a segal model for modular operads

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based on
1906.01143 a graphical category for higher modular operads
1906.01144 modular operads and the nerve theorem


fringes of msri

Outline:

- main theorems
- graphs, embeddings
- the graph category $\mathcal{U}$ + additional structure
- presheaves and the Segal condition

Motivation: Boavida - Horel - Robertson
profinite completion of GT

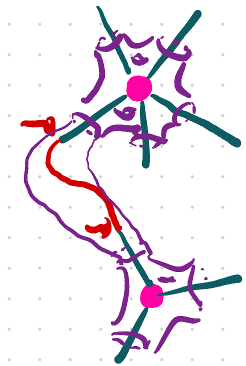
What is a modular operad?

$M(0), M(1), M(2), \ldots$

cyclic operad $\quad$ jk maps

$$M(j)\times M(k) \longrightarrow M(j+k-2)$$

pick



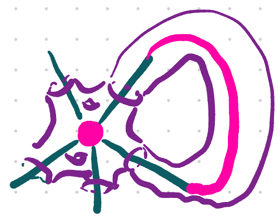
$\in M(5)$

$\in M(3)$

$\rightsquigarrow M(6)$

$+\ j(j-1)$ **contractions** $\quad M(j) \to M(j-2)$

or $\binom{j}{2}$?



$+$ symm. gp actions, units, colorings

Theorem The inclusion of $\mathcal{U}$ into modular operads (= compact symmetric multicategories) induces

$$\mathrm{MO} \xrightarrow[N]{} [\mathcal{U}^{op}, \mathrm{Set}]$$

which is fully-faithful and whose essential image consists of Segal presheaves.

Corollary (Joyal-Kock '11 ; HRY '19 ; Raynor '19)

The full subcategory inclusion $\overline{\mathcal{U}} \subseteq \mathrm{MO}$ induces

$$\mathrm{MO} \xrightarrow[N]{} [\overline{\mathcal{U}}^{op}, \mathrm{Set}]$$

which is fully-faithful and whose essential image consists of Segal presheaves.

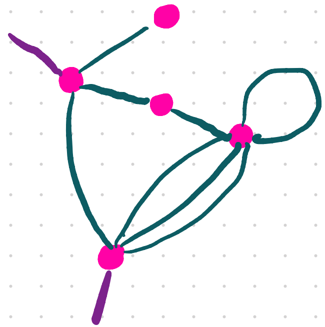
Theorem The category of simplicial $\mathcal{U}$-presheaves, $[\mathcal{U}^{op}, sSet]$ admits a model structure so that X is fibrant just when

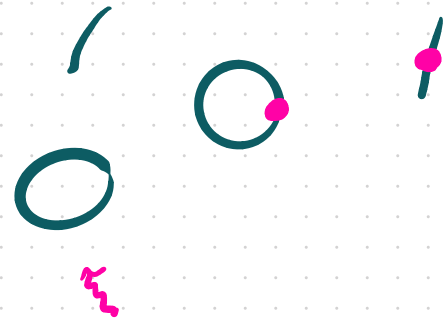
- X is Reedy fibrant
- $X(e) \simeq *$
- X is Segal

$X(\emptyset) = X(\bullet)$

$X(1)$
$\parallel$
set of colors

Graphs





(X, V)

X: space

V: finite discrete subspace

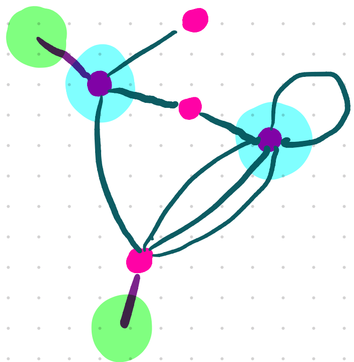
$X - V$ is a 1-mfld with finitely many components.

$E = \pi_0(X - V)$

Information from a graph

set of edges with orientation

$$A = \partial E \not\longrightarrow V$$



$$i \circlearrowright A \xleftarrow{s} D \xrightarrow{t} V$$

$t^{-1}(v) =: nb(v)$

$$\partial = A - sD$$

$$\mathcal{G} = \{ G\bullet \longleftarrow \bullet \longrightarrow \bullet \}$$

graphs $\subsetneq$ Ob $[\mathcal{G}, \text{FinSet}]$

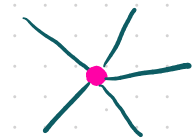
i is fixed-point free

s is monomorphism.

