Boundaries and 3-dimensional topological field theories

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Joint work with Constantin Teleman

Local elliptic boundary conditions

Second order elliptic operator (Laplace):

$$\Delta f = -\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} \qquad \text{on} \quad D = \left\{ (x, y) \in \mathbb{A}^2 : x^2 + y^2 \leq 1 \right\}$$

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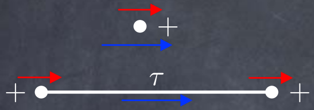
Instead of *conditions* one can have boundary *data* (degrees of freedom)

Boundary theories in QFT

Example (quantum mechanics): $F\colon \mathrm{Bord}_1^{\mathrm{Riem},w_1} \longrightarrow \mathrm{Vect_{top}}$



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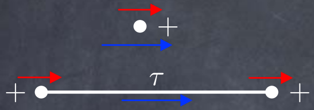

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Boundary data: $\xi \in \mathcal{H}$, $\theta \in \mathcal{H}^*$

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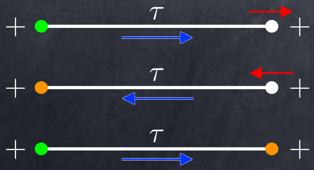

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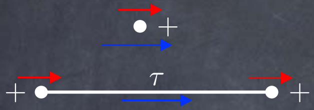

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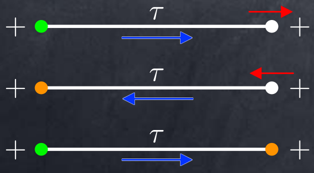

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In higher dimensions we use boundaries in *space* rather than time

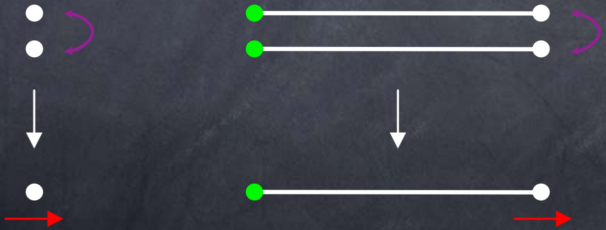
Toy example

$\mathrm{Bord}^{\mathrm{dc}}_{\langle 0,1\rangle}$ manifolds with a double cover

$F\colon \mathrm{Bord}^{\mathrm{dc}}_{\langle 0,1\rangle} \to \mathrm{Vect}_{\mathbb{C}}$ invertible field theory

$F\big(\mathrm{pt} \times \{\pm 1\} \to \mathrm{pt}\big) = \mathbb{C}$ with sign representation

$\widetilde{F}$ extension with boundary theory



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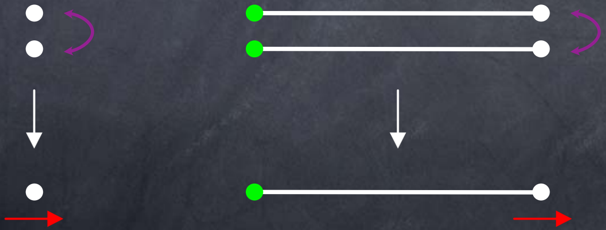

1-morphism evaluates to a sign-invariant element of $\mathbb{C}$, hence vanishes

Reshetikhin-Turaev theories

Witten: Path integral from classical Chern-Simons invariant

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Reconcile through notion of *extended* field theory

$$F\colon\ \mathrm{Bord}^{\mathrm{fr}}_{\langle 1,2,3\rangle} \longrightarrow \mathrm{Cat}_{\mathbb{C}}$$

$\mathrm{Cat}_{\mathbb{C}}$ is a 2-category of complex linear categories (to be specified)

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Remark: In the 3-framed case there are two circles: S^1_b, S^1_n
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Problem: Does this F admit a nonzero boundary theory?

Fully extended TFT

Full locality is encoded in a fully extended theory:

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$\mathrm{Hom}_{\mathcal{C}}\big(1, F(+)\big)$ = boundary theories (it may be zero)

Tensor categories

Deligne, Franco, Etingof-Gelaki-Nikshych-Ostrik, Haugseng, Johnson-Freyd-Scheimbauer, Douglas-Schommer-Pries-Snyder, Ben-Zvi-Brochier-Jordan, BJS, BJ-Safronov-S, . . .

