Joyal's cylinder conjecture (arXiv:1911.02631)

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Plan

1. Background to Joyal's conjecture
2. Overview of proof
3. Application to covariant equivalences

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The collage of a profunctor

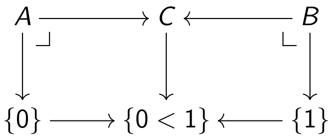
Let A and B be a pair of categories. A **profunctor** from A to B is a functor $M\colon A^{\mathrm{op}} \times B \longrightarrow \mathbf{Set}$. The **collage** of the profunctor M is the category $coll(M)$ with set of objects $\mathrm{ob}\, coll(M) = \mathrm{ob}\, A + \mathrm{ob}\, B$, with hom-sets

$$coll(M)(x,y) = \begin{cases} A(x,y) & \text{if } x, y \in A \\ B(x,y) & \text{if } x, y \in B \\ M(x,y) & \text{if } x \in A,\ y \in B \\ \emptyset & \text{if } x \in B,\ y \in A, \end{cases}$$

and whose identities and composition are defined in the evident way by those of the categories A and B, and by the action of M on morphisms.

There is a unique functor $coll(M) \longrightarrow \{0 < 1\}$ whose fibres above 0 and 1 are A and B respectively.

The collage construction defines an equivalence between the category of profunctors from A to B and the category of categories C over $\{0 < 1\}$ with:



Cylinders

Now, let A and B be a pair of simplicial sets. A **cylinder** (or **correspondence**) from A to B is a simplicial set X over $\Delta[1]$ with:

$$\begin{array}{ccccc} A & \longrightarrow & X & \longleftarrow & B \\ \big\downarrow & \lrcorner & \big\downarrow & \llcorner & \big\downarrow \\ \{0\} & \longrightarrow & \Delta[1] & \longleftarrow & \{1\} \end{array}$$

The category $\mathbf{Cyl}(A, B)$ of cylinders from A to B is defined to be fibre of the functor

$$(\partial_0, \partial_1)\colon \mathbf{sSet}/\Delta[1] \longrightarrow \mathbf{sSet} \times \mathbf{sSet}$$

over the object (A, B).
The initial and terminal objects of $\mathbf{Cyl}(A, B)$ are the disjoint union $A + B$ and the join $A \star B$, equipped with the manifest structure maps. Hence, for each cylinder $X \in \mathbf{Cyl}(A, B)$, there are canonical morphisms $A + B \longrightarrow X$ and $X \longrightarrow A \star B$.

$$(A \star B)_n = A_n + \sum_{i+j=n-1} A_i \times B_j + B_n$$

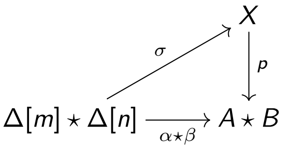
Cylinders as presheaves

For each simplicial set C, let Δ/C denote the **category of simplices** of C. Joyal observed that, for each pair of simplicial sets A and B, the category $\mathbf{Cyl}(A, B)$ is equivalent to the category of presheaves over $\Delta/A \times \Delta/B$.

$$\mathbf{Cyl}(A, B) \simeq [(\Delta/A \times \Delta/B)^{\mathrm{op}}, \mathbf{Set}]$$

Under this equivalence, a cylinder $X \in \mathbf{Cyl}(A, B)$ with structure map $p\colon X \longrightarrow A \star B$ corresponds to the presheaf on $\Delta/A \times \Delta/B$ given by

$$(\alpha \in A_m, \beta \in B_n) \mapsto \{\sigma \in X_{m+1+n} \mid p(\sigma) = \alpha \star \beta\}$$



N.B. $\Delta[m] \star \Delta[n] \cong \Delta[m+1+n]$.
This equivalence yields further useful equivalences, e.g.

$$\mathbf{Cyl}(A, B) \simeq [(\Delta/A)^{\mathrm{op}}, \mathbf{sSet}/B] \qquad \mathbf{Cyl}(A, B) \simeq \mathbf{ssSet}/(A \boxtimes B)$$

Model structures

A **model structure** on a (complete and cocomplete) category $\mathcal{C}$ consists of three classes of morphisms in $\mathcal{C}$ – called **weak equivalences**, **cofibrations**, and **fibrations** – subject to axioms. A model structure on $\mathcal{C}$ enables one to "do homotopy theory" in $\mathcal{C}$. A category equiped with a model structure is called a **model category**.

