A universal state sum

K.W. – 2020-05-21

Very brief history:

∃ earlier?

1982 – Kauffman (Alexander-Conway polynomial)

1990 – Dijkgraaf-Witten ($n+1$-dim'l, finite group)

1992 – Turaev-Viro (2+1 dim'l oriented TQFT)
(also Barret-Westbury 1996)

1993 – Crane-Yetter (3+1 dim'l)

2018 – Douglas-Reutter (3+1 dim'l)

State sum, rough idea:

combinatorial description of W^{n+1}

e.g. triangulation or handle decomposition

$\rightsquigarrow$ set of labelings or "states"

$\rightsquigarrow$ local weights

$$\rightsquigarrow Z(W) := \sum_{\text{labelings}} \prod_{\text{weights}} w_i$$

$\rightsquigarrow$ equivalent to tensorial contraction

TQFT $\rightsquigarrow$ cut W into (combinatorial) pieces

$\rightsquigarrow$ reassemble pieces $\rightsquigarrow$ tensor contraction

$\rightsquigarrow$ state sum

$$\boxed{Z(W^{n+1}) = \sum_{\beta \in \mathcal{L}(\mathcal{H})} \prod_{j=0}^{n+1} \prod_{\beta \in j\text{-handles}} \frac{ev(\beta(\partial h))}{N(\beta(h))}}$$

H: SO or O or $Spin$ or $Pin_\pm$

W: $n+1$-dim'l H-manifold

$\mathcal{H}$: handle/cell decomposition of W

C: (a) H-pivotal n-category ($\mathbb{K}$-linear)

(b) equipped with "conjugation" map

(c) finite, semisimple

(d) evaluation map $ev: A(S^n) \longrightarrow \mathbb{K}$

which induces <u>non-degenerate pairings</u> on n-morphisms

See below

string diagrams on S^n

$$Z(W^{n+1}) = \sum_{\beta \in \mathcal{L}(\mathcal{H})} \prod_{j=0}^{n+1} \prod_{\beta \in j\text{-handles}} \frac{ev(\beta(\partial h))}{N(\beta(h))}$$

$\mathcal{L}(\mathcal{H})$: the set of labelings of i-handles of $\mathcal{H}$ by minimal $(n+1-i)$-morphisms of C, compatibly with adjacent $(j>i)$-handles

minimal k-morphism y: $\mathrm{End}(id^{n-1-k}(y))$ is a simple algebra. WLOG assume that any k-morphism of C is isomorphic to a sum of minimal k-morphisms ("weakly complete")

or minimal idem if $k=n-1$

$e \sim f \iff \exists\, u,v \ni \; e\,f\,u = e, \quad f\,e\,v = f$

$$Z(W^{n+1}) = \sum_{\beta \in \mathcal{L}(\mathcal{H})} \prod_{j=0}^{n+1} \prod_{\beta \in j\text{-handles}} \frac{ev(\beta(\partial h))}{N(\beta(h))}$$

$ev(\beta(\partial h))$ — cells $\cap\ \partial h \rightsquigarrow$ cell complex

$\beta \rightsquigarrow$ labeled cell complex $=: \beta(\partial h)$

(Now for the interesting part....)

$$N(x) := \sum_{\substack{y \text{ minimal} \\ y : x \to x}} \frac{tr_s(y)^2}{N(y)}$$

k-morphism

(inductive def'n)

when $k=n$, $N(x) := \langle x, x\rangle^2 = tr_s(x)^2$

$$tr_s(y) := ev\left[y \times S^{n-k-1} \cup (\partial y) \times \mathbb{B}^{n-k} \right]$$

k+1-morphism

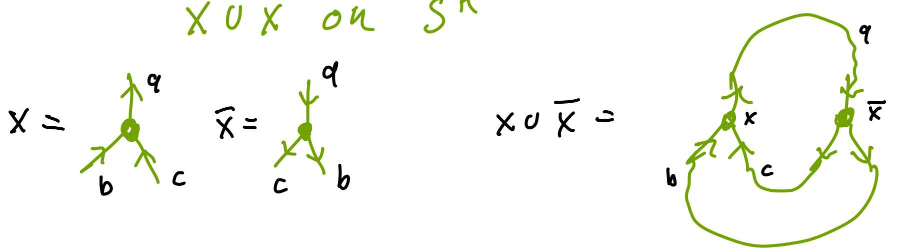
(examples below)