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Definition: $\mathrm{Cat}_{\mathbb{C}}$ is the symmetric monoidal 2-category of finitely cocomplete $\mathbb{C}$-linear categories under Deligne-Kelly $\boxtimes$

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Definition: $\mathrm{FSCat} \subset \mathrm{Cat}_{\mathbb{C}}$ full subcategory of finite semisimple abelian

Fusion category: finite semisimple rigid abelian tensor cat

$\mathrm{Fus} \subset E_1(\mathrm{Cat}_{\mathbb{C}})$ symmetric monoidal subcategory of fusion categories and finite semisimple bimodule categories

Cobordism hypothesis

Theorem (**Hopkins-Lurie**, **Lurie**): Evaluation at a point

$$\begin{aligned} \mathrm{Hom}(\mathrm{Bord}_n^{\mathrm{fr}}, \mathcal{C}) &\longrightarrow (\mathcal{C}^{\mathrm{fd}})^{\sim} \\ F &\longmapsto F(+) \end{aligned}$$

is a homotopy equivalence of spaces

$\mathcal{C}^{\mathrm{fd}}$ is the fully dualizable subcategory of $\mathcal{C}$
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Examples: Chern-Simons for finite groups, or special tori and levels

Main Theorem

Let $\mathcal{C}$ be a symmetric monoidal 3-category whose fully dualizable part $\mathcal{C}^{\mathrm{fd}}$ contains the 3-category Fus of fusion categories as a full subcategory. Let $F\colon \mathrm{Bord}_3^{\mathrm{fr}} \to \mathcal{C}$ be a 3-framed topological field theory such that

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Then $F(S^1_b)$ is braided tensor equivalent to the Drinfeld center of a fusion category Φ_0 with simple unit

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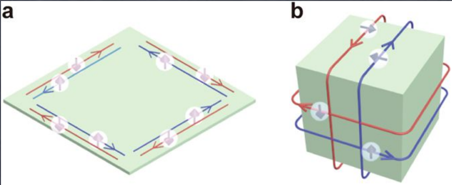
Usefulness of regular Φ-module: see also Section 6 of `arXiv:1806.00008`

Application to physics

New materials which insulate in the bulk and conduct on the boundary



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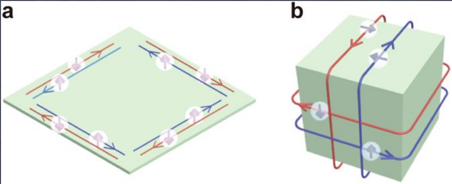

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Question: When is conduction forced on the boundary?

The main theorem gives a criterion in $2+1$ dimensions, if we are willing to make a few jumps

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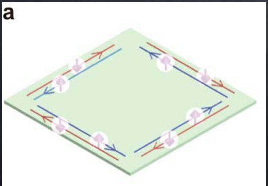

Apply main theorem to $2+1$ dim'l system:

Gapped interior (bulk) $\Longrightarrow$ 3d TFT F

If gapped boundary theory exists, then F admits a nonzero boundary theory β

For many F the theorem implies no such β

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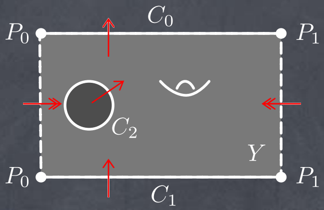
Fix $n \in \mathbb{Z}^{>0}$, $k \in \{0, \dots, n\}$, $d \in \{0, \dots, k\}$

A *k-morphism of depth d in* Bord_n is a compact k-manifold with corners of depth $\leq d$ with extra data: highly structured boundary, arrows of time, tangential structure

Bord_2:



$P \in \mathrm{Bord}_2$ $\qquad$ $C: P_0 \amalg P_1 \longrightarrow \emptyset^0$ $\qquad$ $Y: C_0 \Rightarrow C_1 \amalg C_2 : P_0 \amalg P_1 \to \emptyset^0$

$$P_0 \amalg P_1 \underset{C_1 \amalg C_2}{\overset{C_0}{\rightrightarrows}} \emptyset^0 \quad (\Uparrow Y)$$

