Torsion and homotopical invariants from TFTs

Mark Penney
joint w/ Sergei Gukov and Jeff Meier

MSRI Higher Categories
Perimeter Insitute
UWaterloo

7 May, 2020

(π_*-) finite homotopy type

A space X is of finite homotopy type if:

- $|\pi_n(X, x)| < \infty$ for $n \geq 0$ and $x \in X$; and,
- $\pi_n(X, x) = 0$ for $n >> 0$.

(π_*-) finite homotopy type

A space X is of finite homotopy type if:

- $|\pi_n(X, x)| < \infty$ for $n \geq 0$ and $x \in X$; and,
- $\pi_n(X, x) = 0$ for $n >> 0$.

Homotopy cardinality

$\#(\cdot) : \{\text{Finite homotopy types}\} \to \mathbb{Q}$
Characterized by:

- $\#(*) = 1$;
- If $X \sim Y$ then $\#X = \#Y$;
- $\#\left(X \amalg Y\right) = \#X + \#Y$; and,
- $\#E = \#F \cdot \#B$ for fibration $F \hookrightarrow E \twoheadrightarrow B$ with B connected

Eilenberg-MacLane spaces

$X = K(A, n)$ for $|A| < \infty$.

Eilenberg-MacLane spaces

$X = K(A, n)$ for $|A| < \infty$.
$\#BA = |A|^{-1}$ since $A \hookrightarrow EA \twoheadrightarrow BA$

Eilenberg-MacLane spaces

$X = K(A, n)$ for $|A| < \infty$.
$\#BA = |A|^{-1}$ since $A \hookrightarrow EA \twoheadrightarrow BA$
$\Rightarrow \#K(A, n) = \#B^n(A) = |A|^{(-1)^n}$

Examples of finite homotopy types

Eilenberg-MacLane spaces

$X = K(A, n)$ for $|A| < \infty$.
$\#BA = |A|^{-1}$ since $A \hookrightarrow EA \twoheadrightarrow BA$
$\Rightarrow \#K(A, n) = \#B^n(A) = |A|^{(-1)^n}$

Homotopy quotients

G a finite group and X finite homotopy type with G-action

Examples of finite homotopy types

Eilenberg-MacLane spaces

$X = K(A, n)$ for $|A| < \infty$.
$\#BA = |A|^{-1}$ since $A \hookrightarrow EA \twoheadrightarrow BA$
$\Rightarrow \#K(A, n) = \#B^n(A) = |A|^{(-1)^n}$

Homotopy quotients

G a finite group and X finite homotopy type with G-action
$X \hookrightarrow X /\!\!/ G \twoheadrightarrow BG$ so $\#(X /\!\!/ G) = \frac{\#X}{|G|}$

Examples of finite homotopy types

Eilenberg-MacLane spaces

$X = K(A, n)$ for $|A| < \infty$.
$\#BA = |A|^{-1}$ since $A \hookrightarrow EA \twoheadrightarrow BA$
$\Rightarrow \#K(A, n) = \#B^n(A) = |A|^{(-1)^n}$

Homotopy quotients

G a finite group and X finite homotopy type with G-action
$X \hookrightarrow X /\!\!/ G \twoheadrightarrow BG$ so $\#(X /\!\!/ G) = \frac{\#X}{|G|}$

Mapping spaces

M CW complex with finitely many cells and X finite homotopy type

Examples of finite homotopy types

Eilenberg-MacLane spaces

$X = K(A, n)$ for $|A| < \infty$.
$\#BA = |A|^{-1}$ since $A \hookrightarrow EA \twoheadrightarrow BA$
$\Rightarrow \#K(A, n) = \#B^n(A) = |A|^{(-1)^n}$

Homotopy quotients

G a finite group and X finite homotopy type with G-action
$X \hookrightarrow X /\!\!/ G \twoheadrightarrow BG$ so $\#(X /\!\!/ G) = \frac{\#X}{|G|}$

Mapping spaces

M CW complex with finitely many cells and X finite homotopy type
$\mathcal{M}\mathrm{ap}(M, X)$ is a finite homotopy type

Homotopy cardinality formula

Postnikov tower

Tower of fibrations $X \to \cdots \twoheadrightarrow X_2 \twoheadrightarrow X_1$ with:

- $\pi_n(X_k) = 0$ for $n > k$, i.e., X_k is a *k*-type; and,
- $X \to X_k$ is a $\pi_{\leq k}$-iso.