An object X of a model category is **fibrant** if the unique morphism $X \to 1$ is a fibration.

Lemma

A model structure on a category is determined by either:

1. *its cofibrations and weak equivalences, or*
2. *its cofibrations and fibrant objects.*

The Joyal model structure on **sSet**

A simplicial set X (resp. a morphism of simplicial sets $p\colon X \to Y$) is said to be a **quasi-category** (resp. an **inner fibration**) if it has the RLP wrt to the inner horn inclusions, i.e. for every $n \geq 2$ and $0 < k < n$:

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & X \\ \big\downarrow & \overset{\exists}{\nearrow} & \\ \Delta[n] & & \end{array} \qquad\qquad \begin{array}{ccc} \Lambda^k[n] & \longrightarrow & X \\ \big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow{\scriptstyle p} \\ \Delta[n] & \longrightarrow & Y \end{array}$$

Joyal's model structure for quasi-categories

There exists a unique model structure on the category **sSet** whose cofibrations are the monomorphisms and whose fibrant objects are the quasi-categories.
The weak equivalences in this model structure are called **weak categorical equivalences**.
A morphism $p\colon X \longrightarrow Y$ between quasi-categories is a fibration iff it is an **isofibration**, i.e. an inner fibration with the "isomorphism lifting property": for every object $x \in X$ and isomorphism $g\colon p(x) \to y$ in Y, there exists an isomorphism $f\colon x \to x'$ in X with $p(f) = g$.

Let A and B be a pair of simplicial sets.

The Joyal model structure on $\mathbf{Cyl}(A, B)$

There exists a unique model structure on $\mathbf{Cyl}(A, B)$ created by the forgetful functor $\mathbf{Cyl}(A, B) \longrightarrow \mathbf{sSet}$ from the Joyal model structure for quasi-categories. That is, a morphism in $\mathbf{Cyl}(A, B)$ is a weak equivalence, cofibration, or fibration iff its underlying morphism of simplicial sets is a weak equivalence, cofibration, or fibration respectively in the Joyal model structure on $\mathbf{sSet}$.

N.B. By definition, a cylinder $X \in \mathbf{Cyl}(A, B)$ is a fibrant object in this model structure iff the canonical morphism $X \longrightarrow A \star B$ is a fibration in the model structure for quasi-categories on $\mathbf{sSet}$.
In his notes on quasi-categories, Joyal made the following conjecture about this model structure on $\mathbf{Cyl}(A, B)$:

Joyal's cylinder conjecture

A cylinder $X \in \mathbf{Cyl}(A, B)$ is fibrant if and only if the canonical morphism $X \longrightarrow A \star B$ is an inner fibration, and a morphism between fibrant cylinders in $\mathbf{Cyl}(A, B)$ is a fibration if and only if it is an inner fibration.

The easy case of Joyal's cylinder conjecture

Joyal's cylinder conjecture

A cylinder $X \in \mathbf{Cyl}(A, B)$ is fibrant if and only if the canonical morphism $X \longrightarrow A \star B$ is an inner fibration, and a morphism between fibrant cylinders in $\mathbf{Cyl}(A, B)$ is a fibration if and only if it is an inner fibration.

The special case of Joyal's cylinder conjecture in which A and B are quasi-categories is easy to prove.

Lemma

Suppose A and B are quasi-categories. Then, in the Joyal model structure on $\mathbf{Cyl}(A, B)$, an object X is fibrant if and only if the canonical morphism $X \longrightarrow A \star B$ is an inner fibration, and a morphism between fibrant objects is a fibration if and only if it is an inner fibration.

