition

A **pseudo-monomial** has the form $f = \prod_{i \in \sigma} x_i \prod_{j \in \tau}(1 - x_j)$.

The **neural ideal** is $J_{\mathcal{C}} = \langle \{\rho_v \mid v \notin \mathcal{C}\} \rangle$, where $\rho_v = \prod_{i=1}^{n}(1 - v_i - x_i)$ are the characteristic pseudo-monomials of codewords that are not in $\mathcal{C}$.

Definition

The **canonical form** of the neural ideal, denoted as $CF(J_{\mathcal{C}})$, is the set of all minimal pseudo-monomials of $J_{\mathcal{C}}$.

Canonical Form

Example: The canonical form of $J_{\mathcal{C}}$ where

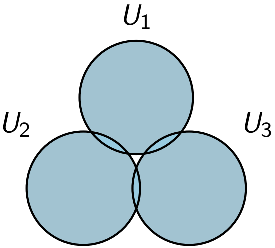
$$\mathcal{C} = \{\mathbf{123}, \mathbf{124}, \mathbf{126}, \mathbf{135}, \mathbf{456}, 12, 13, 4, 5, 6, \emptyset\}:$$

$$CF(J_{\mathcal{C}}) = \{(1-x_2)(1-x_3)(1-x_4)(1-x_5)(1-x_6), (1-x_1)(1-x_4)(1-x_5)(1-x_6), x_1(1-x_2)(1-x_3), x_1(1-x_2)x_4, x_1(1-x_3)x_5, x_4x_5(1-x_6), x_1x_4x_5, x_1(1-x_2)x_6, x_4(1-x_5)x_6, x_1x_4x_6, x_2x_4x_6, (1-x_4)x_5x_6, x_1x_5x_6, (1-x_1)x_2, (1-x_1)x_3, x_3x_4, x_2x_5, x_3x_6\}$$

Receptive Field Relations

1. **Type 1 Relation**:
   The relation $\prod_{i\in\sigma} x_i$ corresponds to $\bigcap_{i\in\sigma} U_i = \emptyset$.

2. **Type 2 Relation**:
   The relation $\prod_{i\in\sigma} x_i \prod_{i\in\tau}(1 - x_i)$ corresponds to $\bigcap_{i\in\sigma} U_i \subseteq \cup_{i\in\tau} U_i$.



$U_1 \cap U_2 \cap U_3 = \emptyset$ corresponds to a Type 1 relation.

Stanley-Reisner Ideal

Definition

Let $\mathcal{C}$ be a code. The ideal generated by the Type 1 relations of $CF(J_{\mathcal{C}})$ is the **Stanley-Reisner ideal** of $\Delta(\mathcal{C})$.

$$I_\Delta \overset{def}{=} \langle x_\sigma | \sigma \notin \Delta(\mathcal{C})\rangle$$

Note that $I_\Delta \in CF(J_{\mathcal{C}})$. To show an example, recall

$$CF(J_{\mathcal{C}}) = \{(1-x_2)(1-x_3)(1-x_4)(1-x_5)(1-x_6), (1-x_1)(1-x_4)(1-x_5)(1-x_6), x_1(1-x_2)(1-x_3), x_1(1-x_2)x_4, x_1(1-x_3)x_5, x_4x_5(1-x_6), x_1x_4x_5, x_1(1-x_2)x_6, x_4(1-x_5)x_6, x_1x_4x_6, x_2x_4x_6, (1-x_4)x_5x_6, x_1x_5x_6, (1-x_1)x_2, (1-x_1)x_3, x_3x_4, x_2x_5, x_3x_6\}$$

And so, $I_\Delta = \langle x_1x_4x_5, x_1x_4x_6, x_2x_4x_6, x_3x_4, x_2x_5, x_3x_6\rangle$