More on n-categories

- "H-pivotal" = string diagrams on H-manifolds make sense (strict version of "$H(n)$-fixed point", presumably)

- "finite, semisimple" = $\dim(A(M^n;c)) < \infty \quad \forall M, c$
  $A(Y^{n-1}; c)$ semisimple $\forall Y, c$

- "conjugation": reflection in "time" direction. If x is a k-morphism, then can form string diagram $x \cup \bar{x}$ on S^k



- pairing $A(B^n;c) \otimes A(B^n;c) \to \mathbb{k}, \quad x \otimes y \mapsto ev(\bar{x} \cup y)$
  assumed non-degenerate

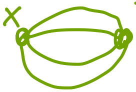
n	SO-pivotal	also finite, semisimple, etc.
2	• pivotal $\otimes$-cat • pivotal 2-cat	• fusion cat, • subfactor planar alg. • multi-fusion cat
3	• ribbon cat • $Rep_q(\mathfrak{g})$, q generic • contact 3-cat	• premodular cat • Fusion 2-cat (Douglas-Reutter)
4	• Kh-4-cat (Morrison-Wedrich-W)	
n	• $\pi_{\leq n}(X)$, X any space • symmetric monoidal ribbon cat • disk-like n-cat (Morrison-W)	• $\pi_{\leq n}(X)$, $\lvert\pi_i(X)\rvert < \infty \ \forall i$ • $Rep(G)$, G: finite group

C: n-cat, $x \in C^k$

$tr_s(x)$

$n=2$, $k=0$ $\quad ev(\emptyset_x)$

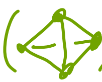
$n=2$, $k=1$ $\quad ev($$_x) = d_x$

$n=2$, $k=2$ $\quad ev(x$$\bar{x}) = \langle x, x \rangle$

$n=3$, $k=0$ $\quad ev(\emptyset_x)$

$n=3$, $k=1$ $\quad ev($$_x) = ev(S^2_x)$

$n=3$, $k=2$ $\quad ev$["spun  "]

$n=3$, $k=3$ $\quad ev$[double-cone()]

$$N(x) = \sum_{\substack{y : x \to x \\ y \text{ minimal}}} \frac{tr_s(y)^2}{N(y)}$$

$\boxed{C : n\text{-cat},\ x \in C^k}$

$k = n$ $\quad N(x) = ev(x \cup \bar{x})^2$

$k = n-1$ $\quad N(x) = \dim(End(x))$ $\leftarrow \begin{cases} = 1 \text{ for normal simple obj} \\ = 2 \text{ for Majorana simple obj.} \end{cases}$

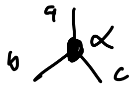
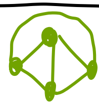
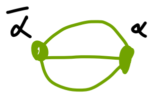
$k = n-2$ $\quad N(x) = GD(End(x)) = \sum_y \frac{d_y^2}{N(y)}$ $\leftarrow$ fm $ev(S_x^2)^2$

$k = n-3$ $\quad N(x) = \sum_y \frac{1}{GD(End(y))}$

Turaev-Viro

input: fusion category C

assume generic cell decomp.

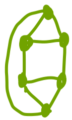
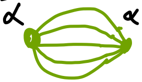
	0h	1h	2h	3h
label	—		simple ob a	$* \in C^0$
$ev(\beta(\partial h))$	 $=$ Tet	 $= \Theta_{abc\alpha}$	 $= d_a$	$\emptyset = 1$
$N(\beta(h))$	1	$\Theta_{abc\alpha}^2$	1	$GD = \sum_x d_x^2$

$$Z(M^3) = \sum_{\text{labelings}} \prod_{0h} ev\left(\text{Tet}\right) \prod_{1h} \frac{1}{\Theta_{abc\alpha}} \prod_{2h} d_a \prod_{3h} \frac{1}{GD}$$

Turaev-Viro

input: fusion category $\mathcal{C}$

general cell decomp.

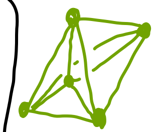
	0h	1h	2h	3h
label	—	 α	simple ob a	$* \in \mathcal{C}^0$
$ev(\beta(\partial h))$	 $= Link(h)$	$\bar{\alpha}$  α $= \Theta_\alpha$	 a $= d_a$	$\emptyset = 1$
$N(\beta(h))$	1	Θ_α^2	1	$GD = \sum_x d_x^2$

$$Z(M^3) = \sum_{\text{labelings}} \prod_{0h} ev(Link(h)) \prod_{1h} \frac{1}{\Theta_\alpha} \prod_{2h} d_a \prod_{3h} \frac{1}{GD}$$

Crane-Yetter

input: premodular cat C

assume generic cell decomp.