Postnikov tower

Tower of fibrations $X \to \cdots \twoheadrightarrow X_2 \twoheadrightarrow X_1$ with:

- $\pi_n(X_k) = 0$ for $n > k$, i.e., X_k is a k-type; and,
- $X \to X_k$ is a $\pi_{\leq k}$-iso.

Observe: Fibres of $X_k \twoheadrightarrow X_{k-1}$ are $K(\pi_k(X, x), k)$

Cardinality formula

$$\#X = \sum_{x \in \pi_0(X)} \prod_{n=1}^{\infty} |\pi_n(X, x)|^{(-1)^n}$$

Pullback formula

For homotopy pullback squares

$$\begin{array}{ccc} X \times_B Y & \rightarrow & Y \\ \downarrow & & f\downarrow \\ X & \xrightarrow[g]{} & B \end{array}$$

one has

$$\#(X \times_B Y) = \sum_{b \in \pi_0(B)} \#B_b \#(f^{-1}(b)) \#(g^{-1}(b))$$

where B_b is the connected component of $b \in \pi_0(B)$.

Original DW invariants

M a compact 3-manifold and G a finite group

$$\mathrm{DW}_G(M) = \#\mathcal{M}\mathrm{ap}\,(M, BG) = \prod_{m \in \pi_0(M)} \#\mathrm{Hom}\,(\pi_1(M, m), G) /\!\!/ G$$

Original DW invariants

M a compact 3-manifold and G a finite group

$$\mathrm{DW}_G(M) = \#\mathcal{M}\mathrm{ap}\,(M, BG) = \prod_{m \in \pi_0(M)} \#\mathrm{Hom}\,(\pi_1(M, m), G) \,/\!\!/\, G$$

Geometric interpretation

Enumerates G-bundles on M since $\mathcal{B}\mathrm{un}_G(M) = \mathcal{M}\mathrm{ap}(M, BG)$

$$\#\mathcal{B}\mathrm{un}_G(M) = \sum_{P \to M} \frac{1}{|\mathrm{Aut}(P)|}$$

Higher invariants

Let M be a compact d-manifold and X a finite homotopy type
$\mathrm{DW}_X(M) = \#\mathcal{M}\mathrm{ap}\,(M, X)$

Higher invariants

Let M be a compact d-manifold and X a finite homotopy type
$\mathrm{DW}_X(M) = \#\mathcal{M}\mathrm{ap}\,(M, X)$

Special case: n-groups

- $\mathcal{G}$ an $(n-1)$-type with E_1-structure. $B\mathcal{G}$ is a connected n-type
- X a connected n-type $\Leftrightarrow X \sim B\mathcal{G}$
- $\mathrm{DW}_{\mathcal{G}}(M) := \mathrm{DW}_{B\mathcal{G}}(M)$ enumerates $\mathcal{G}$-bundles on M

DW TFT: Two perspectives

Cutting and gluing

$$\begin{array}{ccc} N^{d-1} & \overset{\partial}{\hookrightarrow} & M_1^d \\ {\scriptstyle\partial}\downarrow & & \downarrow \\ M_2^d & \hookrightarrow & M^d \end{array} \quad \mapsto \quad \begin{array}{ccc} \mathcal{M}\mathrm{ap}\,(M,X) & \longrightarrow & \mathcal{M}\mathrm{ap}\,(M_1,X) \\ \downarrow & & \downarrow \\ \mathcal{M}\mathrm{ap}\,(M_2,X) & \longrightarrow & \mathcal{M}\mathrm{ap}\,(N,X) \end{array}$$

