Structures in Hochschild cohomology

Pablo S. Ocal

April 22, 2020

Abstract

The Hochschild cohomology is a tool for studying associative algebras that has a lot of structure: it is a Gerstenhaber algebra. However, computing this structure is difficult. We will give a mild introduction to this cohomology, we will justify its importance by computing some of the lower degrees, and we will emphasize the limitations of restricting ourselves to one resolution. We will conclude by presenting some of the recent developments that advance the understanding of its structure, and by mentioning applications to quantum symmetries. This is joint work with Tekin Karadag, Dustin McPhate, Tolulope Oke, and Sarah Witherspoon.

Contents

1 Motivation **1**

2 Hochschild cohomology **1**

3 Twisted tensor product of algebras **3**

4 Applications and future work **5**

1 Motivation

Before jumping into the definitions and the abstract concepts, let me tell you about the result motivating some of this work: In 2014 Le and Zhou proved that $\mathrm{HH}^*(A \otimes B) \cong \mathrm{HH}^*(A) \otimes \mathrm{HH}^*(B)$ under some finiteness conditions. It concerns *Hochschild cohomology* and the usual tensor product of k-algebras, which is commutative. There are generalizations of this tensor product to the non-commutative case, like the *twisted tensor product*, and it would be nice to have a similar behavior over it, something like "$\mathrm{HH}^*(A \otimes_\tau B) \cong \mathrm{HH}^*(A) \otimes_\tau \mathrm{HH}^*(B)$". Unfortunately, that symbol-by-symbol translation cannot possibly be correct: the left hand side is a graded commutative algebra, while the right hand side is (in general) a non-commutative algebra.

What is the correct translation of Le and Zhou's result to the non-commutative case? What techniques does it involve? Here we will attempt to address some of these questions.

2 Hochschild cohomology

Definition 1. *Let A be a k-algebra (our algebras are unital and associative, over a field k. We can also work over a ring k, but things will change substantially. We define the Hochschild* *cohomology as):* $\mathrm{HH}^n(A) = \mathrm{Ext}^n_{A^e}(A, A)$ *where* $A^e = A \otimes A^{op}$ *(is called the enveloping algebra of* A*).*

This is a rather clean definition, where the fact that k is a field is heavily used. However when doing computations, defining operations, and proving properties about the Hochschild cohomology, it is useful and common to pick a projective resolution. A common one (and historically relevant) is the *bar resolution.*

Definition 2. *For any* $n \in \mathbb{N}$*, consider* $A^{\otimes(n+2)}$ *as an* A^e*-module (by multiplication on the outermost factors) and the sequence:*

$$\cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \xrightarrow{\mu_A} A$$

with

$$d_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_{n+1}$$

for all $a_0, \ldots, a_{n+1} \in A$*. This is (called) the (augmented) bar resolution of* A*.*

When working with a special algebra or family of algebras, they frequently have a special resolution that are more convenient to use. For example if A is Koszul we can use the Koszul resolution, or if A is a complete intersection we can use finite free resolutions.

Definition 3. $\mathrm{HH}^*(A)$ *comes with two operations (natively defined on the cochains of the bar resolution):*

$$\smile : \mathrm{HH}^m(A) \times \mathrm{HH}^n(A) \longrightarrow \mathrm{HH}^{m+n}(A),$$
$$[-,-] : \mathrm{HH}^m(A) \times \mathrm{HH}^n(A) \longrightarrow \mathrm{HH}^{m+n-1}(A).$$

We call $\smile$ the *cup product* and $[-,-]$ the *Gerstenhaber bracket.* The cup product gives $(\mathrm{HH}^*(A), \smile)$ the structure of a graded commutative algebra. The Gerstenhaber bracket gives $(\mathrm{HH}^*(A), [-,-])$ the structure of a graded Lie algebra. Together with some compatibility conditions, they give $(\mathrm{HH}^*(A), \smile, [-,-])$ the structure of a Gerstenhaber algebra, also known as Poisson 2-algebra, or a Poisson algebra with Poisson bracket of degree -1.

To fix ideas, this last structure can be thought of as a graded Lie algebra coming from an associative algebra. There are some key differences, since to be precise the degree of $\mathrm{HH}^*(A)$ with respect to $[-,-]$ (also called the Lie degree) is one less than the degree of $\mathrm{HH}^*(A)$ with respect to $\smile$ (also called the homological degree).