	0h	1h	2h	3h	4h
label	–	 α (Tet)	simple obj a	$*_1 \in C^1$	$*_0 \in C^0$
$ev(\beta(\partial h))$	 4-simplex	 α $\bar{\alpha}$	 a $= d_a$	$\emptyset = 1$	$\emptyset = 1$
$N(\beta(h))$	1	θ_α^2	1	$GD(C)$	$GD(C)^{-1}$

$$Z(W^4) = \sum_{\text{labelings}} \prod_{0h} ev(4\text{-simplex}) \prod_{1h} \frac{1}{\theta_\alpha} \prod_{2h} d_a \prod_{3h} \frac{1}{GD} \prod_{4h} GD$$

Crane-Yetter

input: premodular cat $\mathcal{C}$

$W^4 = 0h \cup \{2h\}$

along framed link $L \subset S^3$

	0	2
labels	—	simple obj a
$ev(\beta(\partial h))$	$J(L,\beta)$	$\bigcirc_a = d_a$
$N(\beta(h))$	1	1

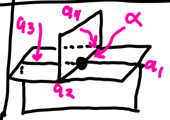
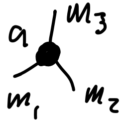
$$Z(W^4) = \sum_{\substack{\text{labelings}\\ \beta}} J(L,\beta) \cdot \prod_{2h} d_{\beta(h)}$$

RT Dehn surgery formula, up to Euler char. normalization

$\boxed{\mathcal{C} \text{ modular} \Rightarrow \text{depends only on } \partial W}$

Douglas-Reutter

input: fusion 2-cat
(finite s.s. 3-cat, one 0-mor.)
generic cell decomp.

	0h	1h	2h	3h	4h
label	–	 (Tet)	 simple obj a	minimal $m \in C^1$	$* \in C^0$
$ev(\beta(\partial h))$	4-simplex	d-cone(Tet)	span($\ominus$)	$= S^2_m$	$\emptyset = 1$
$N(\beta(h))$	1	d-cone(Tet)^2	1	$GD_m \cdot ev(S^2_m)^2$	$\underbrace{\sum_m \frac{1}{GD_m}}_{GD_3(C)}$

$$Z(W^4) = \sum_{\text{labelings}} \prod_{0h} ev(\text{4-simplex}) \prod_{1h} \frac{1}{ev(\text{d-cone(Tet)})} \prod_{2h} ev(\text{span}(\ominus))$$

$$\prod_{3h} \frac{1}{GD_m \, ev(S^2_m)} \prod_{4h} \frac{1}{GD_3(C)}$$

Proof

Outline:

1. $(n+\varepsilon)$-dim'l TQFT

2. Inductive construction of path integral

$$\cdots \to Z(S^k \times B^{n-k+1}) \rightsquigarrow A(S^k \times B^{n-k})\ \text{pairing}$$
$$\rightsquigarrow A(S^k \times B^{n-k})\ \text{copairing} \rightsquigarrow Z(S^{k+1} \times B^{n-k}) \rightsquigarrow \cdots$$

3. Compute P.I. $Z(W^{n+1})$ in terms of handle decomposition $\mathcal{H}$

4. Gluing associativity lemma $\Rightarrow$ indep. of choice of $\mathcal{H}$

5. Observe that 3 is a state sum

$(n+\varepsilon)$-dim'l TQFT

- $\mathcal{C}$: H-pivotal n-category

- $\mathcal{C}(X^k; b)$: $\mathcal{C}$-string diagrams on X, $0 \le k \le n$.
  with $\partial = b$

- $A(M^n; b)$: $\quad \mathbb{k}[\mathcal{C}(M; b)]/\sim$ "(generalized) skein module"

- $A(Y^{n-1}; b)$: 1-cat $\begin{cases} \text{0-mor:} & \mathcal{C}(Y; b) \\ \underset{x \to y}{\text{1-mor}} & A(Y \times I; \bar{x} \cup y) \end{cases}$

