Structures in Hochschild cohomology

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1. Motivation.

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Le and Zhou proved: $\boxed{HH^*(A \otimes B) \cong HH^*(A) \otimes HH^*(B)}$

usual Hochschild cohomology

usual tensor product of algebras: $A \otimes B \cong B \otimes A$.

1. Motivation:

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There is a non-commutative tensor product of algebras: $\otimes_\tau$.

We would like to have: $\boxed{HH^*(A \otimes_\tau B) \cong HH^*(A) \otimes_\tau HH^*(B).}$

⚠ unfortunately, <u>not</u> true.

What is the correct translation?

2. Hochschild cohomology

④

<u>Definition:</u> Let A be a k-algebra: $A^e := A \otimes A^{op}$

$$\boxed{\begin{aligned} HH^n(A) &:= \mathrm{Ext}^n_{A^e}(A,A) \\ HH^*(A) &:= \bigoplus_{n\in\mathbb{N}} \mathrm{Ext}^n_{A^e}(A,A). \end{aligned}}$$

The working mathematician needs: $\left\{\begin{array}{l}\text{a resolution.} \\ \text{operations.}\end{array}\right.$
to compute it.

2. Hochschild cohomology: bar resolution

⑤

For every $n \in \mathbb{N}$, $A^{\otimes(n+2)} = A \otimes A^{\otimes n} \otimes A$ is an A^e-module.

The bar resolution of A is:

$$\boxed{\cdots \longrightarrow A^{\otimes(n+2)} \xrightarrow{d_n} \cdots \longrightarrow A\otimes A\otimes A \xrightarrow{d_1} A\otimes A \xrightarrow{m_A} A}$$

$$d_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

2. Hochschild cohomology: operations

⑥

<u>Cup product:</u> $\smile : HH^m(A) \times HH^n(A) \longrightarrow HH^{m+n}(A).$

<u>Gerstenhaber bracket:</u> $[-,-] : HH^m(A) \times HH^n(A) \longrightarrow HH^{m+n-1}(A).$

Natively defined on the bar resolution.

2. Hochschild cohomology: structures.

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$$HH^*(A) \longrightarrow \begin{cases} \cup & : \text{ graded commutative algebra.} \\ [-,-] & : \text{ graded Lie algebra.} \\ \cup, [-,-] & : \text{ Gerstenhaber algebra.} \end{cases}$$

2. Hochschild cohomology: degree 0.

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$HH^0(A) = \operatorname{Ker}(d_1^*)$, pick:

$$\alpha \in \operatorname{Ker}(d_1^*) \subseteq \operatorname{Hom}_{A^e}(A \otimes A, A)$$

Then for all $a \in A$.

$$0 = d_1^*(\alpha)(1 \otimes a \otimes 1) = a \cdot \alpha(1 \otimes 1) - \alpha(1 \otimes 1) a$$

Any $z \in Z(A)$ defines $\alpha_z \in \operatorname{Ker}(d_1^*)$:

$$\alpha_z(a \otimes b) = azb,$$

for all $a, b \in A$.

Hence: $\boxed{HH^0(A) \cong Z(A).}$

2. Hochschild cohomology: degree 1.

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Pick $\alpha \in \operatorname{Ker}(d_2^*)$, then for all $a, b \in A$:

$$0 = d_2^*(\alpha)(1 \otimes a \otimes b \otimes 1) = a\,\alpha(1 \otimes b \otimes 1)$$
$$- \alpha(1 \otimes ab \otimes 1) + \alpha(1 \otimes a \otimes 1) b.$$

So: $\operatorname{Ker}(d_2^*) = \operatorname{Der}(A, A)$.

Pick $\alpha \in \operatorname{Im}(d_1^*)$, so $\alpha = d_1^*(\beta)$:

$$\alpha(1 \otimes a \otimes 1) = d_1^*(\beta)(1 \otimes a \otimes 1) =$$
$$= a \cdot \beta(1 \otimes 1) - \beta(1 \otimes 1) \cdot a$$

for all $a \in A$.

So: $\operatorname{Im}(d_1^*) = \operatorname{InnDer}(A, A)$.

Hence: $\boxed{HH^1(A) \cong \operatorname{OutDer}(A, A).}$

2. Hochschild cohomology: degree 2.

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$\mathrm{Ker}(d_3^*)$: infinitesimal deformations of A.

$\mathrm{Im}(d_2^*)$: infinitesimal deformations giving an algebra isomorphic to A.

So $HH^2(A)$ encodes the "important" deformations.

Theorem: [Le-Zhou 2014] Let A, B be k-algebras, at least one finite dimensional. Then: (11)

$$HH^*(A \otimes B) \cong HH^*(A) \otimes HH^*(B) \quad \text{as Gerstenhaber algebras.}$$

Proof: Cumbersome. $\square$.

3. Twisted tensor product.

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Definition: Let A, B be k-algebras, $\tau: B \otimes A \longrightarrow A \otimes B$ a bijective k-linear map such that: τ "commutes" with m_A and m_B.

