Convexity of 4-Maximal Neural Codes

Analyzing conditions for open convexity

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Contents

Motivation

Biologists have observed neurons in some animal brains called **place cells**, which act as position sensors. They fire at high rates when the animal is inside the cell's preferred region of the environment, called its **place field**.

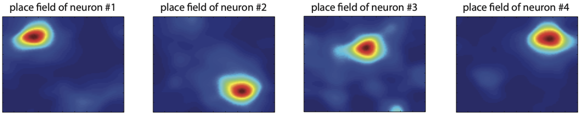

Figure: Place fields of neurons in a rat's hippocampus. Note that these place fields are approximately convex.

Motivation

The intersections of place fields generate a neural code that helps the brain determine an animal's location at a given time. We model these codes to understand their structure. One thing we wish to understand is which of these neural codes can arise from convex place fields – those observed experimentally.

Neural Codes

Definition

A **neural code** (or **code**) $\mathcal{C}$ on n neurons is a collection of **codewords** $\sigma \subseteq [n]$ such that $\mathcal{C} \subseteq 2^{[n]}$. Elements in $\mathcal{C}$ that are maximal with respect to set inclusion are called **maximal codewords**, which we write in bold.

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Example

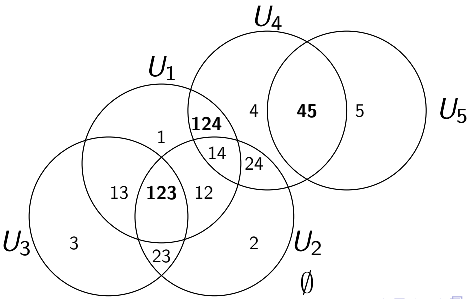
$$\mathcal{C} = \{\mathbf{123}, \mathbf{12}, \mathbf{24}, 45, 1, 2, \emptyset\}.$$

Generating a Neural Code

Example

Consider the family $\mathcal{U} = \{U_1, U_2, U_3, U_4, U_5\}$ of subsets of $\mathbb{R}^2$ shown below. Then by definition,

$$\mathcal{C} = \mathcal{C}(\mathcal{U}) = \{\mathbf{123}, \mathbf{124}, \mathbf{45}, 12, 13, 23, 14, 24, 1, 2, 3, 4, 5, \emptyset\}.$$



Generating a Neural Code

Definition

Let $\mathcal{U}$ be some family of sets $\{U_1, \dots, U_n\}$ with $U_i \subset \mathbb{R}^d$ for some $d \geq 1$. The **code generated by** $\mathcal{U}$ is

$$\mathcal{C}(\mathcal{U}) := \left\{ \sigma \subseteq [n] : \left(\bigcap_{i \in \sigma} U_i \right) \setminus \left(\bigcup_{j \notin \sigma} U_j \right) \neq \emptyset \right\}.$$

In particular, if the family $\mathcal{U}$ is composed by open convex sets, we say that $\mathcal{C} = \mathcal{C}(\mathcal{U})$ is an **(open) convex code**.

Simplicial Complex

Definition

Given a code $\mathcal{C}$ on n neurons, we define a **simplicial complex**, $\Delta(\mathcal{C}) := \{\omega \subseteq [n] : \omega \subseteq \sigma \text{ for some } \sigma \in \mathcal{C}\}$.

Note: $\Delta(\mathcal{C})$ is closed under taking subsets.

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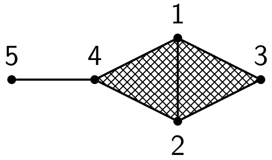

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Given a simplicial complex Δ, its **facets** are the maximal sets with respect to set inclusion in Δ.

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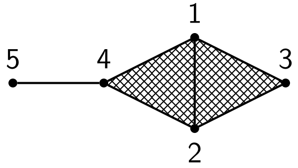
Example

For $\Delta(\mathcal{C}) = \{\mathbf{123}, \mathbf{124}, \mathbf{45}, 12, 13, 14, 23, 24, 1, 2, 3, 4, 5, \emptyset\}$, the set of facets is $\mathcal{F} = \{123, 124, 45\}$.



Nerve Complex

Definition

For a collection of subsets $\mathcal{W} = \{W_1, W_2, \cdots, W_n\}$ of a set X, the **nerve of** $\mathcal{W}$ is the simplicial complex,

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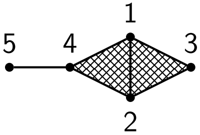
$\Delta(\mathcal{C})$ 

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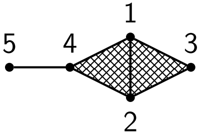

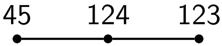

$\mathcal{N}(\mathcal{F})$

Links

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For a $\Delta(\mathcal{C})$ on n neurons the **link** of σ in Δ is

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For $\Delta(\mathcal{C}) = \{\mathbf{123}, \mathbf{124}, \mathbf{45}, 12, 13, 23, 14, 24, 1, 2, \ 3, 4, 5, \emptyset\}$,
$Lk_{\Delta(\mathcal{C})}(3) = \{12, 1, 2\}$.