Examples

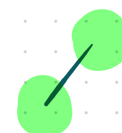
- Z a finite set

$$\star_Z : \quad G\; \underset{\substack{\| \\ \{z,z'\}}}{2Z} \hookleftarrow Z \longrightarrow 1$$

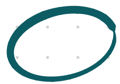


$Z=5$

- e : $\quad G\; 2 \hookleftarrow \emptyset \longrightarrow \emptyset$



Non-example:



should have

$G\partial \longleftrightarrow \emptyset \longrightarrow \emptyset$

but that was already e

Example: $m \geq 1$

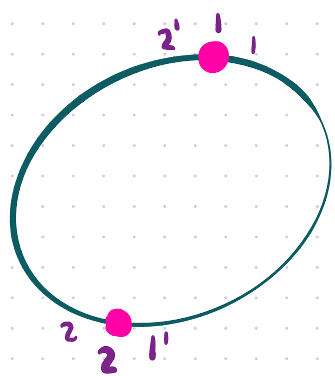
$$\partial m \longleftarrow \partial m \xrightarrow{t} m$$

$$k \longmapsto k$$

$$k' \longmapsto k+1$$

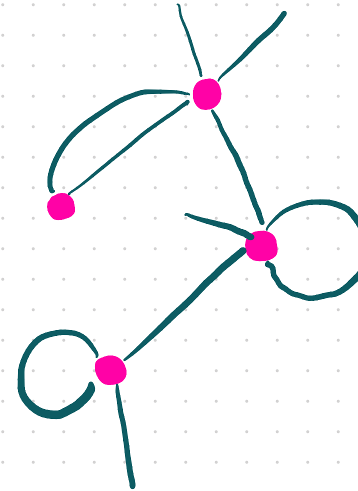
$$m' \longmapsto 1$$

loop with m-vertices



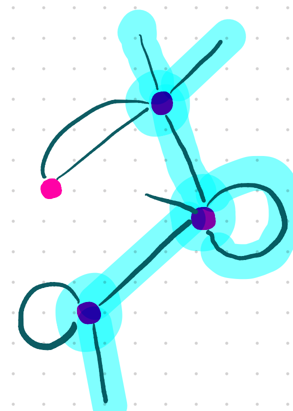
embedding of connected graphs

$$\begin{tikzcd} G \arrow[d] \\ G' \end{tikzcd} \qquad \begin{tikzcd} \circlearrowleft A \arrow[d] & D \arrow[l, hook'] \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & V \arrow[d, hook] \\ \circlearrowleft A' & D' \arrow[l, hook'] \arrow[r] & V' \end{tikzcd}$$



Write $\mathrm{Emb}(G')$ for embeddings with codomain G'

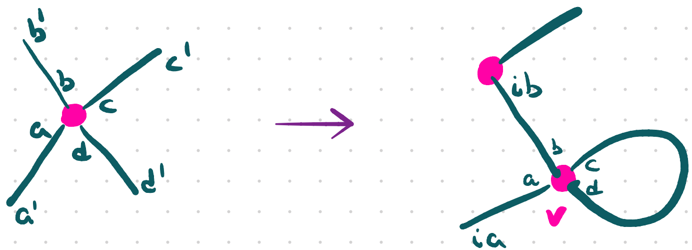
$\mathcal{U}_{emb}$ for the category of embeddings.



Warning: embeddings are not always monomorphisms.