Example 4. *To see what information does Hochschild cohomology encode, we can manually compute its lower degrees.*

- *Degree 0: we have* $\mathrm{HH}^0(A) = \ker(d_1^*)$*, so pick any* $\alpha \in \hom_{A^e}(A \otimes A, A)$ *also in* $\ker(d_1^*)$*, we have for all* $a \in A$*:*

  $$0 = d_1^*(\alpha)(1 \otimes a \otimes 1) = \alpha(d_1(1 \otimes \alpha \otimes 1)) = \alpha(a \otimes 1 - 1 \otimes a) = a\alpha(1 \otimes 1) - \alpha(1 \otimes 1)a$$

  Notice that since α *is determined by its value* $\alpha(1 \otimes 1) \in A$*, this is enough. Conversely, any element of the algebra* A *defines a function in* $\hom_{A^e}(A \otimes A, A)$*, and additionally for all* $z \in Z(A)$ *and* $a, b \in A$ *setting* $\alpha_z(a \otimes b) = azc$ *gives a function* $\alpha_z \in \hom_{A^e}(A \otimes A, A)$ *also in* $\ker(d_1^*)$*. Thus as* k*-modules we have that* $\mathrm{HH}^0(A) \cong Z(A)$*.*

- *Degree 1: we have* $\mathrm{HH}^1(A) = \ker(d_2^*)/\operatorname{im}(d_1^*)$. *Doing an analogous analysis to the above, we obtain that as* k*-modules* $\ker(d_2^*) \cong \mathrm{Der}(A, A)$ *the space of* k*-derivations from* A *to* A, $\operatorname{im}(d_1^*) \cong \mathrm{InnDer}(A, A)$ *the space of inner* k*-derivations from* A *to* A. *Hence as a* k*-module* $\mathrm{HH}^1(A) \cong \mathrm{OutDer}(A, A)$ *the space of outer* k*-derivations from* A *to* A.

- *Degree 2: we have* $\mathrm{HH}^2(A) = \ker(d_3^*)/\operatorname{im}(d_2^*)$. *Doing a similar analysis as done above, we obtain that as* k*-modules* $\ker(d_3^*)$ *are the infinitesimal deformations of* A*, and* $\operatorname{im}(d_2^*)$ *are the infinitesimal deformations of* A *that give an algebra isomorphic to the original* A. *Hence we can think of* $\mathrm{HH}^2(A)$ *as encoding the "important" infinitesimal deformations. I will not define what an infinitesimal deformation is, but the name is quite suggestive.*

This hints at $\mathrm{HH}^*(A)$ encoding algebraically some form of infinitesimal information of A. Notice how the above computations rely heavily on inside knowledge of the resolution being used. We can now revisit Le and Zhou's result.

Theorem 5 (Le-Zhou 2014)**.** *Let* A *and* B *be* k*-algebras, at least one of them finite dimensional. Then (as Gerstenhaber algebras):*

$$\mathrm{HH}^*(A \otimes B) \cong \mathrm{HH}^*(A) \otimes \mathrm{HH}^*(B).$$

This was proven by working over the bar resolution and using the cumbersome Alexander-Whitney and Eilenberg-Zilber maps.

3 Twisted tensor product of algebras

Definition 6. *Let* A *and* B *be* k*-algebras, a twisting map* $\tau : B \otimes A \to A \otimes B$ *is a bijective* k*-linear map (with the conditions* $\tau(1_B \otimes a) = a \otimes 1_B$, $\tau(b \otimes 1_A) = 1_A \otimes b$ *for all* $a \in A$, $b \in B$, *and:*

$$\tau \circ (m_B \otimes m_A) = (m_A \otimes m_B) \circ (1 \otimes \tau \otimes 1) \circ (\tau \otimes \tau) \circ (1 \otimes \tau \otimes 1)$$

meaning that twisting and multiplication "commute". Equivalently

$$\begin{array}{ccccc}
B \otimes B \otimes A \otimes A & \xrightarrow{1\otimes\tau\otimes 1} & B \otimes A \otimes B \otimes A & \xrightarrow{\tau\otimes\tau} & A \otimes B \otimes A \otimes B \\
\downarrow{\scriptstyle m_B \otimes m_A} & & & & \downarrow{\scriptstyle 1\otimes\tau\otimes 1} \\
B \otimes A & \xrightarrow{\quad\tau\quad} & A \otimes B & \xleftarrow{m_A \otimes m_B} & A \otimes A \otimes B \otimes B
\end{array}$$

is a commutative diagram). The twisted tensor algebra $A \otimes_\tau B$ *is* $A \otimes B$ *(as a vector space) with (as it turns out associative) multiplication:*

$$m_{A\otimes_\tau B} : A \otimes B \otimes A \otimes B \xrightarrow{1\otimes\tau\otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_A\otimes m_B} A \otimes B.$$