Wheels

Wheels

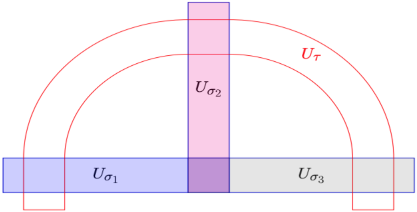
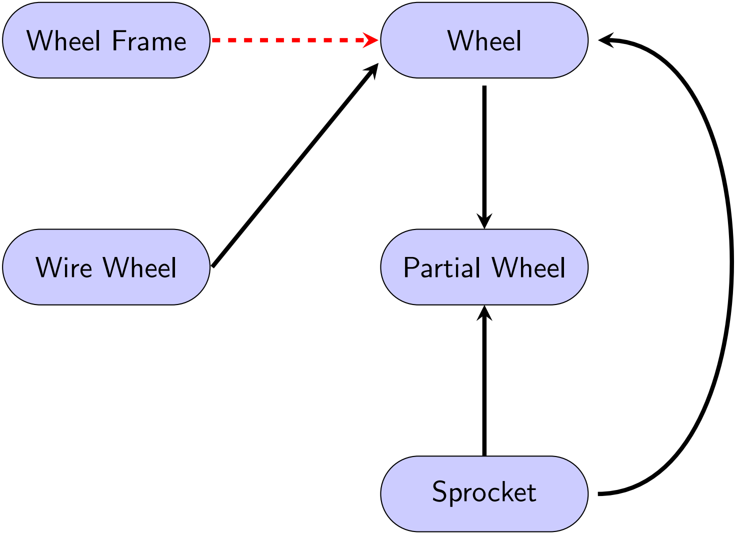


Figure: Intuition behind a wheel from [4]

Why do we care about wheels?

- **Wheels are markers for non-convexity.**
- It is a useful combinatorial tool to test for open convexity on neural codes.

Classifications of Wheels



Goal

Example of an algebraic signature:

Curto et al. (2019)

If a code $\mathcal{C}$ contains $x_\sigma(1-x_i)(1-x_j)$ and $x_\sigma x_i x_j$ in $CF(J_\mathcal{C})$, then $\mathcal{C}$ is guaranteed to be non-convex.

Goal

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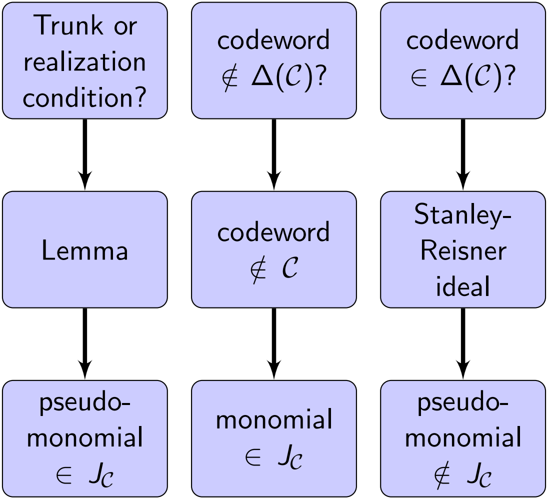
If a code $\mathcal{C}$ contains $x_\sigma(1-x_i)(1-x_j)$ and $x_\sigma x_i x_j$ in $CF(J_\mathcal{C})$, then $\mathcal{C}$ is guaranteed to be non-convex.

Goal: We wish to find algebraic signatures for wheels within the neural ideal, $J_\mathcal{C}$.

Translation Process



Partial Wheels

Definition

A tuple $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau) \in (\Delta(\mathcal{C}))^4$ is a **partial wheel** of a neural code $\mathcal{C}$ if it satisfies the following conditions:

P(1) $\sigma_1 \cup \sigma_2 \cup \sigma_3 \in \Delta(\mathcal{C})$, and
$Tk_{\mathcal{C}}(\sigma_j \cup \sigma_k) = Tk_{\mathcal{C}}(\sigma_1 \cup \sigma_2 \cup \sigma_3)$ for every $1 \leq j < k \leq 3$,

P(2) $\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau \notin \Delta(\mathcal{C})$, and

P(3) $\sigma_j \cup \tau \in \Delta(\mathcal{C})$ for $j \in \{1, 2, 3\}$.

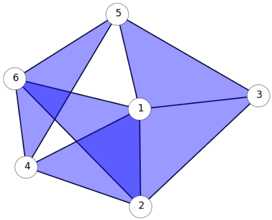
Example: Let $\mathcal{C} = \{\mathbf{123}, \mathbf{124}, \mathbf{126}, \mathbf{135}, \mathbf{456}, 12, 13, 4, 5, 6, \emptyset\}$. We show that $(5, 4, 6, 1) = (\sigma_1, \sigma_2, \sigma_3, \tau)$ is a partial wheel of $\mathcal{C}$

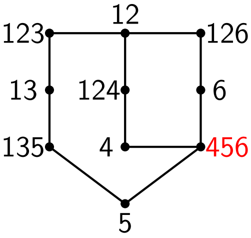
Partial Wheel Condition 1

Recall $\mathcal{C} = \{\mathbf{123}, \mathbf{124}, \mathbf{126}, \mathbf{135}, \mathbf{456}, 12, 13, 4, 5, 6, \emptyset\}$ and $(5, 4, 6, 1) = (\sigma_1, \sigma_2, \sigma_3, \tau)$.