Example G graph, $v \in V$ $\quad \bigstar_v := \bigstar_{nb(v)}$ $\quad \bigstar_v \hookrightarrow G$

étale



$V \hookrightarrow Emb(G)$

Example $\{a, ia\} \in E$ $\quad e \cong \left(\{a, ia\} \hookleftarrow \emptyset \to \emptyset\right) \longrightarrow G$ $\quad E \hookrightarrow Emb(G)$

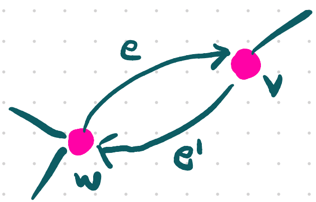
edge = i-orbit

<u>Observation</u> G connected with V nonempty

We can present G as a coequalizer in $[\mathcal{G}, \mathrm{Set}]$

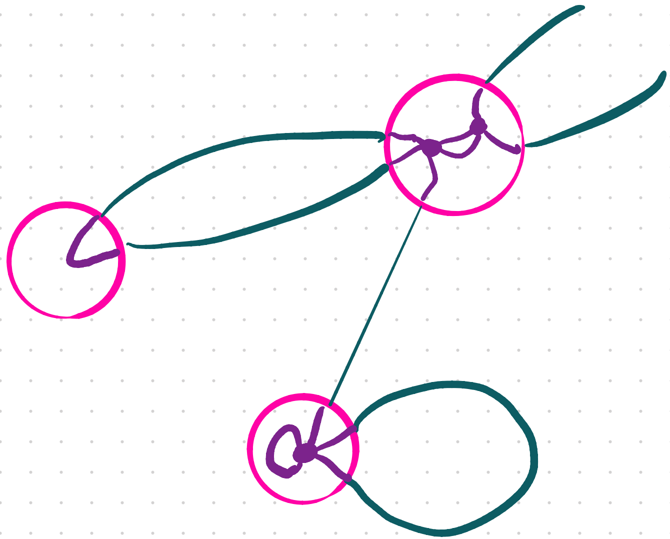
$$\coprod_{\substack{\text{internal}\\\text{edges}}} e \rightrightarrows \coprod_{V} \star_v \longrightarrow G$$

by choosing an orientation for each internal edge.



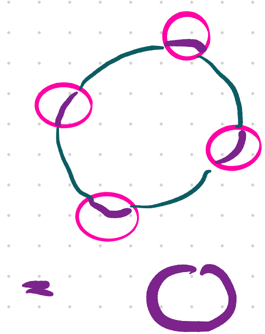
$$\begin{tikzcd}
e \arrow[r] \arrow[dr] & \star_v & \\
e' \arrow[ur] \arrow[r] & \star_w & \arrow[from=1-2, to=2-3, phantom, "\longrightarrow"] G
\end{tikzcd}$$

Graph Substitution



erase pink circles
$\rightsquigarrow$ new graph

⚠ Not all subs are valid in our formalism



Observation G connected with V nonempty

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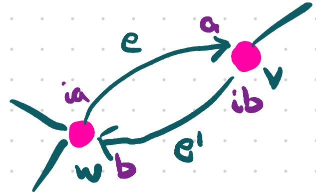
Graph Substitution

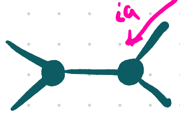
$$\begin{tikzcd}
 & A(G) & A(G) \arrow[l, "i"'] \\
\partial(H_v) \arrow[r, "\cong"] & i(nb(v)) \arrow[u, hook] & nb(v) \arrow[l, "\cong"'] \arrow[u, hook] \\
 & \partial(\star_v) \arrow[u, phantom, "\cong" rotate=90] &
\end{tikzcd}$$

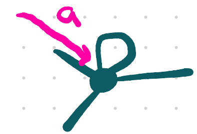
$$\coprod_{\substack{\text{internal}\\ \text{edges}}} e \rightrightarrows \coprod_{V} H_v \longrightarrow G\{H_v\}$$

Works except in the bad case from previous page



$$
\begin{tikzcd}
e \arrow[r] \arrow[dr] & \star_v \\
e' \arrow[ur] \arrow[r] & \star_w
\end{tikzcd}
\longrightarrow G
$$

$H_w =$ 

$H_v =$ 

$$
\begin{tikzcd}
e \arrow[r] \arrow[dr] & H_v \\
e' \arrow[ur] \arrow[r] & H_w
\end{tikzcd}
$$

Graphical Map $\varphi : G \longrightarrow G'$

Data:
- map of involutive sets $A \xrightarrow{\varphi_0} A'$
- for each $v \in V$, $(\varphi_v : H_v \hookrightarrow G') \in \mathrm{Emb}(G')$

(embedding: $V \to V' \subseteq \mathrm{Emb}(G')$)