Pullback formula $\Rightarrow$ cutting and gluing formula for DW_X

DW TFT: Two perspectives

Cutting and gluing

$$\begin{array}{ccc} N^{d-1} & \overset{\partial}{\hookrightarrow} & M_1^d \\ {\scriptstyle\partial}\downarrow & & \downarrow \\ M_2^d & \hookrightarrow & M^d \end{array} \quad \mapsto \quad \begin{array}{ccc} \mathcal{M}\mathrm{ap}\,(M, X) & \longrightarrow & \mathcal{M}\mathrm{ap}\,(M_1, X) \\ \downarrow & & \downarrow \\ \mathcal{M}\mathrm{ap}\,(M_2, X) & \longrightarrow & \mathcal{M}\mathrm{ap}\,(N, X) \end{array}$$

Pullback formula $\Rightarrow$ cutting and gluing formula for DW_X

Extended functorial field theory

$$\mathcal{Z}_X : \ \mathcal{B}\mathrm{ord}_{d,d-1,d-2} \xrightarrow{\mathcal{M}\mathrm{ap}(-,X)} \mathcal{S}\mathrm{pan}_2\, \mathcal{S}^{\mathrm{fin}} \xrightarrow{\mathcal{L}\mathrm{oc}} \mathcal{L}\mathrm{incat}$$

DW_X is the partition function of $\mathcal{Z}_X$.

Connected 2-types

X a CW complex, connected 2-type

$$0 \longrightarrow \pi_2(X) \longrightarrow \pi_2\left(X, X^{(1)}\right) \xrightarrow{\partial} \pi_1\left(X^{(1)}\right) \longrightarrow \pi_1(X) \longrightarrow 1$$

∂ and action of $\pi_1\left(X^{(1)}\right)$ on $\pi_2\left(X, X^{(1)}\right)$ determine X up to homotopy

Connected 2-types

X a CW complex, connected 2-type

$$0 \longrightarrow \pi_2(X) \longrightarrow \pi_2\left(X, X^{(1)}\right) \xrightarrow{\partial} \pi_1\left(X^{(1)}\right) \longrightarrow \pi_1(X) \longrightarrow 1$$

∂ and action of $\pi_1\left(X^{(1)}\right)$ on $\pi_2\left(X, X^{(1)}\right)$ determine X up to homotopy

$\partial : \pi_2\left(X, X^{(1)}\right) \to \pi_1\left(X^{(1)}\right)$ is a crossed module

Connected 2-types

X a CW complex, connected 2-type

$$0 \longrightarrow \pi_2(X) \longrightarrow \pi_2\left(X, X^{(1)}\right) \xrightarrow{\partial} \pi_1\left(X^{(1)}\right) \longrightarrow \pi_1(X) \longrightarrow 1$$

∂ and action of $\pi_1\left(X^{(1)}\right)$ on $\pi_2\left(X, X^{(1)}\right)$ determine X up to homotopy

$\partial : \pi_2\left(X, X^{(1)}\right) \to \pi_1\left(X^{(1)}\right)$ is a crossed module

Theorem[Noohi, 2007]

Crossed modules completely describe connected 2-types and their mapping spaces

Example: Finite gerbes

Automorphism crossed module

Γ a finite group

$$0 \longrightarrow Z(\Gamma) \longrightarrow \Gamma \xrightarrow{\mathrm{Conj}} \mathrm{Aut}(\Gamma) \longrightarrow \mathrm{Out}(\Gamma) \longrightarrow 1$$

Corresponding 2-type $B_g\Gamma$ is classifying space for Γ-gerbes

Example: Finite gerbes

Automorphism crossed module

Γ a finite group

$$0 \longrightarrow Z(\Gamma) \longrightarrow \Gamma \xrightarrow{\mathrm{Conj}} \mathrm{Aut}(\Gamma) \longrightarrow \mathrm{Out}(\Gamma) \longrightarrow 1$$

Corresponding 2-type $B_g\Gamma$ is classifying space for Γ-gerbes

Mapping spaces out of 1-types

Homotopy groups of $\mathcal{M}\mathrm{ap}\left(BG, B_g\Gamma\right)$:

- $\pi_0 \simeq \left\{1 \to \Gamma \to \hat{\Gamma} \to G \to 1\right\} /\mathrm{equiv}$;
- π_1 at $\hat{\Gamma}$ is automorphism group of the extension
- π_2 is always $Z(\Gamma)$.