P(i): $\sigma_1 \cup \sigma_2 \cup \sigma_3 \in \Delta(\mathcal{C})$, and
$Tk_{\mathcal{C}}(\sigma_j \cup \sigma_k) = Tk_{\mathcal{C}}(\sigma_1 \cup \sigma_2 \cup \sigma_3)$ for every $1 \leq j < k \leq 3$

- $\sigma_1 \cup \sigma_2 \cup \sigma_3 = 456 \in \Delta(\mathcal{C})$.
- $\mathrm{Tk}(\sigma_1 \cup \sigma_2) = \mathrm{Tk}(\sigma_1 \cup \sigma_3) = \mathrm{Tk}(\sigma_2 \cup \sigma_3) = \mathrm{Tk}(\sigma_1 \cup \sigma_2 \cup \sigma_3) = 456$.





Translation for Condition 1

Recall $(5, 4, 6, 1) = (\sigma_1, \sigma_2, \sigma_3, \tau)$.

Theorem (LG, LT, EW 2022)

(a) $\displaystyle\prod_{i \in \sigma_1 \cup \sigma_2 \cup \sigma_3} x_i \notin J_{\mathcal{C}}$

(b) $\displaystyle\prod_{a \in \sigma_j \cup \sigma_k} x_a \prod_{i \in \sigma_\ell \setminus (\sigma_j \cup \sigma_k)} (1 - x_i) \in J_{\mathcal{C}}$, for distinct $j, k, \ell \in \{1, 2, 3\}$.

$$CF(J_{\mathcal{C}}) = \{(1-x_2)(1-x_3)(1-x_4)(1-x_5)(1-x_6),\ (1-x_1)(1-x_4)(1-x_5)(1-x_6), x_1(1-x_2)(1-x_3), x_1(1-x_2)x_4,\ x_1(1-x_3)x_5, x_4x_5(1-x_6), x_1x_4x_5, x_1(1-x_2)x_6, x_4(1-x_5)x_6,\ x_1x_4x_6, x_2x_4x_6, (1-x_4)x_5x_6, x_1x_5x_6, (1-x_1)x_2, (1-x_1)x_3, x_3x_4, x_2x_5, x_3x_6\}$$

We see that $x_4x_5x_6 \notin J_{\mathcal{C}}$ by (a) and $x_4x_5(1-x_6)$, $(1-x_4)x_5x_6$, $x_4(1-x_5)x_6 \in J_{\mathcal{C}}$ by (b).

Partial Wheel Condition 2

Recall $\mathcal{C} = \{\mathbf{123}, \mathbf{124}, \mathbf{126}, \mathbf{135}, \mathbf{456}, 12, 13, 4, 5, 6, \emptyset\}$ and $(5, 4, 6, 1) = (\sigma_1, \sigma_2, \sigma_3, \tau)$.

P(ii): $\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau \notin \Delta(\mathcal{C})$

- $\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau = 1456 \notin \Delta(\mathcal{C})$.

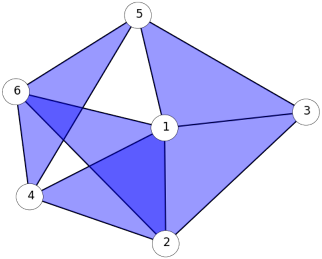


Figure: The simplicial complex $\Delta(\mathcal{C})$

Translation for Condition 2

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Theorem (LG, LT, EW 2022)

$$P(ii) : \prod_{i \in (\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau)} x_i \in J_{\mathcal{C}}$$

$$CF(J_{\mathcal{C}}) = \{(1-x_2)(1-x_3)(1-x_4)(1-x_5)(1-x_6), (1-x_1)(1-x_4)(1-x_5)(1-x_6), x_1(1-x_2)(1-x_3), x_1(1-x_2)x_4, x_1(1-x_3)x_5, x_4x_5(1-x_6), x_1x_4x_5, x_1(1-x_2)x_6, x_4(1-x_5)x_6, x_1x_4x_6, x_2x_4x_6, (1-x_4)x_5x_6, x_1x_5x_6, (1-x_1)x_2, (1-x_1)x_3, x_3x_4, x_2x_5, x_3x_6\}$$

We see that $x_1x_4x_5x_6 \in J_{\mathcal{C}}$.

