MSRI online seminar
– Spring 2020 –
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Algebras in Group-Theoretical Fusion Categories

arXiv: 2001.03837

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Goal: To construct Morita equivalence class representatives of indecomposable, semisimple algebras in GTFCs

Outline

I. Why care about this goal?

II. How we achieve the goal.

III. What next...

Feel free to ask questions
I can't see the chat
I might stop occasionally to ask questions
because talking to a laptop gets boring

I. Why care? Replacing rings with reps —

__Defn__ We say two rings R & S are __Morita equivalent__ if $\exists$ equivalence: Mod-R $\sim$ Mod-S

__Thm__ [Morita, late 50's] equivalent to

R-Mod $\sim$ S-Mod

$S \cong e\,\mathrm{Mat}_n(R)\,e$ for some $n \geq 1$, $e \in \mathrm{Mat}_n(R)$ full idempotent

$S \cong \mathrm{End}(P_R)$ for some $P_R \in$ Mod-R progenerator

$\exists$ bimodules ${}_SP_R$ & ${}_RQ_S$ so that $P \otimes_R Q \cong S$ as S-bimod & $Q \otimes_S P \cong R$ as R-bimod

Loads of properties are __Morita invariant__ [property that holds for both R & S if $R \underset{\text{Morita}}{\sim} S$]

left/right Noetherian

(semi)hereditary

(semi)prime

(semi)primitive

__EX.__ R is Morita equivalent to $\mathrm{Mat}_n(R)$

Morita equivalence appears in many fields

Analysis

Thm [Brown-Green-Rieffel, 1977]
Two (seperable or unital) C^*-algebras R and S are strongly Morita equivalent

[$\exists$ (R,S)-bimodule X so that
X is a $\begin{cases} \text{left } R\text{-Hilbert mod} \\ \text{right } S\text{-Hilbert mod} \end{cases}$
with additional conditions]

$\Updownarrow$

R and S are stably equivalent

[$R \otimes K \simeq S \otimes K$, for K some algebra of compact operators on a separable Hilbert space]

Geometry

Thm [Xu, 1990-1992]
Take P a regular Poisson manifold with symplectic fibration $\pi: P \to Q$

Then P is Morita equiv to $(Q, \{,\}_{zero})$

[$\exists$ symplectic manifold X "equiv. bimod" with complete Poisson morphisms $X \longrightarrow P$ & $X \longrightarrow (Q, -\{,\})$ satisfying certain conditions]

$\Updownarrow$

all symplectic leaves of P are connected & simply connected & the fundamental class vanishes.

Intersection of Algebra & Physics

specifically in $\underbrace{\text{rational conformal field theory}}$ (RCFTs)

Classical field thy | special relativity | quantum mechanics

— invariant under conformal transformations —

— dimensions in $\mathbb{Q}$ —

algebraic structures are used to understand RCFTs:

$\underbrace{\text{modular tensor categories}}$

Ex. Rep (vertex operator alg)

- monoidal categ $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$
- with dual object $X^*, {}^*X$
- with a braiding $X \otimes Y \longrightarrow Y \otimes X$
- semisimple $X = \bigoplus$ simple objects
- ⋮

- An <u>algebra in $\mathcal{C}$</u> is a triple $(A, m: A \otimes A \longrightarrow A, u: \mathbb{1} \longrightarrow A)$ satisfying associativity & unitality constraints

- Say two algs A, B in $\mathcal{C}$ are <u>Morita equivalent</u> if $\mathcal{C}_A \sim \mathcal{C}_B$ as $\mathcal{C}$-mod. categs [$\exists$ equiv of categ $\mathcal{C}_A \longrightarrow \mathcal{C}_B$ compatible with $\mathcal{C}$-action]

- A <u>right A-module in $\mathcal{C}$</u> is a pair $(M, \rho_M: M \otimes A \longrightarrow M)$ satisfying associativity & unitality constraints

Denote the collection by $\boxed{\mathcal{C}_A}$

Studying "boundary conditions" of RCFTs boils down to studying nice algebras in MTCs up to Morita equivalence [Fuchs, Runkel, Schweigert...]

- In general, it's easier to work with explicit algebras (that represent Morita equivalence classes) than the module categories themselves

  Determining this can be tough.

- Convenient to have nice Morita equiv. class representatives of algebras (A, m, u) in monoidal categories $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$

Property-wise

A is $\left\{\begin{array}{l}\underline{\text{indecomposable}}\\ \underline{\text{semisimple}}\end{array}\right.$ algebra in $\mathcal{C}$ $\overset{def}{\Longleftrightarrow}$ $\mathcal{C}_A$ is an $\left\{\begin{array}{l}\text{indecomposable}\\ \text{semisimple}\end{array}\right.$ $\mathcal{C}$-module category

Structure-wise

$(A, m: A\otimes A \to A, u: \mathbb{1} \to A)$ algebra in $\mathcal{C}$ is Frobenius
$\overset{def}{\Longleftrightarrow}$ $\exists$ morphisms $\Delta: A \to A\otimes A$ & $\varepsilon: A \to \mathbb{1}$ in $\mathcal{C}$ so that