Of course, the idea of working with $\mathrm{HH}^*(A \otimes_\tau B)$ is not to treat $A \otimes_\tau B$ as a given algebra, but to use some information about A and B that is already known. For example, if we have tailored resolutions to specific algebras. Teorem 7 allows computing the Gerstenhaber bracket in the Hochschild cohomology of a twisted tensor product $A \otimes_\tau B$, a notoriously difficult task, as long as we know the Gerstenhaber bracket in the respective Hochschild cohomologies of A and B. This has applications in, for example, deformations of algebras.

Theorem 7 (KMOOW)**.** *Let $P \to A$, $Q \to B$ (projective bimodule resolutions of A, B respectively) such that $P \otimes_\tau Q \to A \otimes_\tau B$ is nice (a counital differential graded coalgebra) and $\sigma : (P \otimes_\tau Q) \otimes_{A \otimes_\tau B} (P \otimes_\tau Q) \to (P \otimes_A P) \otimes_\tau (Q \otimes_B Q)$ is (also) nice (a chain map isomorphism satisfying some technical conditions). Then the (Gerstenhaber) bracket (is given) explicitly.*

The tools used to prove Theorem 7 are constructions of resolutions given by Shepler and Witherspoon, and homotopy lifting techniques by Negron and Witherspoon which essentially remain a black box. Volkov generalized this black box to a tool that can always be applied to any resolution, that does not require inside knowledge of the specifics of the resolution, and that induces the Gerstenhaber bracket in cohomology! This is absolutely fantastic.

Theorem 8 (Shepler-Witherspoon 2019)**.** *Under some compatibility conditions, given $P \to A$, $Q \to B$ (projective bimodule resolutions of A, B respectively, and a twisting map $\tau : B \otimes A \to A \otimes B$), we can construct $P \otimes_\tau Q \to A \otimes_\tau B$ (a projective bimodule resolution of $A \otimes_\tau B$).*

Definition 9 (Volkov 2016)**.** *(Given A a k-algebra,) let $\mu_P : P \to A$ be a resolution of A-bimodules, $\Delta_P : P \to P \otimes_A P$ a diagonal map, and $\alpha \in \hom_{A^e}(P_m, A)$ a cocycle. A homotopy lifting (of α with respect to Δ_P) is (an A-bimodule chain homomorphism) $\psi_\alpha : P \to P[1-m]$ satisfying (some very) technical conditions (depending only on the augmentation map μ_P, the diagonal map Δ_P, and the cocycle α:*

$$d(\psi_\alpha) = (\alpha \otimes 1_P - 1_P \otimes \alpha)\Delta_P, \quad \text{and} \quad \mu_P \psi_\alpha \text{ is cohomologous to } (-1)^{m-1}\alpha\psi$$

for some A-bimodule chain map $\psi : P \to P[1]$ for which $d(\psi) = (\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P$).

Theorem 10 (Volkov 2016)**.** *The bracket (given at the chain level by):*

$$[\alpha, \beta] = \alpha\psi_\beta - (-1)^{(m-1)(n-1)}\beta\psi_\alpha$$

induces the Gerstenhaber bracket (on Hochschild cohomology).

Sometimes, we can do more than compute the Gerstenhaber bracket.