Axioms:

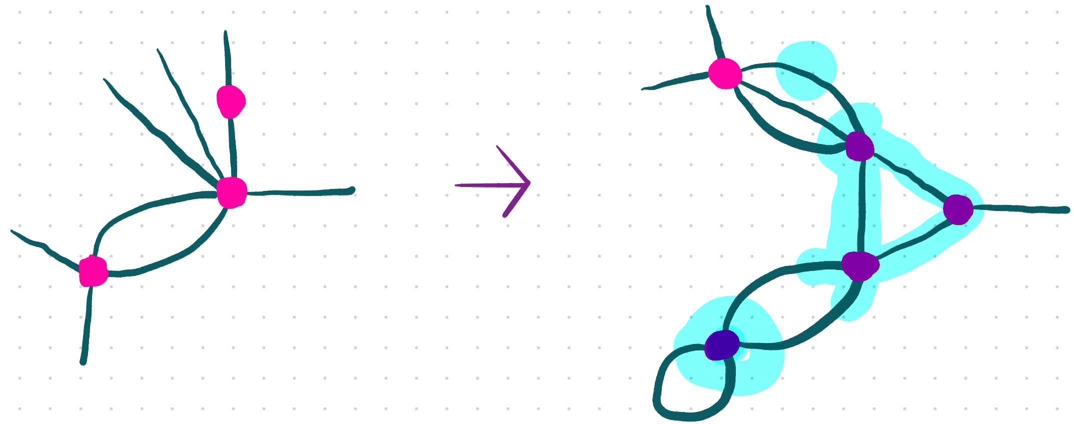
(i) each $w \in V'$ appears in at most one H_v

generalizes $V \hookrightarrow V'$

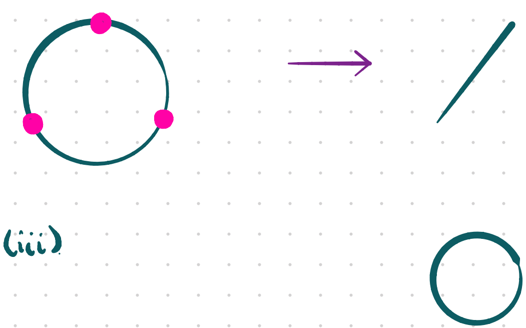
(ii)
$$\begin{tikzcd}
nb(v) \arrow[r, "\cong"] & i(nb(v)) \arrow[r] \arrow[d, "\exists\ \cong"'] & A \arrow[d, "\varphi_0"] \\
 & \partial(H_v) \arrow[r, hook] & A'
\end{tikzcd}$$

(iii) if $\partial(G) = \emptyset$, then some $H_v \neq e$

Example:

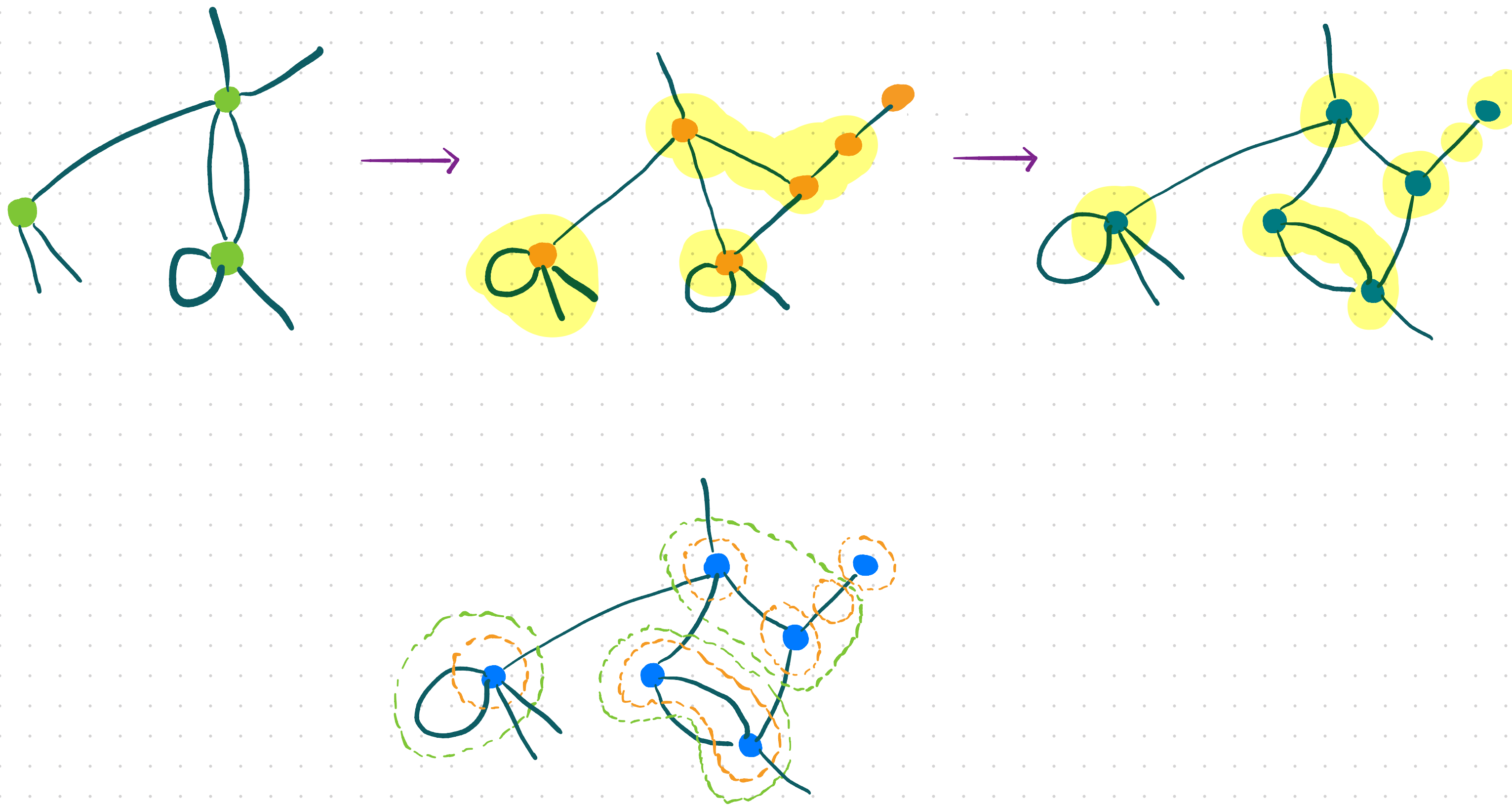


Non-example:



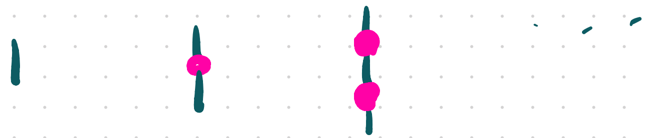
prohibited by (iii)

Composition via graph substitution



$\Delta \hookrightarrow \mathcal{U}$ faithful.

[0] [1] [2]



each comes with an orientation

maps in Δ are the ones that preserve this.

Maps in $\mathcal{U}$ between linear graphs are order preserving <u>or</u> reversing functions $[k] \to [\ell]$ and they know which one they are.

For instance, $\mathrm{Aut}(1) = C_2$

Result: Category $\mathcal{U}$ of undirected connected graphs

Interesting subcategories (containing all isomorphisms)

- $\mathcal{U}_{emb}$ ~ inert
- $\mathcal{U}_{act}$ = active/ boundary preserving maps $\partial(G) \xrightarrow{\cong} \partial(G')$
- $\mathcal{U}^- \subseteq \mathcal{U}_{act}$ $\varphi_0 : A \longrightarrow A'$ surjective
- $\mathcal{U}^+_{act} \subseteq \mathcal{U}_{act}$ $\varphi_0 : A \longrightarrow A'$ injective
- $\mathcal{U}^+$ generated by $\mathcal{U}_{emb}$ and $\mathcal{U}^+_{act}$

φ_0 need not be injective.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[dr] & G' \\ & G\{H_v\} \arrow[u] \end{tikzcd}$$

$\varphi_v : H_v \to G'$

Proposition 1 $(\mathcal{U}_{act}, \mathcal{U}_{emb})$

is an orthogonal factorization system on $\mathcal{U}$

Consequence: Apply 'algebraic patterns' machinery of Chu–Haugseng '19 elementaries = $\star$'s or $\star$'s & edges.