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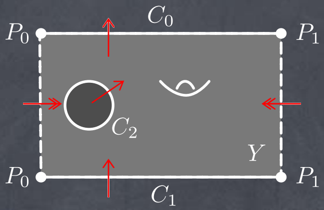

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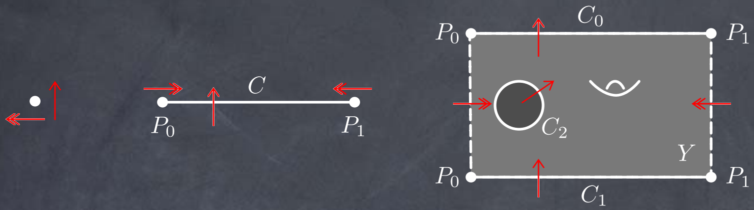

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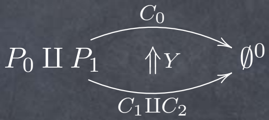
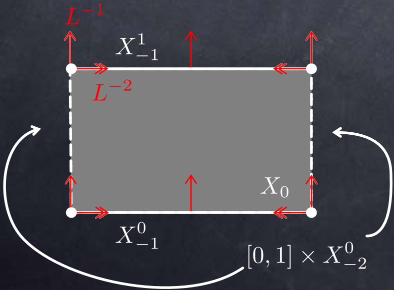
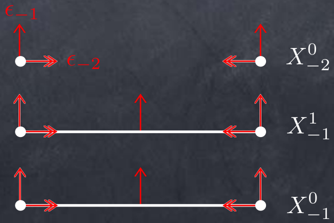
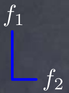
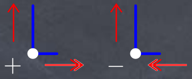


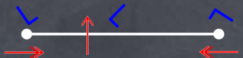




$\mathrm{Bord}_2^{\mathrm{fr}}$:







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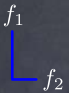

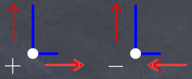

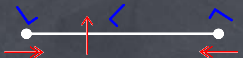

$\mathrm{Bord}_3^{\mathrm{fr}}$: f_0 and 0^{th} arrow of time point *into* the screen

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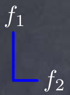

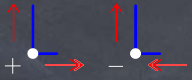

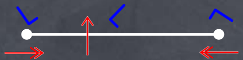

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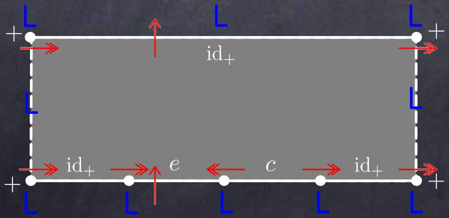
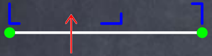
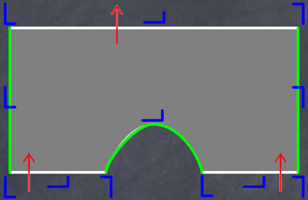


Adjoints exist:

$\text{Bord}^{\text{fr}}_{2,\partial}$: domain of $\widetilde{F} = (F, \beta)$

Same objects as $\text{Bord}^{\text{fr}}_2$, but new 1- and 2-morphisms





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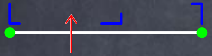

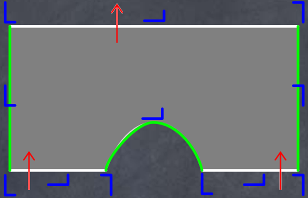

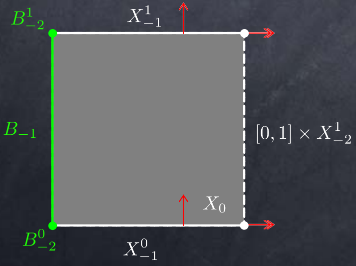

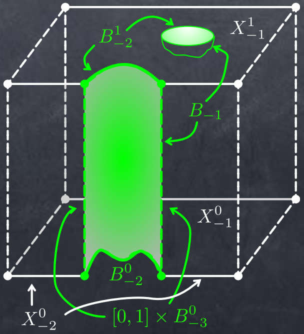

Preliminary 2: internal homs

$\mathcal{C}$	symmetric monoidal 2-category
$x \xrightarrow{f} y$	1-morphism with adjoints

