Mathematical Science Research Institute

Morphisms and Neural Codes

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Motivation

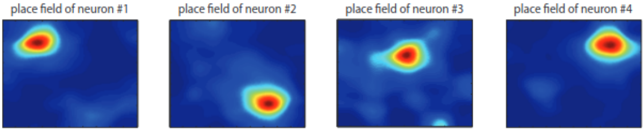

Figure: *Red areas indicate the place cells firing at a higher rate compared to the blue areas. Place fields are of four neurons in rat hippocampus. Data was provided by the Pastalkova lab. Giusti et al. 2015.*

- Morphisms and open convexity
- Explore other properties

Neural Codes

Definition

A *code* $\mathcal{C}$ is a collection $\mathcal{C} \subseteq 2^{[n]}$, where $[n] = \{1, \ldots, n\}$. The elements of $\mathcal{C}$ are called *codewords*.

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Notation

12 represents $\{1, 2\}$.

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12 represents $\{1, 2\}$.

Example

For the code C $= \{\varnothing, 1, 2\}$

1 is a codeword $\qquad$ 123 is not a codeword

Neural Codes

Definition

Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ where $U_1, \ldots, U_n \subseteq \mathbb{R}^d$. The *neural code* generated by $\mathcal{U}$ is given by

$$\mathcal{C}(\mathcal{U}) = \left\{ \sigma \subseteq [n] \mid \left(\bigcap_{i \in \sigma} U_i \right) \setminus \left(\bigcup_{j \in [n] \setminus \sigma} U_j \right) \neq \varnothing \right\}.$$

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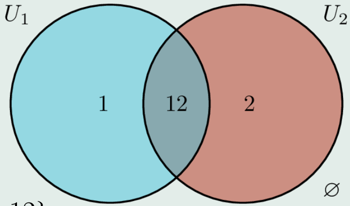
Example

Let $\mathcal{U} = \{U_1, U_2\}$ be such that



- $C(\mathcal{U}) = \{\varnothing, 1, 2, 12\}$.

Maximal Codewords

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Example

Consider $\mathcal{C} = \{\emptyset, 12, 23, \underline{123}, \underline{234}\}$.

- Maximal codewords are 123 and 234.
- 12 is not maximal since it is contained in 123.

Trunks and Morphisms

Definition

Let $\mathcal{C}$ be a neural code on n neurons and let $\sigma \subseteq [n]$. The *trunk* of σ in $\mathcal{C}$ is the set of all codewords of $\mathcal{C}$ that contain σ:

$$Tk_{\mathcal{C}}(\sigma) := \{c \in \mathcal{C} \mid \sigma \subseteq c\}.$$

Trunks and Morphisms

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Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be neural codes.

- A function $f : \mathcal{C} \to \mathcal{D}$ is a *morphism* if for each trunk $Tk_{\mathcal{D}}(\sigma)$ in $\mathcal{D}$ the preimage $f^{-1}(Tk_{\mathcal{D}}(\sigma))$ is also a trunk in $\mathcal{C}$ for all $\sigma \in \mathcal{D}$.
- A morphism f is an *isomorphism* if f is a bijection and f^{-1} is also a morphism.

Trunks and Morphisms

$$Tk_{\mathcal{C}}(\sigma) := \{c \in \mathcal{C} \mid \sigma \subseteq c\}.$$

Example

Consider $\mathcal{C} = \{\emptyset, 1, 3, 12\}$ and $\mathcal{D} = \{\emptyset, 1, 2\}$. Define the function $f : \mathcal{C} \to \mathcal{D}$ by:

$$f(\emptyset) = \emptyset \qquad f(1) = 1 \qquad f(3) = 2 \qquad f(12) = 1.$$

The trunks of $\mathcal{D}$ are

$$Tk_{\mathcal{D}}(\emptyset) = \mathcal{D} \quad Tk_{\mathcal{D}}(1) = \{1\} \quad Tk_{\mathcal{D}}(2) = \{2\} \quad Tk_{\mathcal{D}}(12) = \emptyset.$$

The pre-images of the trunks of $\mathcal{D}$ are as follows:

$$f^{-1}(\mathcal{D}) = \mathcal{C} \quad f^{-1}(\{1\}) = \{1, 12\} \quad f^{-1}(\{2\}) = \{3\} \quad f^{-1}(\emptyset) = \emptyset.$$

Closed Convexity

Definition

A code $\mathcal{C}$, where $\mathcal{C} = \mathcal{C}(\mathcal{U})$, is *closed convex* if there is a realization $\mathcal{U}$ such that the U_i's can be written as closed convex sets in some $\mathbb{R}^d$.

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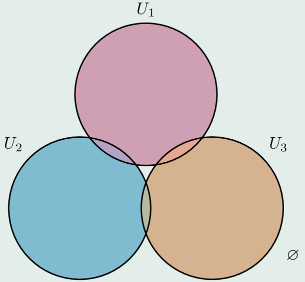
Example

The following realization shows that $\mathcal{C} = \{\varnothing, 1, 2, 3, \underline{12}, \underline{13}, \underline{23}\}$ is closed convex.



Results

The following is a generalization of a result in Jeffs 2021.

Theorem (Maldonado, Morales, Shaw 2022)

Let P be a property of a code C such that P is closed under intersection. The image under a morphism of a code with property P also has property P.