Proof.

Suppose $p \colon X \longrightarrow A \star B$ is an inner fibration. Since A and B are quasi-categories, so is their join $A \star B$. So it suffices to show that $p \colon X \longrightarrow A \star B$ is an isofibration. But any isomorphism in $A \star B$ belongs either to A or B, and so *uniquely* lifts to an isomorphism in X. □

The general case of Joyal's cylinder conjecture

The easy case of Joyal's cylinder conjecture (in which A and B are quasi-categories) was so easy to prove because we have an easy-to-check explicit description of the fibrations between quasi-categories in the Joyal model structure on **sSet**. Unfortunately no such useful description is known for the fibrations between general simplicial sets.

Remark

For many years, it was an open question whether every monic bijective-on-0-simplices weak categorical equivalence is inner anodyne. Were this so, the general case of Joyal's conjecture would be as easy to prove as the special case in which A and B are quasi-categories. However, I recently proved that this is not so (in my short paper *A counterexample in quasi-category theory*, published in Proc. AMS).

Hence a different argument is required to prove Joyal's conjecture. I will now give an overview of my proof. For details, see my preprint *Joyal's cylinder conjecture* (arXiv:1911.02631).

Plan

Step 1: the ambivariant model structure

I will outline the proof of Joyal's cylinder conjecture in 5 steps:

1. We construct a model structure on $\mathbf{Cyl}(A, B)$, which we call the **ambivariant model structure**, which has the same cofibrations as the Joyal model structure (i.e. the monomorphisms), but whose (fibrations between) fibrant objects are as described in Joyal's cylinder conjecture:

Theorem (the ambivariant model structure on $\mathbf{Cyl}(A, B)$)

There exists a unique model structure on $\mathbf{Cyl}(A, B)$ whose cofibrations are the monomorphisms and whose fibrant objects are those cylinders $X \in \mathbf{Cyl}(A, B)$ for which the canonical morphism $X \longrightarrow A \star B$ is an inner fibration. A morphism between fibrant objects in $\mathbf{Cyl}(A, B)$ is a fibration if and only if it is an inner fibration.

There are many ways to construct this model structure. In the preprint, I (left-and-right) transfer this model structure from the "parametrised Joyal model structure" on $\mathbf{sSet}/(A \star B)$. [For each simplicial set C, the **parametrised Joyal model structure** on $\mathbf{sSet}/C$ has cofibrations the monomorphisms, and fibrant objects the **inner fibrations** with codomain C. See the Appendix of the preprint.]

Steps 2 & 3: model structure observations

2. We observe that Joyal's conjecture is therefore equivalent to the statement that, on the category $\mathbf{Cyl}(A, B)$, the Joyal model structure and the ambivariant model structure coincide. In particular, we know that these two model structures do coincide if A and B are quasi-categories (by the previous Lemma).

Recall that a model structure is determined by (i) its cofibrations and weak equivalences, or alternatively (ii) its cofibrations and fibrant objects. Since the Joyal and ambivariant model structures on $\mathbf{Cyl}(A, B)$ have the same cofibrations (viz. the monomorphisms), we may argue:

3. Since every fibration in the Joyal model structure is in particular an inner fibration, it follows that every ambivariant equivalence in $\mathbf{Cyl}(A, B)$ is a weak categorical equivalence. It remains to prove the converse: that every weak categorical equivalence in $\mathbf{Cyl}(A, B)$ is an ambivariant equivalence.

Step 4: change of base

By step 3, it remains to prove that every weak categorical equivalence in **Cyl**(A, B) is an ambivariant equivalence. We will prove this using the following technical results.

Pushforward of cylinders

$$\begin{array}{ccc} A + B & \xrightarrow{u+v} & A' + B' \\ \downarrow & & \downarrow \\ X & \longrightarrow & (u,v)_!(X) \end{array}$$

4. For each pair of weak categorical equivalences $u \colon A \longrightarrow A'$ and $v \colon B \longrightarrow B'$ in **sSet**, we prove that the pushforward functor

$$(u,v)_! \colon \mathbf{Cyl}(A,B) \longrightarrow \mathbf{Cyl}(A',B')$$

(a) preserves weak categorical equivalences, and
(b) reflects ambivariant equivalences.