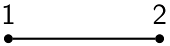

Figure: $\mathrm{Lk}_{\Delta}(3) = \{12, 1, 2\}$

Local Obstructions

Definition

A neural code $\mathcal{C}$ has a **local obstruction at $\boldsymbol{\sigma}$** if there exists a nonempty face $\sigma \in \Delta(\mathcal{C})$ such that:

- σ is an intersection of at least two facets of $\Delta(\mathcal{C})$
- $\sigma \notin \mathcal{C}$
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Recall: A set U of a topological space X is **contractible** if it is homotopy equivalent to a point.

Local Obstructions and Convexity

It is known that convex codes do not have local obstructions [3]. However, for some codes, the converse is also true.

Theorem (Johnston, Shiu, Spinner 2020)

Let $\mathcal{C}$ be a code with at most 3 maximal codewords. Then $\mathcal{C}$ is convex if and only if it has no local obstructions.

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What happens if we consider codes on 4 maximal codewords?

Example

The code $\mathcal{C}^* = \{\mathbf{2345}, \mathbf{123}, \mathbf{134}, \mathbf{145}, 13, 14, 23, 34, 45, 4, 5, \emptyset\}$ is a locally good, non-convex code [5].

Wheels

A **wheel** is a configuration of sets in Euclidean space that intersect in a specific way that forces one of the sets to "bend" around an intersection of the others, and hence be non-convex [6].

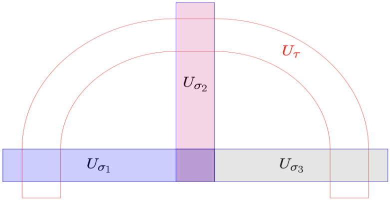

Figure: A conceptual wheel. Note how $U_{\sigma_1}, U_{\sigma_2}$, and U_{σ_3} are convex, but U_τ bends around the 3 sets.

Convex codes do not have wheels [6].

Goals

Recall that convex codes lack local obstructions and wheels ([3], [6]). We investigate the converse for codes with 4 facets.

Conjecture (R. Amzi Jeffs)

Let $\mathcal{C}$ be a code with exactly 4 maximal codewords. Then $\mathcal{C}$ is (open) convex if and only if

- *$\mathcal{C}$ has no local obstructions, and*
- *$\mathcal{C}$ has no wheels.*

Methods

- The nerve lemma allows us to investigate convexity by looking at the nerve of the facets of a code.

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- This splits our conjecture into cases based on the 20 simplicial complexes on 4 vertices, where each vertex represents a facet.
- We only consider the 14 connected complexes, since if a simplicial complex is disconnected we can treat each connected component separately.

Methods

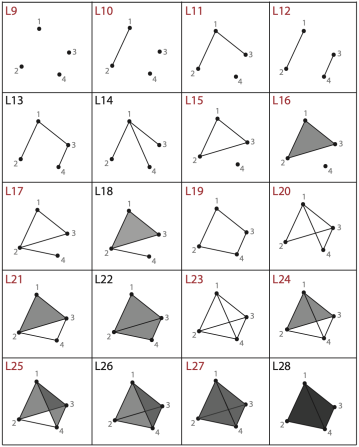

Figure: The simplicial complexes on up to 4 vertices [2].

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A code $\mathcal{C}$ is **max-$\cap$-complete** if it contains every intersection of at least two facets of $\mathcal{C}$.

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Example

The neural code $\mathcal{C}(\mathcal{U}) = \{\mathbf{123}, \mathbf{124}, \mathbf{45}, 12, 13, 23, 14, 24, 1, 2, 3, 4, 5, \emptyset\}$ has the set of facets $\mathcal{F} = \{123, 124, 45\}$. $\mathcal{C}(\mathcal{U})$ is max-$\cap$-complete.

Results

Max-$\cap$-complete codes are locally good (since they are convex) [1]. We proved the converse in the following special case:

Theorem (Flores, Isekenegbe, Perez 2022)

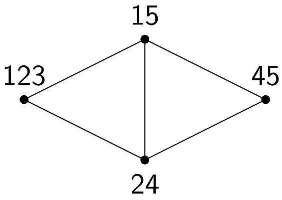
Let $\mathcal{C}$ be a neural code and denote by $\mathcal{F}$ the set of facets of $\mathcal{C}$. If $\mathcal{N}(\mathcal{F})$ contains no 2-simplices, then the following are equivalent:

- *$\mathcal{C}$ has no local obstructions*
- *$\mathcal{C}$ is max-$\cap$-complete*
- *$\mathcal{C}$ is convex*

This proves our conjecture in 6 of the 14 cases!