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If $h = g = f$ (so $z = x$ and $w = y$), then have algebra objects

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Proposition [DSS]: $A, B \in \mathrm{Fus}$, $M\colon A \to B$ a (B,A)-bimodule category. Then M has adjoints and

$$\underline{\mathrm{End}}^R(M) = \mathrm{End}_B(M)$$
$$\underline{\mathrm{End}}^L(M) = \mathrm{End}_A(M)$$

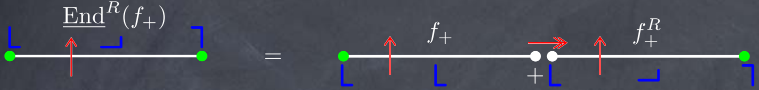
Recall the theorems

Theorem: $\mathcal{C}$ symmetric monoidal 3-category
$\mathcal{C}^{\mathrm{fd}} \supset \mathrm{Fus}$ *full* subcategory
$F\colon \mathrm{Bord}_3^{\mathrm{fr}} \to \mathcal{C}$
$\widetilde{F}\colon \mathrm{Bord}_{3,\partial}^{\mathrm{fr}} \to \mathcal{C}$ such that $\beta\colon 1 \to \tau_{\leq 2}F$ is nonzero
$\dots$
$\Longrightarrow$ …of a fusion category Φ_0 with simple unit

Theorem: Let $\Phi \in E_1(\mathrm{Cat}_{\mathbb{C}})$ be a tensor category. Then Φ is a fusion category if and only if

(i) Φ is 3-dualizable in $E_1(\mathrm{Cat}_{\mathbb{C}})$, and
(ii) Φ is 2-dualizable as a left Φ-module

The fusion category Φ



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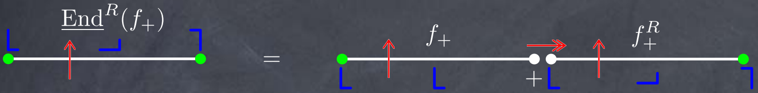

$$\Phi = \widetilde{F}\big(\underline{\mathrm{End}}^R(f_+)\big)$$
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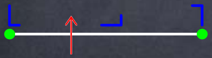
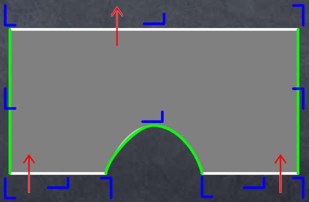


$$\Phi \otimes \Phi \longrightarrow \Phi$$



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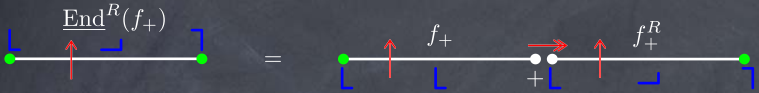

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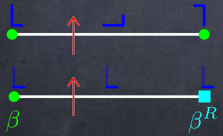

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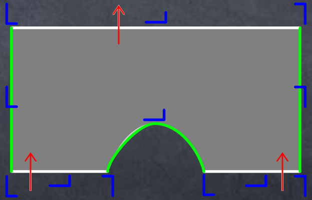

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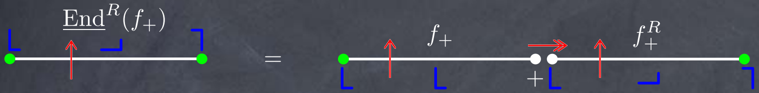

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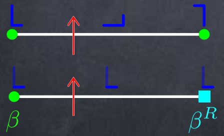

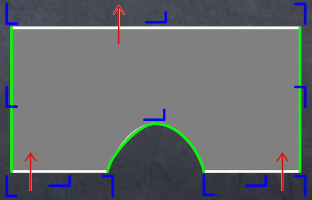

Lemma: Φ is a finite semisimple abelian category

$\overline{F}\colon \mathrm{Bord}_2^{\mathrm{fr}} \longrightarrow \mathrm{Cat}_{\mathbb{C}}$ (dimensional reduction)