- $A(V^{n-2}; b)$: 2-cat $\begin{cases} \text{0-mor} & \mathcal{C}(V; b) \\ \underset{x \to y}{\text{1-mor}} & \mathcal{C}(V \times I; \bar{x} \cup y) \\ \underset{p \to q}{\text{2-mor}} & A(V \times B^2; \bar{p} \cup q) \end{cases}$

$\vdots$

- $A(X^{n-k}; b)$: k-cat with j-morphisms $\begin{cases} \mathcal{C}(X \times B^j; \ldots) & j < k \\ A(X \times B^k; \ldots) & j = k \end{cases}$

Path Integral Axioms

$\boxed{0}$ $Z(W^{n+1}) : A(\partial W) \longrightarrow \Bbbk$

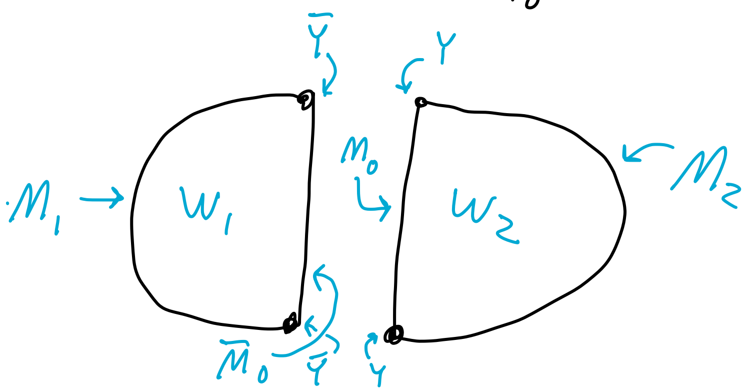
$\boxed{1}$ pairing $P_{(M,b)} : A(\bar{M};\bar{b}) \otimes A(M;b) \longrightarrow \Bbbk$

assumed non-degenerate

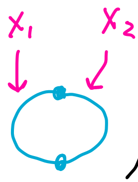
$\bar{x} \otimes y \longmapsto Z(M \times I)(\bar{x} \cup y)$

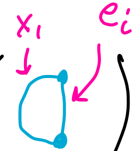
$\Rightarrow$ copairing $Q_{(M,b)} : \Bbbk \to A(M;b) \otimes A(\bar{M},\bar{b})$

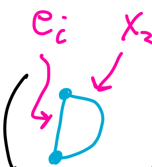
$\boxed{2}$ gluing $W = W_1 \cup_{Y_0} W_2$



$$\xrightarrow{A(M_1)} Z(W) \xleftarrow{A(M_2)} \;=\; \xrightarrow{A(M_1)} Z(W_1) \xleftarrow{A(\bar{M}_0)} Q_{M_0} \xrightarrow{A(M_0)} Z(W_2) \xleftarrow{A(M_2)}$$

$Z(W)\Big($ 

$\Big) = \sum_i Z(W_1)\Big($ 

$\Big) \cdot Z(W_2)\Big($ 

$\Big) \cdot \dfrac{1}{\langle \bar{e}_i, e_i \rangle}$

where $b = \partial x_2 = \overline{\partial x_1}$, $\{e_i\}$ orthog basis of $A(M_0, \bar{b})$

$$p_{M_0,\bar{b}}(\bar{e}_i, e_j) =: \langle \bar{e}_i, e_j \rangle = \delta_{ij} \cdot \lambda_i$$

Def. "(m,k)-handlebody": an m-manifold built out of $(0 \le i \le k)$-handles

Inductive assumptions (k)

1. $Z(W^{n+1})$ defined $\forall$ $(n+1,k)$-handlebodies invariant under handle slides and index $\leq k$ handle cancellations
2. Pairings for (M,b) nondegenerate $\forall$ b and $\forall$ (n,k)-hbodies M

Start of induction $(k=0)$

$$Z(B^{n+1}) = ev : A(S^n) \longrightarrow \mathbb{K}$$

I.P. $\quad A(\overline{B^n};\bar{c}) \otimes A(B^n;c) \longrightarrow \mathbb{K}$