The proof of (a) is easy. The proof of (b) is **much harder**!

Step 5: putting it together

It then remains to argue as follows:

5. Let $u\colon A \longrightarrow A'$ and $v\colon B \longrightarrow B'$ be weak categorical equivalences in **sSet** such that A' and B' are quasi-categories. For any morphism f in $\mathbf{Cyl}(A, B)$, f is a weak categorical equivalence
$\implies$ $(u, v)_!(f)$ is a weak categorical equivalence (by step 4(a))
$\implies$ $(u, v)_!(f)$ is an ambivariant equivalence (by step 2, since A' and B' are quasi-categories)
$\implies$ f is an ambivariant equivalence (by step 4(b)).

This completes the proof.

Theorem (Joyal's cylinder conjecture)

Let A and B be a pair of simplicial sets. In the Joyal model structure on $\mathbf{Cyl}(A, B)$*, an object X is fibrant if and only if the canonical morphism $X \longrightarrow A \star B$ is an inner fibration, and a morphism between fibrant objects is a fibration if and only if it is an inner fibration.*

Cylinders as functors I

Combining Joyal's equivalence between $\mathbf{Cyl}(A, B)$ and the category of presheaves over $\Delta/A \times \Delta/B$ with the equivalence $\mathbf{sSet}/B \simeq [(\Delta/B)^{\mathrm{op}}, \mathbf{Set}]$ yields the equivalence

$$\mathbf{Cyl}(A, B) \simeq [(\Delta/A)^{\mathrm{op}}, \mathbf{sSet}/B].$$

Under this equivalence, the ambivariant model structure on $\mathbf{Cyl}(A, B)$ corresponds to some model structure on the functor category $[(\Delta/A)^{\mathrm{op}}, \mathbf{sSet}/B]$.
The following explicit description of this corresponding model structure is key to the proof of step 4(b) (*pushforward of cylinders along a pair of weak categorical equivalences reflects ambivariant equivalences*). First, we recall:

The covariant model structure

The **covariant model structure** on $\mathbf{sSet}/B$ has cofibrations the monomorphisms, weak equivalences the **covariant equivalences**, and fibrant objects the left fibrations with codomain B.

Final vertex inclusions

For each m-simplex α in A, we have the **final vertex inclusion** $i_m \colon (\Delta[0], \alpha_m) \longrightarrow (\Delta[m], \alpha)$ in Δ/A.

Cylinders as functors II

Theorem

Under the equivalence $\mathbf{Cyl}(A, B) \simeq [(\Delta/A)^{\mathrm{op}}, \mathbf{sSet}/B]$*, the ambivariant model structure on* $\mathbf{Cyl}(A, B)$ *corresponds to the Bousfield localisation of the Reedy covariant model structure on* $[(\Delta/A)^{\mathrm{op}}, \mathbf{sSet}/B]$ *w.r.t. the final vertex inclusions in* Δ/A*.*

Proof.

A key ingredient in the proof is Stevenson's result that the class of right anodyne extensions is the smallest weakly saturated class of monos in **sSet** satisfying the right cancellation property and which contains the final vertex inclusions $i_m \colon \Delta[0] \longrightarrow \Delta[m]$. □

We prove step 4(b) – pushforward $(u, v)_! \colon \mathbf{Cyl}(A, B) \longrightarrow \mathbf{Cyl}(A', B')$ reflects ambivariant equivalences – using the above theorem together with the following theorem of Joyal. See the preprint for the detailed argument.