Lemma: Φ is rigid (has internal left and right duals)

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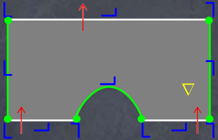
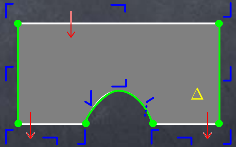
$\Phi \in E_1(\mathrm{Cat}_{\mathbb{C}})$. The unit η and multiplication ∇



have right adjoints $\varepsilon : \Phi \to \mathbf{1}$ and $\Delta : \Phi \to \Phi \boxtimes \Phi$ (Frobenius data)



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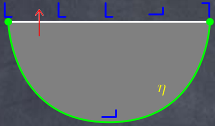

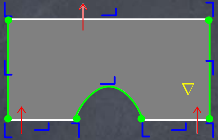

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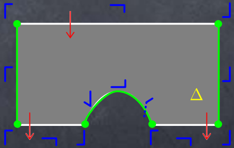

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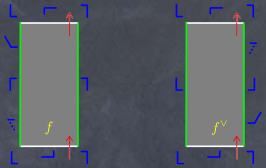
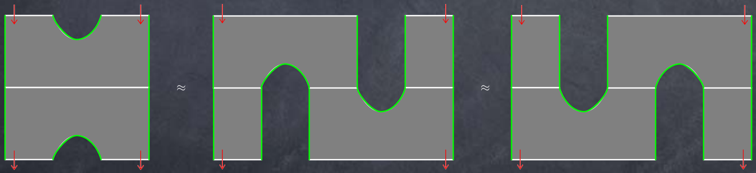
Theorem: A finite semisimple tensor category Φ is rigid if (i) $f, f^\vee$ are isomorphisms, and (ii) Δ is a Φ-Φ bimodule map

(i) and (ii) have "picture proofs" in the extended field theory $\widetilde{F}$:





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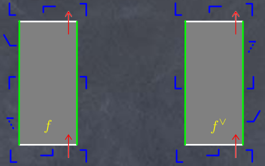

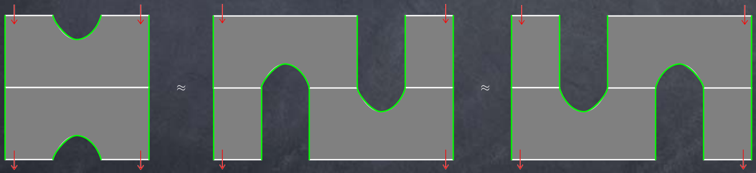

Theorem: Φ is a fusion category

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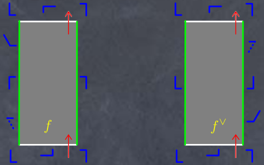
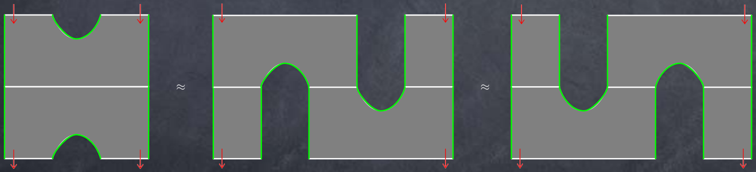




Theorem: Φ is a fusion category

Cobordism hypothesis $\Longrightarrow$ $T\colon \mathrm{Bord}_3^{\mathrm{fr}} \to E_1(\mathrm{Cat}_{\mathbb{C}})$ with $T(+) = \Phi$

Main Theorem

Let $\mathcal{C}$ be a symmetric monoidal 3-category whose fully dualizable part $\mathcal{C}^{\mathrm{fd}}$ contains the 3-category $T\mathrm{Cat}_{\mathbb{C}}$ of finite tensor categories as a full subcategory. Let $F\colon \mathrm{Bord}_3^{\mathrm{fr}} \to \mathcal{C}$ be a 3-framed topological field theory such that

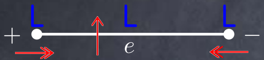
(a) $F(S^0)$ is isomorphic in $\mathcal{C}$ to a multifusion category, and

(b) $F(S_b^1)$ is invertible as an object in the 4-category $E_2(\Omega\mathcal{C})$ of braided tensor categories.