Proposition 2 Define $\deg(G) = \#V + \#$ internal edges.

The data $(\mathcal{U}^-, \mathcal{U}^+, \deg : Ob(\mathcal{U}) \to \mathbb{N})$

gives $\mathcal{U}$ the structure of a (dualizable)

generalized Reedy category.

Consequence: get a model structure on $[\mathcal{U}^{op}, \mathcal{M}]$

nice enough

(Berger–Moerdijk)

Recall:

Observation G connected with V nonempty

We can present G as a coequalizer in $[\mathcal{G}, \mathrm{Set}]$

$$\coprod_{\substack{\text{internal}\\ \text{edges}}} e \rightrightarrows \coprod_{V} \bigstar_v \longrightarrow G$$

Definition (Segal Core) G connected with V nonempty

Coequalizer in $[\mathcal{U}^{op}, \mathrm{Set}]$

$$\coprod_{\substack{\text{internal}\\ \text{edges}}} U[e] \rightrightarrows \coprod_{V} U[\bigstar_v] \longrightarrow Sc[G]$$

$$Sc[G] \longrightarrow \underset{\substack{\| \\ \hom(-,G)}}{U[G]}$$ is Segal core inclusion at G

Definition $X \in [\mathcal{U}^{op}, Set]$ is Segal when

$$X(G) = \hom(U[G], X) \longrightarrow \hom(Sc[G], X) = \text{equalizer}$$

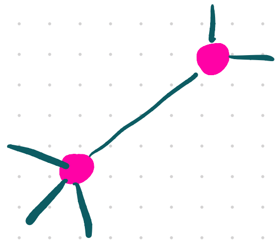
is a bijection for each $G \neq e$

Special Case $X(e) = *$

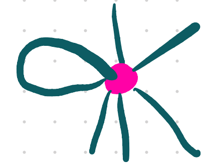
$$\begin{tikzcd} X(G) \arrow[r, "\cong"] \arrow[d] & \prod_v X(\star_v) \\ X(\star_{\#\partial G}) & \end{tikzcd}$$

$$M(n) = X(\star_n)$$

Graphs with one internal edge



or



$$\begin{tikzcd} X(G) \arrow[r, "\cong"] \arrow[d] & X(\bigstar_4) \times X(\bigstar_3) \arrow[dl] \\ X(\bigstar_5) & \end{tikzcd} \qquad \begin{tikzcd} X(G) \arrow[r, "\cong"] \arrow[d] & X(\bigstar_7) \arrow[dl, "\text{contraction}"] \\ X(\bigstar_5) & \end{tikzcd}$$

Associativity $\rightsquigarrow$ come from graphs with <u>two</u> internal edges.

Simplicial Sets

$X \in [\mathcal{U}^{op}, sSet]$ is Segal when

$$X(G) = \mathrm{map}(U[G], X) \longrightarrow \mathrm{map}(Sc[G], X)$$

← still a coequalizer

is a weak equivalence for each $G \neq e$

---

when $X(e) = *$:

$$\begin{tikzcd} X(G) \arrow[r, "\simeq"] \arrow[d] & \prod X(\star_v) \arrow[dl, dashed] \\ X(\star_{\#26}) & \end{tikzcd}$$

Proof sketch of second theorem

Use the Reedy model structure on $[\mathcal{U}^{\infty}, sSet]$

Left Bousfield localization at $Sc[G] \to U[G]$

and $\emptyset \to U[e]$

and interpret what "local object" means.

Other Questions

- take topological picture more seriously.

  is there some topological category $\mathcal{C}$ with $\pi_0(\mathcal{C}) = \mathcal{U}$?

  Getzler - Operads revisited?

- make this work over other ground categories $\mathcal{V}$

  (Gepner - Haugseng)

  (Chu - Haugseng)

- Ben Ward's approach to ∞-modular operads.

  (See Getzler–Kapranov 1998)

- $C_*(\overline{\mathcal{M}}_{*,*})$

Directed version:

book with Marcy & Donald. - wheeled proprads

Merkulov - wheeled props