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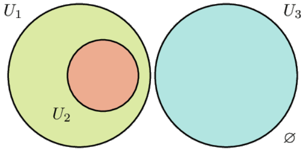
Let P be a property of a code $\mathcal{C}$ such that P is closed under intersection. The image under a morphism of a code with property P also has property P.

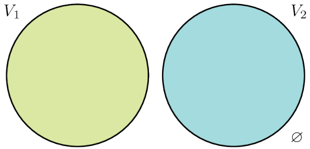
Example

Consider $\mathcal{C} = \{\varnothing, 1, \underline{3}, \underline{12}\}$ and $\mathcal{D} = \{\varnothing, \underline{1}, \underline{2}\}$. Consider the surjective morphism $f : \mathcal{C} \to \mathcal{D}$ given by:

$$f(\varnothing) = \varnothing \qquad f(1) = 1 \qquad f(3) = 2 \qquad f(12) = 1.$$



Closed Convex Realization of $\mathcal{C}$



Closed Convex Realization of $\mathcal{D}$

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The image under a morphism of a code with k maximal codewords has up to k maximal codewords.

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Consider $\mathcal{C} = \{\varnothing, 1, \underline{3}, \underline{12}\}$ and $\mathcal{D} = \{\varnothing, \underline{1}, \underline{2}\}$. Consider the following surjective morphism $f : \mathcal{C} \to \mathcal{D}$

$$\begin{aligned} f(\varnothing) &= \varnothing \\ f(1) &= 1 \\ f(3) &= 2 \\ f(12) &= 1. \end{aligned}$$

Both $\mathcal{C}$ and $\mathcal{D}$ have 2 maximal codewords.

Results

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- Isomorphism invariant

Minors

Definition

A code $\mathcal{D}$ is a *minor* of a code $\mathcal{C}$, denoted by $\mathcal{D} \leq \mathcal{C}$, if there exists a surjective morphism $f : \mathcal{C} \to \mathcal{D}$. A *proper minor* of $\mathcal{C}$ is a minor that is not isomorphic to $\mathcal{C}$.

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A code $\mathcal{D}$ is a *minor* of a code $\mathcal{C}$, denoted by $\mathcal{D} \leq \mathcal{C}$, if there exists a surjective morphism $f : \mathcal{C} \to \mathcal{D}$. A *proper minor* of $\mathcal{C}$ is a minor that is not isomorphic to $\mathcal{C}$.

Example

Let $\mathcal{C} = \{\emptyset, 1, 3, 12\}$ and $\mathcal{D} = \{\emptyset, 0, 1, 2\}$, and let $f : \mathcal{C} \to \mathcal{D}$ given by

$$f(\emptyset) = \emptyset$$
$$f(1) = 1$$
$$f(3) = 2$$
$$f(12) = 1.$$

Note, f is a surjective morphism, yet f fails to be injective. Therefore $\mathcal{D}$ is a proper minor of $\mathcal{C}$.

Minimally Non-Closed Convex

Definition

A neural code $\mathcal{C}$ is *minimally non-closed convex* if $\mathcal{C}$ is not a closed convex code, but every proper minor of $\mathcal{C}$ is closed convex.

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Example/Proposition (Maldonado, Morales, Shaw 2022)

The following code is not minimally non-closed convex.

$$C = \{\underline{2345}, \underline{123}, \underline{124}, \underline{145}, 12, 14, 23, 24, 45, 2, 4, \varnothing, \underline{237}, \underline{238}, \underline{367}, \underline{678}, \underline{26}, 37, 67, 6, 8\}$$

Results

Remark

No code was known to be minimally non-closed, but open convex.

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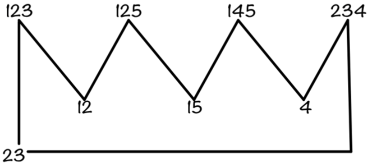
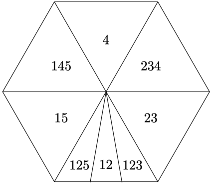
Theorem (Maldonado, Morales, Shaw 2022)

The code

$$C_6 = \{\underline{125}, \underline{234}, \underline{145}, \underline{123}, 4, 23, 15, 12, \varnothing\}$$

is minimally non-closed convex.





$C6$

Acknowledgements

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Questions

Questions?

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Additional Results

Lemma (Maldonado, Morales, Shaw 2022)

Let $f : \mathcal{C} \to \mathcal{D}$ be a surjective morphism and $d \in \mathcal{D}$ be a maximal codeword. Then there exists a maximal codeword $c \in \mathcal{C}$ such that $c \in f^{-1}(\{d\})$.

Lemma (Maldonado, Morales, Shaw 2022)

Let $f : \mathcal{C} \to \mathcal{D}$ be a surjective morphism such that $\mathcal{C}$ and $\mathcal{D}$ have the same amount of maximal codewords. Then for maximal $d \in \mathcal{D}$, $f(c) = d$ for a unique maximal codeword $c \in \mathcal{C}$.

Lemma (Maldonado, Morales, Shaw 2022)

Let $f : C_6 \to D$ be a surjective morphism, where D has four maximal codewords and $C_6 = \{\underline{125}, \underline{234}, \underline{145}, \underline{123}, 4, 23, 15, 12, \varnothing\}$. Then, for non-maximal $\sigma \in \mathcal{C}_6$, $f(\sigma) \in \mathcal{D}$ is non-maximal.