Applying the Theorem

Consider the code given by $\mathcal{C}_2 = \{\mathbf{123}, \mathbf{45}, \mathbf{24}, \mathbf{15}, \dots\}$ such that $123 = F_1, 15 = F_2, 24 = F_3$, and $45 = F_4$. $\mathcal{N}(\mathcal{F})$ is drawn below. Since it has no filled-in triangles, we conclude by the theorem that it is convex exactly when $1, 2, 5 \in \mathcal{C}_2$.



Results

Theorem (Flores, Isekenegbe, Perez 2022)

Let $\mathcal{C}$ be a 4-maximal code with set of facets $\mathcal{F}$ such that $\mathcal{N}(\mathcal{F})$ is the simplicial complex L18. Then $\mathcal{C}$ is convex if and only if it contains no local obstructions.

Results

Theorem (Flores, Isekenegbe, Perez 2022)

Let $\mathcal{C}$ be a 4-maximal code with set of facets $\mathcal{F}$ such that $\mathcal{N}(\mathcal{F})$ is the simplicial complex L18. Then $\mathcal{C}$ is convex if and only if it contains no local obstructions.

Example

$\mathcal{C}_3 = \{\mathbf{123}, \mathbf{124}, \mathbf{125}, \mathbf{56}, 12, 1, 2, 5, \emptyset\}$ has facets $\mathcal{F} = \{123, 124, 125,\ 56\}$, which have an L18 nerve. Observe that the faces that are intersections of more than one facet are 12 and 5, which are both in $\mathcal{C}$. So $\mathcal{C}_3$ is convex by our theorem.

Next Steps

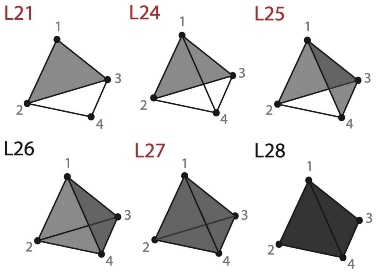
We are looking into properties of morphisms, a concept from [4].

Definition

Let $\mathcal{C} \subseteq 2^{[n]}$ be a code, and let $\gamma \subseteq [n]$. Then the **restriction morphism** defined by γ is $\pi_\gamma : \mathcal{C} \to 2^{[n]}$ given by $\pi_\gamma(c) = c \cap \gamma$. We will use $\mathcal{C}|_\gamma$ to denote $\pi_\gamma(\mathcal{C})$.

Next Steps

- Prove the L21 case using morphisms
- Investigate codes with wheels

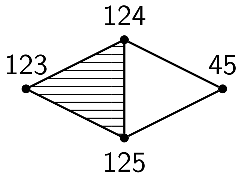


Next Steps

Example

$\mathcal{C}_4 = \{\mathbf{123}, \mathbf{124}, \mathbf{125}, \mathbf{45}, 1, 2, \emptyset\}$ is in the L21 case.

$\Delta(\mathcal{C}_4)$



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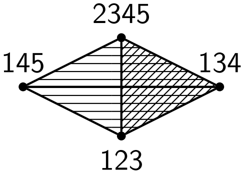
Example

$\mathcal{C}^* = \{\mathbf{2345}, \mathbf{123}, \mathbf{134}, \mathbf{145}, 13, 14, 23, 34, 45, 4, 5, \emptyset\}$ is in the L26 case. Recall that this code has a wheel and is locally good.

$\Delta(\mathcal{C}^*)$



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Acknowledgements

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Appendix: Proof of Proposition

Proof.

It suffices to prove the reverse implication, which we do by proving its contrapositive. Suppose that $\mathcal{C}$ is not max $\cap$-complete. Then there exists some $\sigma \notin \mathcal{C}$ and $F_i, F_j \in \mathcal{F}$ such that $\sigma = F_i \cap F_j$. Since the nerve of the facets of $\mathcal{C}$ contains no 2-simplices, any triple-wise intersection of facets of $\mathcal{C}$ must be empty. In particular, $\sigma \not\subseteq F_k$ for any $k \notin \{i, j\}$. Otherwise, $\emptyset \neq \sigma \subseteq (F_i \cap F_j \cap F_k)$, which would be a contradiction. This implies that $\text{Lk}_\sigma(\Delta(\mathcal{C}))$ is not contractible. By assumption, $\sigma \notin \mathcal{C}$, meaning that $\mathcal{C}$ must have a local obstruction at σ. □