$$\begin{array}{ccc} \Sigma & \overset{\partial}{\hookrightarrow} & H_\alpha \\ {\scriptstyle \partial}\downarrow & & \downarrow \\ H_\beta & \longrightarrow & M \end{array}$$

Heegaard splitting

Every closed, orientable 3-manifold M^3 obtained as pushout with:

- $\Sigma \simeq \sharp^g S^1 \times S^1$ orientable surface of genus g
- $H_i \simeq \natural^g S^1 \times D^2$ bounding handlebodies

Higher DW invariants from Heegaard splittings

$$\begin{array}{ccc} \Sigma & \overset{\partial}{\hookrightarrow} & H_\alpha \\ {\scriptstyle\partial}\big\downarrow & & \big\downarrow \\ H_\beta & \longrightarrow & M \end{array}$$

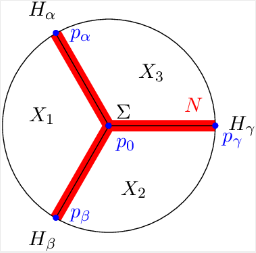
Heegaard splitting

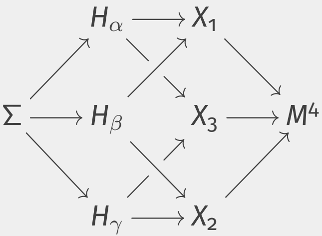
Every closed, orientable 3-manifold M^3 obtained as pushout with:

- $\Sigma \simeq \sharp^g S^1 \times S^1$ orientable surface of genus g
- $H_i \simeq \natural^g S^1 \times D^2$ bounding handlebodies

Computing DW invariants

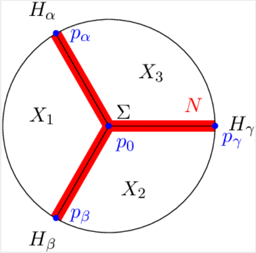
- Pullback formula $\Rightarrow \mathrm{DW}_X(M)$ computed from $\mathcal{M}\mathrm{ap}(\Sigma, X)$ and $\mathcal{M}\mathrm{ap}(H_i, X)$
- When $g > 0$, Σ and H_i are 1-types
- $\mathrm{DW}_{B_g\Gamma}(M)$ computed using extensions of surface groups and free groups

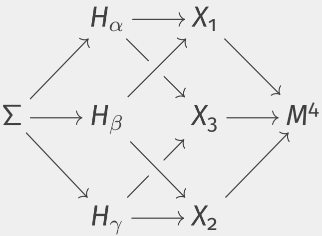




Trisections [Gay–Kirby]

Every closed, oriented 4-manifold M^4 obtained as pushout cube with $X_i \simeq \natural^k S^1 \times D^3$





Trisections [Gay–Kirby]

Every closed, oriented 4-manifold M^4 obtained as pushout cube with $X_i \simeq \natural^k S^1 \times D^3$

Higher DW invariants

M^4 iterated pushout of 1-types
$\Rightarrow \mathrm{DW}_{B_g\Gamma}(M)$ computed in terms of extensions

Bridge splittings / trisections

Given $\mathcal{K} : S^1 \hookrightarrow M^3$ and Heegaard splitting of M^3
Generically, $\mathcal{K} \cap \Sigma = \{\text{points}\}$ and $K \cap H_i = \text{trivial tangle}$
$\Rightarrow$ bridge splitting of $M^3 \setminus \mathcal{K}$

Bridge splittings / trisections

Given $\mathcal{K} : S^1 \hookrightarrow M^3$ and Heegaard splitting of M^3
Generically, $\mathcal{K} \cap \Sigma = \{\text{points}\}$ and $K \cap H_i =$ trivial tangle
$\Rightarrow$ bridge splitting of $M^3 \setminus \mathcal{K}$
Meier–Zupan: Bridge trisections for knotted surfaces in M^4