Example 11. *Let A, B be k-algebras graded by the commutative groups F, G respectively, let $t : F \otimes_{\mathbb{Z}} G \to k^\times$ be a bicharacter. Then $\tau(b \otimes a) = t(|a|, |b|) a \otimes b$ is a twisting map, we denote $A \otimes^t B = A \otimes_\tau B$. It can be checked that $\mathrm{HH}^*(-)$ is bigraded: $\mathrm{HH}^{*,*}(-)$. We denote:*

$$F' = \bigcap_{g \in G} \ker(t(-, g)), \quad G' = \bigcap_{f \in F} \ker(t(f, -)).$$

Theorem 12 (Grimley-Nguyen-Witherspoon 2017, OOW)**.** *As Gerstenhaber algebras (in the twisted tensor product setup, and assuming the necessary finiteness conditions, we have):*

$$\mathrm{HH}^{*,F' \oplus G'}(A \otimes^t B) \cong \mathrm{HH}^{*,F'}(A) \otimes \mathrm{HH}^{*,G'}(B).$$

Proof. (The original proof used extended versions of the Alexander-Whitney and Eilenberg-Zilber maps. We completely avoided them by using) Volkov's homotopy lifting (techniques, as well as a chain isomorphism similar to the aforementioned σ, and a bit of work with the Koszul sign convention). □

4 Applications and future work

We proved that when τ is strongly graded, and both A and B are Koszul algebras, then we can construct a σ satisfying the compatibility conditions of Theorem 8 with respect to the Koszul resolutions of A and B. In turn, this then helps to prove that such σ for Koszul algebras satisfies the conditions of Theorem 7.

The Jordan plane: $k\langle x, y\rangle/(yx - xy - x^2)$ can be seen as $k[x] \otimes_\tau k[y]$ for $\tau(y \otimes x) = x \otimes y + x^2 \otimes 1$. However, this τ is not strongly graded. We can nonetheless manually find a similar σ satisfying the conditions of Theorem 7, and find the explicit Gerstenhaber algebra structure. This poses the question of what conditions over τ are "good enough" to guarantee the compatibility conditions of Theorem 8, that is, how can we relax the strongly graded condition for τ.

In [GNW] they computed the Gerstenhaber algebra structure of the quantum complete intersections $\Lambda_q = k\langle x, y\rangle/(x^2, y^2, xy + qyx)$ for $q \in k^\times$ using the techniques they developed to prove Theorem 12. They found that in many cases $\mathrm{HH}^1(\Lambda_q)$ is a finite dimensional abelian Lie algebra over which $\mathrm{HH}^*(\Lambda_q)$ is a module (the generators being common eigenvectors). One of the exceptions was $q = 1$ and $\mathrm{char}(k) \neq 2$, where $\mathrm{HH}^1(\Lambda_1)$ is isomorphic to the Lie algebra $\mathfrak{gl}_2(k)$. It still acts on $\mathrm{HH}^*(\Lambda_1)$, but in a more complicated way.

The complete Gerstenhaber algebra structure of the Jordan plane was first computed by Lopes and Solotar, using spectral sequences and a lot of machinery. Our computations in [KMOOW] used more elementary and completely different methods, enabled by Volkov's homotopy lifting techniques. These seem to be the way of tackling this type of problems, and they should enable elementary computations of examples like Λ_q.

Thank you for your time!

References

[1] L. Grimley, V. C. Nguyen, and S. Witherspoon, *Gerstenhaber brackets on Hochschild cohomology of twisted tensor products*, J. Noncommutative Geometry 11 (2017), no. 4, 1351–1379.

[2] T. Karadag, D. McPhate, P. S. Ocal, T. Oke, and S. Witherspoon, *Gerstenhaber brackets on Hochschild cohomology of general twisted tensor products*, arXiv:1909.02181.

[3] J. Le and G. Zhou, *On the Hochschild cohomology ring of tensor products of algebras*, J. Pure Appl. Algebra 218 (2014), 1463–1477.

[4] S. A. Lopes and A. Solotar, *Lie structure on the Hochschild cohomology of a family of subalgebras of the Weyl algebra*, arXiv:1903.01226.

[5] C. Negron and S. Witherspoon, *An alternate approach to the Lie bracket on Hochschild cohomology*, Homology, Homotopy and Applications 18 (2016), no. 1, 265-285.

[6] A.V. Shepler and S. Witherspoon, *Resolutions for twisted tensor products*, Pacific J. Math 298 (2019), no. 2, 445–469.

[7] S. Witherspoon, *Hochschild Cohomology for Algebras*, Graduate Studies in Mathematics 204, American Mathematical Society, 2019.

[8] Y. Volkov, *Gerstenhaber bracket on the Hochschild cohomology via an arbitrary resolution*, arXiv:1610.05741.