(A, Δ, ε) is a coassociative, counital coalgebra

and

$$\begin{tikzcd}
 & A\otimes A \arrow[dl, "\Delta\otimes id"'] \arrow[dr, "id\otimes\Delta"] \arrow[d, "m"] & \\
(A\otimes A)\otimes A \arrow[dd, "\alpha"'] & A \arrow[l, phantom, "\circlearrowleft"] \arrow[r, phantom, "\circlearrowleft"] \arrow[dd, "\Delta"] & A\otimes(A\otimes A) \arrow[dd, "\alpha^{-1}"] \\
 & & \\
A\otimes(A\otimes A) \arrow[dr, "id\otimes m"'] & & (A\otimes A)\otimes A \arrow[dl, "m\otimes id"] \\
 & A\otimes A &
\end{tikzcd}$$

[special if $m\Delta = id_A$ & $\varepsilon u = \lambda\, id_{\mathbb{1}}$ for $\lambda \in \Bbbk^{\times}$

Recall goal: To construct Morita equiv. class representatives of indecomp., semisimple algs in GTFCs
... they'll be Frob. too!

Before moving on to part II, need to discuss GTFCs.

A group-theoretical fusion category is a fusion category (monoidal, ss, $\Bbbk$-linear..) of the form

$$\mathcal{C}(\underbrace{G}_{\text{finite group}}, \underbrace{\omega}_{\in H^3(G,\Bbbk^\times)}, \underbrace{K}_{\text{subgroup of } G}, \underbrace{\beta}_{\in C^2(K,\Bbbk^\times) \text{ so that } d\beta = \omega})$$

$$= \left(\underbrace{(\Bbbk K)_\beta}_{\text{algebra in } \mathrm{Vec}_G^\omega} \text{-bimodules in } \underbrace{\mathrm{Vec}_G^\omega}_{\text{category of } G\text{-graded}},\ \otimes_{(\Bbbk K)_\beta},\ (\Bbbk K)_\beta,\ \alpha,\ l,\ r \right)$$

α, l, r induced from Vec_G^ω

Important for testing conjectures about fusion categories

category of G-graded findim'l $\Bbbk$-vs with simple objects $\{\delta_g\}_{g\in G}$ (1-dim'l $\Bbbk$-vs in deg $g \in G$)
with associativity

$$(\delta_g \otimes \delta_{g'}) \otimes \delta_{g''} \xrightarrow[\omega(g,g',g'') \cdot \mathrm{id}_{\delta_g \delta_{g'} \delta_{g''}}]{\sim} \delta_g \otimes (\delta_{g'} \otimes \delta_{g''})$$

algebra in Vec_G^ω = $\bigoplus_{g\in K} \delta_g$ as $\Bbbk$-vs with multiplication
"twisted group alg."

$$\delta_g \otimes \delta_{g'} \longrightarrow \beta(g,g')\, \delta_{gg'}$$

<u>Examples of GTFCs</u>

- $\mathcal{C}(G,\omega,\langle e\rangle,1) \sim \mathrm{Vec}_G^\omega$
- $\mathcal{C}(G,1,\langle e\rangle,1) \sim \mathrm{Vec}_G$
- $\mathcal{C}(G,1,G,1) \sim \mathrm{Rep}(G)$
- $\mathcal{C}(G,1,K,1) \sim \mathrm{Rep}(\Bbbk^N \# \Bbbk K)$ for $N \trianglelefteq G$ & $G = KN$
- $\mathrm{Rep}(\Bbbk^N \#_\sigma^\tau \Bbbk K)$ is GT. (bicrossed product)

II. How goal is achieved (for algs in $\mathcal{C}(G,\omega,K,\beta)$, up to Morita equiv) ($= (\Bbbk K)_\beta$-bimod in Vec_G^ω)

Important Special case —

Theorem [Ostrik 2003]

Every Morita equiv. class of indecomposable, semisimple algebras in Vec_G^ω is represented by a twisted group algebra $(\Bbbk L)_\psi$ for some $L \leq G$, $\psi \in C^2(L, \Bbbk^\times)$ with $d\psi = \omega$

To upgrade to GTFCs —

Theorem [MMPRTW] Take A a special Frobenius in a $\otimes$ category $\mathcal{V}$. We construct a (Frobenius) monoidal structure on the 'free' functor

$$\Phi : \mathcal{V} \longrightarrow {}_A\mathcal{V}_A := A\text{-bimodules in } \mathcal{V}$$
$$X \longmapsto (A \otimes X) \otimes A$$

$\rightsquigarrow$ This sends (Frobenius) algebras in $\mathcal{V}$ to (Frobenius) algebras in ${}_A\mathcal{V}_A$.