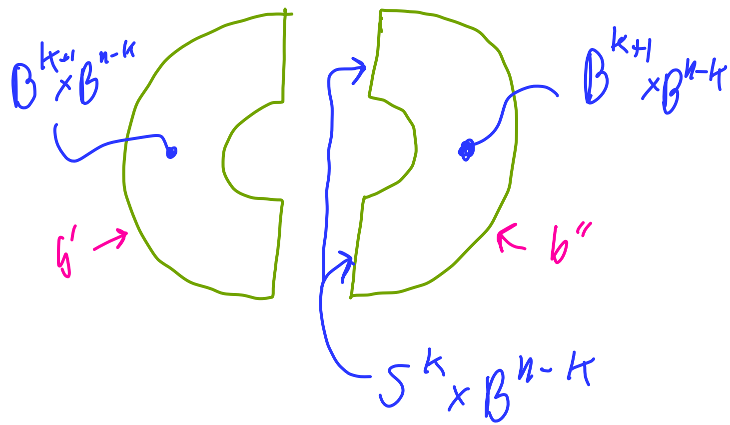
$$\bar{x} \otimes y \longmapsto Z(B^{n+1})(\bar{x} \cup y)$$

assumed nondegenerate $\forall$ $c \in C(\partial B^n)$

Inductive step ($k \to k+1$)

$\boxed{A}$ Compute $Z(S^{k+1} \times B^{n-k})(b)$, $b \in C(\partial(S^{k+1} \times B^{n-k}))$

$b = b' \cup b''$



$$Z(S^{k+1} \times B^{n-k})(b) = \sum_i \frac{Z(B^{k+1} \times B^{n-k})(b' \cup \bar{e}_i) \cdot Z(B^{k+1} \times B^{n-k})(b'' \cup e_i)}{\langle \bar{e}_i, e_i \rangle}$$

Special case: $b = S^{k+1} \times c$, $c \in C(S^{n-k-1})$

$\star$ $\{m_i\}$ orthogonal basis of $A(S^k \times B^{n-k}; S^k \times c)$

$\uparrow$ minimal $(n-k)$-morphisms with $\partial m_i = c$

$$Z(S^{k+1} \times B^{n-k})(S^{k+1} \times c) = \sum_{m_i} \frac{\left[Z(B^{k+1} \times B^{n-k})(B^{k+1} \times c \cup m_i)\right]^2}{\langle \bar{m}_i, m_i \rangle}$$

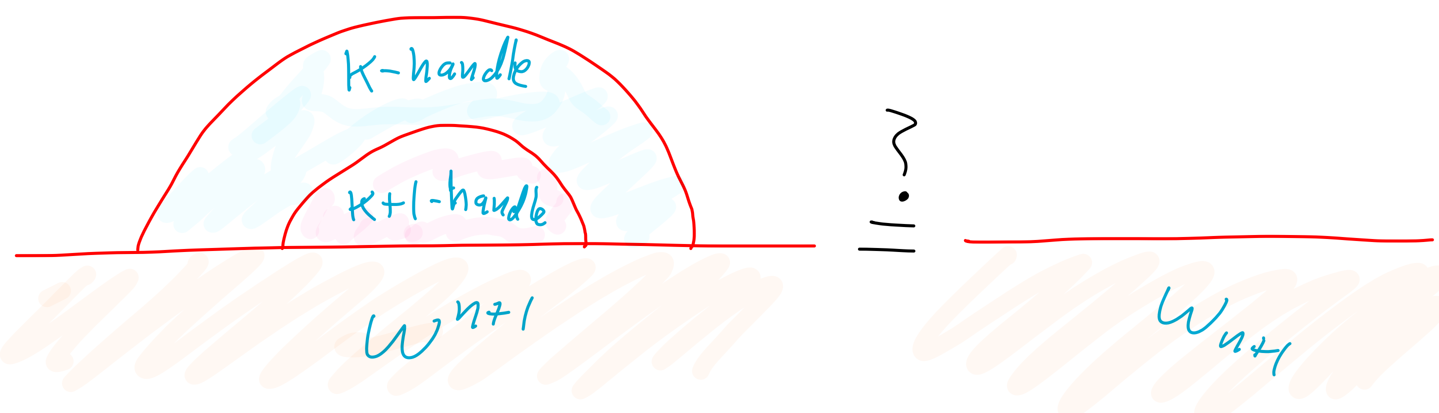
$$= \sum_{m_i} \frac{tr_s(m_i)^2}{\langle \bar{m}_i, m_i \rangle}$$