The algebra $A \otimes_\tau B$ is $A \otimes B$ with multiplication:

$$m_{A \otimes_\tau B}: A \otimes B \otimes A \otimes B \xrightarrow{1 \otimes \tau \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B.$$

$$\begin{tikzcd}
B \otimes B \otimes A \otimes A \arrow[r, "1 \otimes \tau \otimes 1"] \arrow[d, "m_B \otimes m_A"'] & B \otimes A \otimes B \otimes A \arrow[r, "\tau \otimes \tau"] & A \otimes B \otimes A \otimes B \arrow[d, "1 \otimes \tau \otimes 1"] \\
B \otimes A \arrow[r, "\tau"] & A \otimes B & A \otimes A \otimes B \otimes B \arrow[l, "m_A \otimes m_B"']
\end{tikzcd}$$

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Theorem: [KMOOW] Let $P \to A$, $Q \to B$ be bimodule resolutions with:

$P \otimes_\tau Q \longrightarrow A \otimes_\tau B$ nice, and

$\sigma : (P \otimes_\tau Q) \otimes_{A \otimes_\tau B} (P \otimes_\tau Q) \longrightarrow (P \otimes_A P) \otimes_\tau (Q \otimes_B Q)$ well behaved.

Then we give the Gerstenhaber bracket explicitly.

Tools used:
- → Constructing $P \otimes_\tau Q$. [Shepler-Witherspoon].
- → Black box giving the Gerstenhaber bracket. [Negron-Witherspoon].

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Theorem: [Shepler-Witherspoon] Given $P \to A$, $Q \to B$ bimodule resolutions, under some compatibility conditions, we can construct:

$P \otimes_\tau Q \longrightarrow A \otimes_\tau B$ a bimodule resolution. It is a projective resolution whenever $P \to A$, $Q \to B$ are projective resolutions.

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[Negron-Witherspoon] $\xrightarrow{\text{giver}}$ Explicit Gerstenhaber bracket.

user

Koszul-like conditions on the resolutions.

Volkov

Interpretation of the conditions to the cochains.

user

generalizes

[Volkov] $\xrightarrow{\text{giver}}$ Explicit Gerstenhaber bracket.

3. Twisted tensor product: example.

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Let A, B be k-algebras graded by F, G commutative groups.

Let $t : F \otimes_{\mathbb{Z}} G \longrightarrow k^\times$ be a bicharacter. Then:

$$\begin{aligned} \tau : B \otimes A &\longrightarrow A \otimes B \\ b \otimes a &\longmapsto t(|a|,|b|)\, a \otimes b \end{aligned} \qquad \text{induces } A \otimes_\tau B =: \boxed{A \otimes^t B}.$$

Here $HH^*(-)$ is bigraded: $HH^{*,*}(-)$.

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Theorem: [Grimley-Nguyen-Witherspoon, OOW] We have: (under some finiteness conditions).

$$\boxed{HH^{*,F' \oplus G'}(A \otimes^t B) \cong HH^{*,F'}(A) \otimes HH^{*,G'}(B)}$$ as Gerstenhaber algebras.

Where: $F' := \bigcap_{g \in G} \ker(t(-,g))$, $G' := \bigcap_{f \in F} \ker(t(f,-))$.

Proof: Original: cumbersome.

New: avoids that using Volkov's homotopy lifting. $\square$.

4. Applications and future work.

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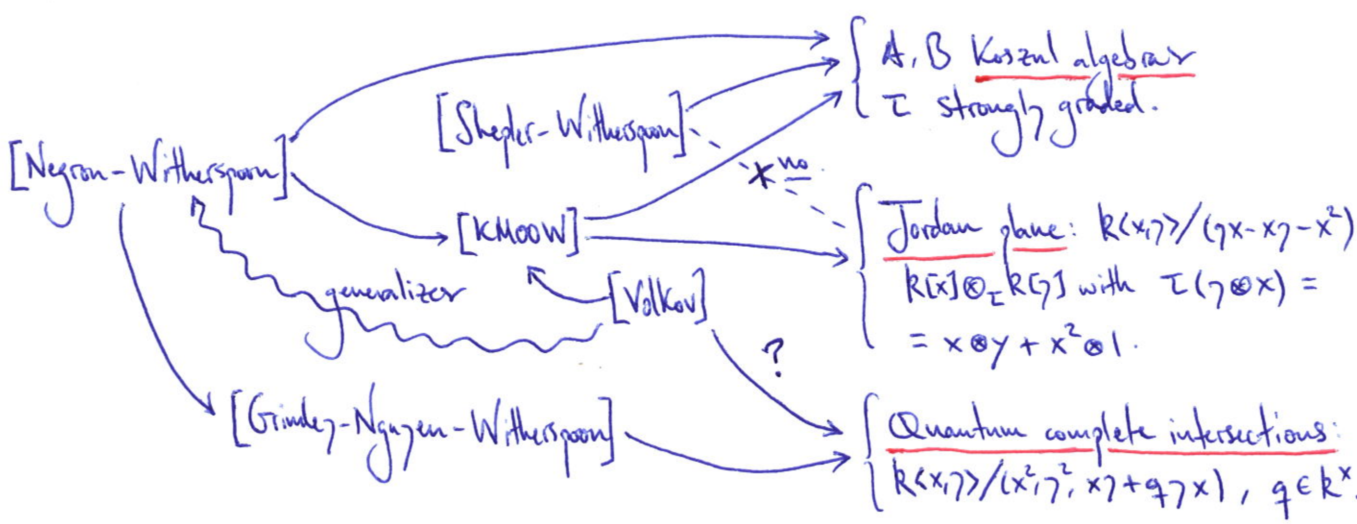

4. Applications and future work.

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[KMOOW] and [OOW] are elementary methods.

In particular, Le and Zhou's result is proven in (almost) exactly the same way as Grimley, Nguyen, and Witherspoon's result.

This is enabled by Volkov's homotopy lifting techniques.

Thank you!

References: (partial list)

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