Bridge splittings / trisections

Given $\mathcal{K} : S^1 \hookrightarrow M^3$ and Heegaard splitting of M^3
Generically, $\mathcal{K} \cap \Sigma = \{\text{points}\}$ and $K \cap H_i = $ trivial tangle
$\Rightarrow$ bridge splitting of $M^3 \setminus \mathcal{K}$
Meier–Zupan: Bridge trisections for knotted surfaces in M^4

Asphericity of (2-) knots

Papakyriokopolous: $E(\mathcal{K}) = S^3 \setminus \mathcal{K}$ is a $K\left(G_{\mathcal{K}}, 1\right)$

Bridge splittings / trisections

Given $\mathcal{K} : S^1 \hookrightarrow M^3$ and Heegaard splitting of M^3
Generically, $\mathcal{K} \cap \Sigma = \{\text{points}\}$ and $K \cap H_i = \text{trivial tangle}$
$\Rightarrow$ bridge splitting of $M^3 \setminus \mathcal{K}$
Meier–Zupan: Bridge trisections for knotted surfaces in M^4

Asphericity of (2-) knots

Papakyriokopolous: $E(\mathcal{K}) = S^3 \setminus \mathcal{K}$ is a $K(G_{\mathcal{K}}, 1)$
Conjecture [Lomonaco]
For $\mathcal{K}, \mathcal{K}' : S^2 \hookrightarrow S^4$, if $E(\mathcal{K})$ and $E(\mathcal{K}')$ have same 2-type then they are homotopic

Bridge splittings / trisections

Given $\mathcal{K} : S^1 \hookrightarrow M^3$ and Heegaard splitting of M^3
Generically, $\mathcal{K} \cap \Sigma = \{\text{points}\}$ and $K \cap H_i = \text{trivial tangle}$
$\Rightarrow$ bridge splitting of $M^3 \setminus \mathcal{K}$
Meier–Zupan: Bridge trisections for knotted surfaces in M^4

Asphericity of (2-) knots

Papakyriokopolous: $E(\mathcal{K}) = S^3 \setminus \mathcal{K}$ is a $K(G_{\mathcal{K}}, 1)$
Conjecture [Lomonaco]
For $\mathcal{K}, \mathcal{K}' : S^2 \hookrightarrow S^4$, if $E(\mathcal{K})$ and $E(\mathcal{K}')$ have same 2-type then they are homotopic

Upshot: DW invariants for gerbes / 2-types possibly interesting for 2-knots

Classification of knots

Peripheral system: $\pi_1(\partial E(\mathcal{K}) \hookrightarrow E(\mathcal{K})) : \nu_{\mathcal{K}} \hookrightarrow G_{\mathcal{K}}$, $\nu_{\mathcal{K}} \simeq \mathbb{Z}^2$

Theorem: $(G_{\mathcal{K}}, \nu_{\mathcal{K}})$ is a complete isotopy invariant

Aside on classification of 2-knots

Classification of knots

Peripheral system: $\pi_1(\partial E(\mathcal{K}) \hookrightarrow E(\mathcal{K})) : \nu_{\mathcal{K}} \hookrightarrow G_{\mathcal{K}}$, $\nu_{\mathcal{K}} \simeq \mathbb{Z}^2$

Theorem: $(G_{\mathcal{K}}, \nu_{\mathcal{K}})$ is a complete isotopy invariant

- Aphericity: $(G_{\mathcal{K}}, \nu_{\mathcal{K}})$ iso
  $\Leftrightarrow$ Exteriors homotopic rel bndry
- Waldhausen: $\Leftrightarrow$ diffeomorphic
- Gordon-Luecke: $\Leftrightarrow$ knots isotopic

Aside on classification of 2-knots

Classification of knots

Peripheral system: $\pi_1(\partial E(\mathcal{K}) \hookrightarrow E(\mathcal{K})) : \nu_{\mathcal{K}} \hookrightarrow G_{\mathcal{K}}$, $\nu_{\mathcal{K}} \simeq \mathbb{Z}^2$

Theorem: $(G_{\mathcal{K}}, \nu_{\mathcal{K}})$ is a complete isotopy invariant

- Aphericity: $(G_{\mathcal{K}}, \nu_{\mathcal{K}})$ iso
  $\Leftrightarrow$ Exteriors homotopic rel bndry
- Waldhausen: $\Leftrightarrow$ diffeomorphic
- Gordon-Luecke: $\Leftrightarrow$ knots isotopic

None of this is true for 2-knots!