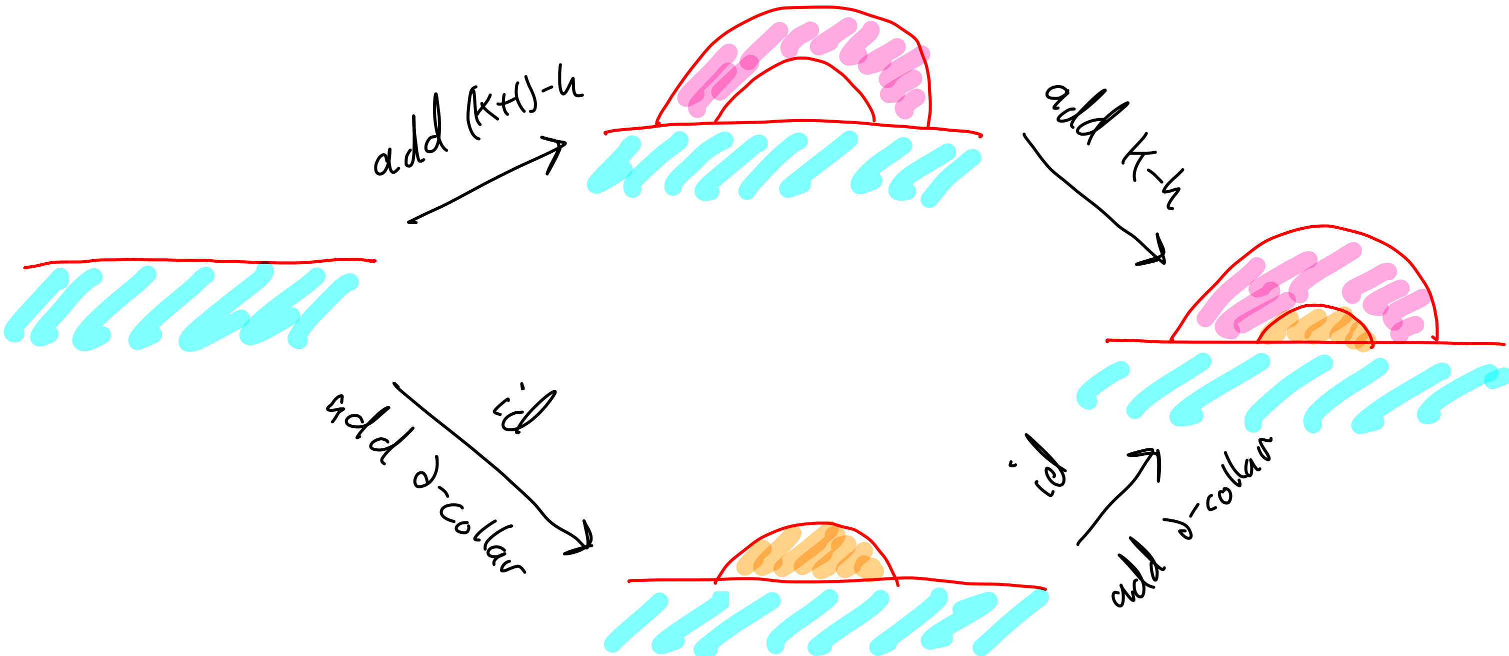
$\Downarrow$

$$\boxed{\underbrace{\langle S^{k+1} \times \bar{x}, S^{k+1} \times x \rangle_{S^{k+1} \times B^{n-k-1}}}_{N(x)} = \sum_{\substack{\text{minimal} \\ y \in End(x)}} \frac{tr_s(y)}{\underbrace{\langle S^k \times \bar{y}, S^k \times y \rangle_{S^k \times B^{n-k}}}_{N(y)}}}$$

$\boxed{B}$ Can now compute $Z(W^{n+1})$ for any $(n+1, k+1)$-handlebody W^{n+1}. Independent of handle decomp?

- Invariance w.r.t. handle slides: trivial
- Invariance w.r.t. hande cancellations:





★ So associativity of gluing formula implies invariance w.r.t. handle cancellation

Thm. (W, 2006) If C is an n-category satisfying (a)-(d) far above (pivotal, finite, semisimple, non-degenerate eval. map), then the "easy" $(n+\varepsilon)$-dim'l TQFT extends uniquely to an $(n+1)$-dim'l TQFT with $Z(B^{n+1}) = ev.$

The proof of the above thm provides an algorithm for computing $Z(W^{n+1})$ in terms of a handle decomposition. To go from the algorithm to the state sum formula, observe that minimal j-morphisms give an orthogonal basis of $A(S^{n-j} \times B^j; S^{n-j} \times c)$.