Proposition [MMPRTW] $(\Bbbk L)_\psi$ has the structure of a special Frobenius algebra in Vec_G^ω

Application $\mathcal{V} = \mathrm{Vec}_G^\omega$, $A = (\Bbbk K)_\beta$

Get Frobenius monoidal functor $\Phi : \mathrm{Vec}_G^\omega \longrightarrow \mathcal{C}(G,\omega,K,\beta)$

Using the Frobenius $\otimes$ functor $\Phi: \mathrm{Vec}_G^\omega \longrightarrow \mathcal{C}(G,\omega,K,\beta)$
$$X \longmapsto \big((\Bbbk K)_\beta \otimes X\big) \otimes (\Bbbk K)_\beta$$

& Ostrik's result that
$\{(\Bbbk L)_\psi\}_{L \le G,\ \psi \in C^2(L,\Bbbk^\times),\ d\psi=\omega}$ represent Morita equiv. classes of algs in Vec_G^ω

$\rightsquigarrow$ Consider the Frobenius algebra in $\mathcal{C}(G,\omega,K,\beta)$:
$$\Phi((\Bbbk L)_\psi) =: A^{K,\beta}(L,\psi)$$
we call this a <u>twisted Hecke algebra</u>.

<u>Theorem</u> [MMPRTW]

① Every Morita equiv. class of indecomposable, semisimple algebras in $\mathcal{C}(G,\omega,K,\beta) =: \mathcal{C}$ is represented by a twisted Hecke algebra $A^{K,\beta}(L,\psi)$ for some $L \le G$, $\psi \in C^2(L,\Bbbk^\times)$ with $d\psi = \omega$.

② $\mathcal{C}_{A^{K,\beta}(L,\psi)} \sim \mathcal{C}_{A^{K,\beta}(L',\psi')}$ as $\mathcal{C}$-module categories
$$\Updownarrow$$
$(\mathrm{Vec}_G^\omega)_{(\Bbbk L)_\psi} \sim (\mathrm{Vec}_G^\omega)_{(\Bbbk L')_{\psi'}}$ as Vec_G^ω-module categories
$$\Updownarrow$$
<u>Theorem</u> [Natale, 2017]

$\exists$ elt $x \in G$ so that $L = xL'x^{-1}$
& the class of a certain 2-cocycle involving ψ, ψ' is trivial in $H^2(L', \Bbbk^\times)$

Examples:

① $\Phi : \mathrm{Vec}_G \longrightarrow {}_{\Bbbk G}(\mathrm{Vec}_G)_{\Bbbk G} = \mathcal{C}(G,1,G,1) \sim \mathrm{Rep}(G)$

$$\{\Bbbk L\}_{L \le G} \longmapsto \{(\Bbbk G \otimes \Bbbk L) \otimes \Bbbk G\}$$

represent morita equiv. classes of algs in Vec_G^ω

represent morita equiv. classes of algs in $\mathcal{C}(G,1,G,1)$

Natale (2002) has explicit $\otimes$ functors for this equivalence, so we can send alg here

② Same for $\Phi : \mathrm{Vec}_G \longrightarrow \mathcal{C}(G,1,K,1) \sim \mathrm{Rep}(\Bbbk^N \# \Bbbk K)$
for $N \trianglelefteq G$ w/ $G = KN$

Proof relied heavily on —

<u>Theorem</u> [MMPRTW] Take $\otimes$ categories $\mathcal{S}$ & $\mathcal{T}$, along with $\otimes$ functor

$$T : \mathcal{S} \longrightarrow \mathcal{T}$$

that preserves epimorphisms & so that $T_{S,S'} : T(S) \otimes_{\mathcal{T}} T(S') \longrightarrow T(S \otimes_{\mathcal{S}} S')$ is epi $\forall S, S' \in \mathcal{S}$

If S, S' are morita equiv. algs in $\mathcal{S}$, then $T(S), T(S')$ are Morita equiv algs in $\mathcal{T}$

- Call T <u>morita preserving</u> in this case

$\rightsquigarrow$ <u>Theorem</u> [MMPRTW] Take $\otimes$ categ $\mathcal{V}$ & A a special Frob. alg in $\mathcal{V}$
Then B, B' are Morita equiv. algs in $\mathcal{V}$ $\Leftrightarrow$ $\Phi(B), \Phi(B')$ are Mor equiv. in ${}_A\mathcal{V}_A$

Pf/ ($\Rightarrow$) showed Φ is morita preserving

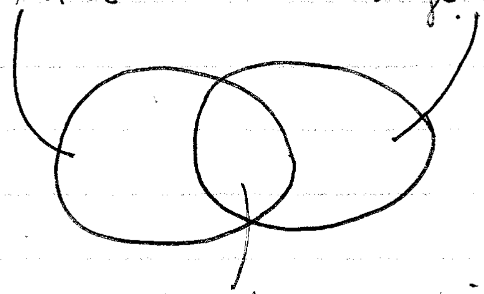
($\Leftarrow$) took <u>loads</u> of work, involved other Morita preserving functors.

III. What now?

Folks care about Morita equivalence in general
- in algebra, it's nice to have <u>explicit</u> Morita equivalence class representatives

① Physical application for algebras in MTCs

Goal achieved for algebras in GTFCs



Could study intersection
(at least the braided GTFCs)

② Many results in the paper about Morita equivalence of algs in $\otimes$ categories are of independent interest.

algebra S →
algebra S' → $\boxed{\text{New Tool}}$ → Morita equiv or not?

New Applications?

Interested in your thoughts about any of the above

Thanks for listening!