- Exteriors are not $K(G_{\mathcal{K}}, 1)$'s
- Homotopic $\not\Rightarrow$ homeomorphic $\not\Rightarrow$ diffeomorphic

$(C, g) \in \mathrm{Ch}^{\mathrm{b}}_{\mathbb{R}}$, C_i equipped with inner product g_i

Analytic torsion of (C, g)

Laplacian: $\Delta : C \to C$, $\Delta = dd^* + d^*d$
Orthogonal decomposition: $C = \mathrm{Ker}(\Delta) \oplus \mathrm{Im}(d) \oplus \mathrm{Im}(d^*)$ and $H(C) = \mathrm{Ker}(\Delta)$

$$T(C, g)^2 := \det\left(\Delta'\right) = \prod_{n \in \mathbb{Z}} \det\left(\Delta'_n\right)^{(-1)^n n}$$

where $\Delta' = \Delta|_{H(C)^\perp}$

Ray–Singer torsion

(M, g) closed manifold with Riemannian metric
(E, ∇) vector bundle with (orthogonal) flat connection

Torsion of (M, ∇)

Laplacian: $\Delta = d^\nabla (d^\nabla)^* + (d^\nabla)^* d^\nabla$
Hodge decomposition: $\Omega(M; E) = \mathrm{Ker}(\Delta) \oplus \mathrm{Im}(d^\nabla) \oplus \mathrm{Im}((d^\nabla)^*)$
and $H_{\mathrm{dR}}(M) = \mathrm{Ker}(\Delta)$

$$T(M; \nabla)^2 = \prod_{n \in \mathbb{Z}} \det_\zeta \left(\Delta'_n\right)^{(-1)^n n}$$

where $\det_\zeta$: regularized determinant using $\zeta_{\Delta_n}(s) = \mathrm{Tr} \Delta_n^{-s}$

Remark: Independent of Riemannian metric

Torsion for rank 2 bundles

Torsion function: $\tau_M = T(M; -) : \{\text{Flat connections}\} \to \mathbb{R}$
Holonomy: $\{\text{Flat connections}\} \leftrightarrow \{H_1(M) \to U(1)\}$

Special case: Rank 2 bundles

Torsion for rank 2 bundles

Torsion function: $\tau_M = T(M; -) : \{\text{Flat connections}\} \to \mathbb{R}$
Holonomy: $\{\text{Flat connections}\} \leftrightarrow \{H_1(M) \to U(1)\}$

Cheeger–Mueller Theorem

- $\tau_M \in \mathrm{Frac}\,(\mathbb{Z}[H_1(M)])$; and,
- τ_M agrees with the Reidemeister torsion

Reidemeister torsion for CW complexes

$\overline{M} \to M$ maximal abelian cover. Lift CW structure from M

Combinatorial torsion of $C_*^{\mathrm{CW}}\left(\overline{M}\right)$ as $\mathrm{Frac}\left(\mathbb{Z}[H_1(M)]\right)$-module

Reidemeister torsion for CW complexes

$\overline{M} \to M$ maximal abelian cover. Lift CW structure from M
Combinatorial torsion of $C_*^{\mathrm{CW}}\left(\overline{M}\right)$ as $\mathrm{Frac}\left(\mathbb{Z}[H_1(M)]\right)$-module

Torsion of knots

$H_1(E(\mathcal{K})) = \mathbb{Z}$ so $\tau_{E(\mathcal{K})} \in \mathbb{Q}(t)$
Proportional to the Alexander polynomial

Special case: Rank 2 bundles

Reidemeister torsion for CW complexes

$\overline{M} \to M$ maximal abelian cover. Lift CW structure from M
Combinatorial torsion of $C_*^{\mathrm{CW}}\left(\overline{M}\right)$ as $\mathrm{Frac}\left(\mathbb{Z}[H_1(M)]\right)$-module

Torsion of knots

$H_1(E(\mathcal{K})) = \mathbb{Z}$ so $\tau_{E(\mathcal{K})} \in \mathbb{Q}(t)$
Proportional to the Alexander polynomial