Theorem (Joyal)

Let $v \colon B \longrightarrow B'$ *be a weak categorical equivalence. Then the pushforward functor* $v_! \colon \mathbf{sSet}/B \longrightarrow \mathbf{sSet}/B'$ *preserves and reflects covariant equivalences.*

The Main Corollary to Joyal's cylinder conjecture

To prove Joyal's cylinder conjecture, we proved that, on the category $\mathbf{Cyl}(A, B)$, the Joyal model structure and the ambivariant model structure coincide.
Therefore, everything we have proved about the ambivariant model structure on $\mathbf{Cyl}(A, B)$ is true also of the Joyal model structure on $\mathbf{Cyl}(A, B)$.
In particular, the previous theorem becomes:

Theorem (the Main Corollary to Joyal's cylinder conjecture)

Under the equivalence $\mathbf{Cyl}(A, B) \simeq [(\Delta/A)^{\mathrm{op}}, \mathbf{sSet}/B]$, the Joyal model structure on $\mathbf{Cyl}(A, B)$ corresponds to the Bousfield localisation of the Reedy covariant model structure on $[(\Delta/A)^{\mathrm{op}}, \mathbf{sSet}/B]$ w.r.t. the final vertex maps in Δ/A.

It seems to me that this Main Corollary is more useful than Joyal's cylinder conjecture!

Plan

Lurie's characterisation of covariant equivalences

Let B be a simplicial set. In HTT Chapter 2, Lurie proves the following characterisation of the covariant equivalences in $\mathbf{sSet}/B$.

The **left cone functor** $C^{\triangleleft}\colon \mathbf{sSet}/B \longrightarrow \mathbf{sSet}$ sends an object $p\colon X \longrightarrow B$ of $\mathbf{sSet}/B$ to the simplicial set $C^{\triangleleft}(p)$ defined by the pushout

$$\begin{array}{ccc} X & \xrightarrow{\quad p \quad} & B \\ \big\downarrow & & \big\downarrow \\ \Delta[0] \star X & \longrightarrow & C^{\triangleleft}(p) \end{array}$$

Theorem (Lurie)

A morphism in $\mathbf{sSet}/B$ is a covariant equivalence iff it is sent by the left cone functor $C^{\triangleleft}\colon \mathbf{sSet}/B \longrightarrow \mathbf{sSet}$ to a weak categorical equivalence.

This characterisation is proved by Lurie as a consequence of the straightening theorem. Using (our Main Corollary to) Joyal's cylinder conjecture, we can give an easier proof of Lurie's characterisation, which avoids the use of the straightening theorem.

An easy proof of Lurie's characterisation

N.B. The left cone functor $C^{\triangleleft}$ factors as $\mathbf{sSet}/B \longrightarrow \mathbf{Cyl}(\Delta[0], B) \longrightarrow \mathbf{sSet}$.

Theorem

The left cone functor $C^{\triangleleft} \colon \mathbf{sSet}/B \longrightarrow \mathbf{Cyl}(\Delta[0], B)$ *is a left Quillen equivalence between the Joyal model structure on* $\mathbf{Cyl}(\Delta[0], B)$ *and the covariant model structure on* $\mathbf{sSet}/B$*.*

Proof.

Under the equivalence $\mathbf{Cyl}(\Delta[0], B) \simeq [\Delta^{\mathrm{op}}, \mathbf{sSet}/B]$, the Joyal model structure on $\mathbf{Cyl}(\Delta[0], B)$ corresponds to the "canonical model structure" on $[\Delta^{\mathrm{op}}, \mathbf{sSet}/B]$ w.r.t. the covariant model structure on $\mathbf{sSet}/B$. (This is the $A = \Delta[0]$ case of the Main Corollary.) Moreover, the left cone functor corresponds to the "constant" functor $\mathbf{sSet}/B \longrightarrow [\Delta^{\mathrm{op}}, \mathbf{sSet}/B]$, which is a left Quillen equivalence (Rezk–Schwede–Shipley). □

Theorem (Lurie)

A morphism in $\mathbf{sSet}/B$ *is a covariant equivalence iff it is sent by the left cone functor* $C^{\triangleleft} \colon \mathbf{sSet}/B \longrightarrow \mathbf{sSet}$ *to a weak categorical equivalence.*

Thank you!