Partial Wheel Condition 3

Recall $\mathcal{C} = \{\mathbf{123}, \mathbf{124}, \mathbf{126}, \mathbf{135}, \mathbf{456}, 12, 13, 4, 5, 6, \emptyset\}$ and $(5, 4, 6, 1) = (\sigma_1, \sigma_2, \sigma_3, \tau)$.

P(iii): $\sigma_j \cup \tau \in \Delta(\mathcal{C})$ for $j \in \{1, 2, 3\}$

- $\sigma_1 \cup \tau = 15 \in \Delta(\mathcal{C})$.
- $\sigma_2 \cup \tau = 14 \in \Delta(\mathcal{C})$.
- $\sigma_3 \cup \tau = 16 \in \Delta(\mathcal{C})$.

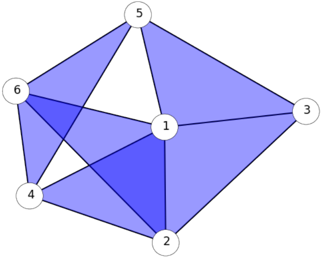


Figure: The simplicial complex $\Delta(\mathcal{C})$

Translation for Condition 3

Recall $\mathcal{C} = \{\mathbf{123}, \mathbf{124}, \mathbf{126}, \mathbf{135}, \mathbf{456}, 12, 13, 4, 5, 6, \emptyset\}$ and $(5, 4, 6, 1) = (\sigma_1, \sigma_2, \sigma_3, \tau)$.

Theorem (LG, LT, EW 2022)

$$P(iii) : \prod_{i \in (\sigma_j \cup \tau)} x_i \notin J_{\mathcal{C}}$$

where $j \in \{1, 2, 3\}$

$$CF(J_{\mathcal{C}}) = \{(1-x_2)(1-x_3)(1-x_4)(1-x_5)(1-x_6), (1-x_1)(1-x_4)(1-x_5)(1-x_6), x_1(1-x_2)(1-x_3), x_1(1-x_2)x_4, x_1(1-x_3)x_5, x_4x_5(1-x_6), x_1x_4x_5, x_1(1-x_2)x_6, x_4(1-x_5)x_6, x_1x_4x_6, x_2x_4x_6, (1-x_4)x_5x_6, x_1x_5x_6, (1-x_1)x_2, (1-x_1)x_3, x_3x_4, x_2x_5, x_3x_6\}$$

We see that $x_1x_5 \notin J_{\mathcal{C}}$, $x_1x_4 \notin J_{\mathcal{C}}$, and $x_1x_6 \notin J_{\mathcal{C}}$.

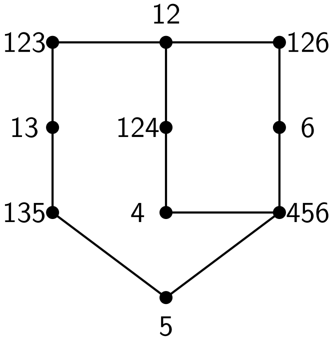
Codeword-Containment Graphs

Codeword-Containment Graphs

Definition

The **codeword-containment graph** of a neural code $\mathcal{C}$ is the (undirected) graph with the vertex set consisting of all codewords of $\mathcal{C} \setminus \{\emptyset\}$ and edge set $\{\sigma, \tau \mid \sigma \subset \tau \text{ or } \sigma \supset \tau\}$.

E.g. let $\mathcal{C} = \{\mathbf{123}, \mathbf{124}, \mathbf{126}, \mathbf{135}, \mathbf{456}, 12, 13, 4, 5, 6\}$. The corresponding codeword-containment graph is



Codeword-Containment Graphs

Question

If a neural code has *no* wheel, then is its codeword-containment graph **planar**?

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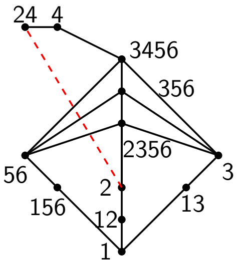
If a neural code has *no* wheel, then is its codeword-containment graph **planar**?

Counter Example

The neural code
$\mathcal{C} = \{\mathbf{12}, \mathbf{13}, \mathbf{24}, \mathbf{156}, \mathbf{2356}, \mathbf{3456}, 4, 356, 3, 1, 56, 2, \emptyset\}$ does not contain a wheel, but its codeword-containment graph is non-planar.



Future Work

- We hope to fully translate all wheel and combinatorial wheel conditions.
- We hope to further explore codeword-containment graphs and how their planarity relates to wheels.

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