Brown–Arf $(Pin_-, n=1)$

$ev(\emptyset) = \lambda \in \mathbb{C}$

input $\mathbb{C}\ell(1)$, s odd, $s^2 = 1$,

reflection of B^1 $\longrightarrow$ $r(s) = \alpha \cdot s$ $\quad \alpha^4 = 1$

	0	1	2
label	$-$	id or S	$*$
$ev(\beta(\partial h))$	$\lambda \cdot \alpha^k$ or zero	λ	λ
$N(\beta(h))$	1	λ^2	2

$$Z(W^2) = \lambda^{\chi(w)} \sum_{\text{labelings}} \prod_{0h} \alpha^k \prod_{2h} \frac{1}{2}$$

$$Z(W^2) = \lambda^{\chi(W)} 2^{-\#2h} \sum_{\text{1-cycles } x} \alpha^{q(x)}$$

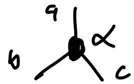
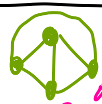
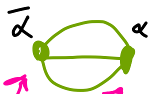
$q(x) \in \mathbb{Z}/4$

cf. Kirby-Taylor

Fermionic TV

input: super fusion cat C

$(n=2, H=Spin)$

	0h	1h	2h	3h
label	—		simple ob a maybe Majorana	$* \in C^0$
$ev(\beta(\partial h))$	 $=$ Tet	 $= \Theta_{abc\alpha}$	 $= d_a$	$\emptyset = 1$
$N(\beta(h))$	1	$\Theta^2_{abc\alpha}$	1 or 2	$GD = \sum_x d_x^2 / N(x)$

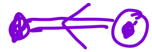
maybe fermionic

Simplest super fusion categories: (2 simple objects)

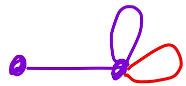
$\{\mathbb{1}, x\}$ $\{\mathbb{1}, m\}$

normal Majorana

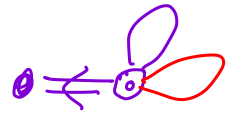
C_2: $m \otimes m \cong \mathbb{C}^{1|1} \cdot \mathbb{1}$



$SO(3)_6/\psi$: $x \otimes x \cong \mathbb{1} \oplus \mathbb{C}^{1|1} \cdot x$



$\frac{1}{2}E_6/\psi$: $m \otimes m \cong \mathbb{C}^{1|1} \cdot \mathbb{1} \oplus \mathbb{C}^{1|1} \cdot m$



If $A(Y^2) \cong \mathbb{C}^{p|q}$, then

$Z(Y \times S^1_B) = p+q$ and $Z(Y \times S^1_N) = p-q$

	C_2	$SO(3)_6/\psi$	$\frac{1}{2}\mathrm{E}_6/y$
$g = 1, \mathrm{Arf} = 0$	$3 \mid 0$	$4 \mid 0$	$3 \mid 0$
$g = 1, \mathrm{Arf} = 1$	$0 \mid 3$	$2 \mid 2$	$1 \mid 2$
$g = 2, \mathrm{Arf} = 0$	$10 \mid 0$	$40 \mid 24$	$19 \mid 8$
$g = 2, \mathrm{Arf} = 1$	$0 \mid 10$	$32 \mid 32$	$11 \mid 16$
$g = 3, \mathrm{Arf} = 0$	$36 \mid 0$	$1184 \mid 1120$	$281 \mid 232$
$g = 3, \mathrm{Arf} = 1$	$0 \mid 36$	$1152 \mid 1152$	$241 \mid 272$
$g = 4, \mathrm{Arf} = 0$	$136 \mid 0$	$51328 \mid 51072$	$5755 \mid 5504$
$g = 4, \mathrm{Arf} = 1$	$0 \mid 136$	$51200 \mid 51200$	$5531 \mid 5728$
$g = 5, \mathrm{Arf} = 0$	$528 \mid 0$	$2368000 \mid 2366976$	$126449 \mid 125056$
$g = 5, \mathrm{Arf} = 1$	$0 \mid 528$	$2367488 \mid 2367488$	$125137 \mid 126368$

Figure 4.5.1: Hilbert space dimensions for closed surfaces in various theories.

For $C = C_2$ above, state sum labelings are

$$\{(\mathbb{Z}/2\text{-2-cycle } S,\ 1\text{-cycle } J \subset S)\}$$

Guess $$Z_{C_2}(M^3) = \frac{1}{2} \sum_{S \in H_2(M;\mathbb{Z}/2)} (-1)^{\mathrm{Arf}(S)}$$