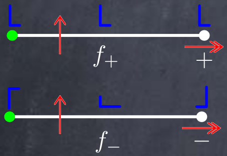
Assume F extends to $\widetilde{F}\colon \mathrm{Bord}_{3,\partial}^{\mathrm{fr}} \to \mathcal{C}$ such that the associated boundary theory $\beta\colon 1 \to \tau_{\leq 2}F$ is nonzero.

Then $F(S_b^1)$ is braided tensor equivalent to the Drinfeld center of a fusion category Φ.

$A \cong F(S^0)$-module categories



$e\colon S^0 \longrightarrow \emptyset^0$



$f_\pm\colon \emptyset^0 \to S^0$

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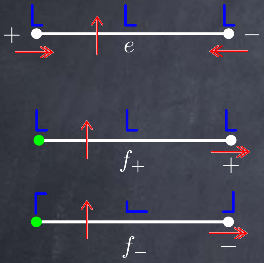


$e\colon S^0 \longrightarrow \emptyset^0$

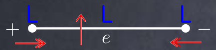
$f_\pm\colon \emptyset^0 \to S^0$

$$M := \left(F(e)\colon \quad A \xrightarrow{\cong} F(S^0) \longrightarrow \mathrm{Vect}_{\mathbb{C}} \right)$$

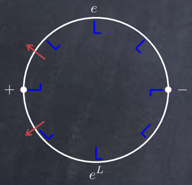
$$N := \left(\widetilde{F}(f_\pm)\colon \mathrm{Vect}_{\mathbb{C}} \longrightarrow F(S^0) \xrightarrow{\cong} A \right)$$

M is a *left* A-module category
N is a *right* A-module category

Morita equivalences



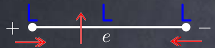
$$M = F(e)$$



$$S^1_b \cong e \circ e^L = \underline{\mathrm{End}}^L(e)$$

$$F(S^1_b) \simeq F\big(\underline{\mathrm{End}}^L(e)\big) \simeq \underline{\mathrm{End}}^L\big(F(e)\big) \simeq \mathrm{End}_A(M)$$

Morita equivalences



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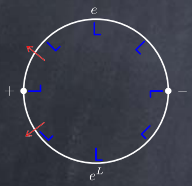
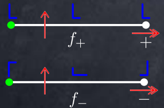


$$S^1_b \cong e \circ e^L = \underline{\mathrm{End}}^L(e)$$

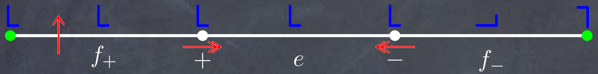
$$F(S^1_b) \simeq F\big(\underline{\mathrm{End}}^L(e)\big) \simeq \underline{\mathrm{End}}^L\big(F(e)\big) \simeq \mathrm{End}_A(M)$$



$$N = \beta(S^0)$$

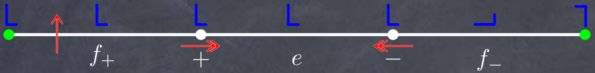
As theories, $T = \underline{\mathrm{End}}^R(\beta)$, and so $T(S^0) \cong \underline{\mathrm{End}}^R(N) \cong \mathrm{End}_A(N)$

Final steps



$$\Phi \simeq M \otimes_A N$$

Final steps



$$\Phi \simeq M \otimes_A N$$

N is invertible as a $\big(A, T(S^0)\big)$-bimodule:

$$\begin{array}{ccccc}
\mathrm{End}_A(M) & \xrightarrow[\simeq]{\otimes N} & \mathrm{End}_{T(S^0)}(\Phi) & = & \mathrm{End}_{\Phi\otimes\Phi^{\mathrm{mo}}}(\Phi) \\
\| & & & & \| \\
F(S^1_b) & & & & Z(\Phi) = T(S^1_b)
\end{array}$$