Torsion of higher knots

$H_1(E(\mathcal{K})) = \mathbb{Z}$ for all $\mathcal{K} : S^n \hookrightarrow S^{n+2}$
Alternating product of higher Alexander polynomials

Gaussian integrals

$A \in M_n(\mathbb{R})$ symmetric, positive definite

$$\int_{x \in \mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}\langle x, Ax \rangle} = \sqrt{\frac{(2\pi)^n}{\det(A)}}$$

Path integrals 101

Gaussian integrals

$A \in M_n(\mathbb{R})$ symmetric, positive definite

$$\int_{x \in \mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}\langle x, Ax \rangle} = \sqrt{\frac{(2\pi)^n}{\det(A)}}$$

"Free" field theories, basic case

(M, g) Riemannian manifold, E vector bundle
$A : \Gamma(M; E) \to \Gamma(M; E)$ differential operator

$$\mathcal{Z}(M) = \int_{V \in \Gamma(M;E)} \mathrm{e}^{-\frac{1}{2} g(V, AV)} := \sqrt{\frac{1}{\det_\zeta(A)}}$$

Birmingham–Blau–Rakowski–Thompson ansatz

$\mathcal{M}(M^d; E)$ moduli space of orthogonal flat connections

$$\mathcal{Z}(M^d; E) = \int_{\nabla \in \mathcal{M}(M;E)} T(M; \nabla)^{-1}$$

is the BF invariant for $(M^d; E)$

BF invariant

Birmingham–Blau–Rakowski–Thompson ansatz

$\mathcal{M}(M^d; E)$ moduli space of orthogonal flat connections

$$\mathcal{Z}(M^d; E) = \int_{\nabla \in \mathcal{M}(M;E)} T(M; \nabla)^{-1}$$

is the BF invariant for $(M^d; E)$

Commentary

- For $\mathrm{Rank}(E) = n$, $\mathcal{M}(M; E)$ is derived manifold $\mathrm{Hom}\,(\pi_1(M), SO(n)) /\!\!/ SO(n)$
  - Unclear (to me) how to integrate
- Suggests that codim-2 operators $\rightsquigarrow$ torsion polynomials

Final remarks

Cattaneo et al: 3D BF theory

Line operators in rank 2 give Alexander polynomial

Final remarks

Cattaneo et al: 3D BF theory

Line operators in rank 2 give Alexander polynomial

Better understanding of 4D BF theory

- Baez: BF + ‘cosmological constant’ = Crane–Yetter for $\mathrm{Rep} U_q \mathfrak{g}$

Final remarks

Cattaneo et al: 3D BF theory

Line operators in rank 2 give Alexander polynomial

Better understanding of 4D BF theory

- Baez: BF + 'cosmological constant' = Crane–Yetter for $\mathrm{Rep} U_q \mathfrak{g}$
  - Deformation of torsion polynomials of 2-knots?
  - Ben-Zvi–Brochier–Gunningham–Jordan–Safronov–Snyder: Factorization homology techniques

Final remarks

Cattaneo et al: 3D BF theory

Line operators in rank 2 give Alexander polynomial

Better understanding of 4D BF theory

- Baez: BF + ‘cosmological constant’ = Crane–Yetter for $\mathrm{Rep} U_q \mathfrak{g}$
  - Deformation of torsion polynomials of 2-knots?
  - Ben-Zvi–Brochier–Gunningham–Jordan–Safronov–Snyder: Factorization homology techniques
- Ben-Zvi–Nadler: Betti geometric Langlands

Final remarks

Cattaneo et al: 3D BF theory

Line operators in rank 2 give Alexander polynomial

Better understanding of 4D BF theory

- Baez: BF + 'cosmological constant' = Crane–Yetter for $\mathrm{Rep} U_q\mathfrak{g}$
  - Deformation of torsion polynomials of 2-knots?
  - Ben-Zvi–Brochier–Gunningham–Jordan–Safronov–Snyder: Factorization homology techniques
- Ben-Zvi–Nadler: Betti geometric Langlands

Thank you!