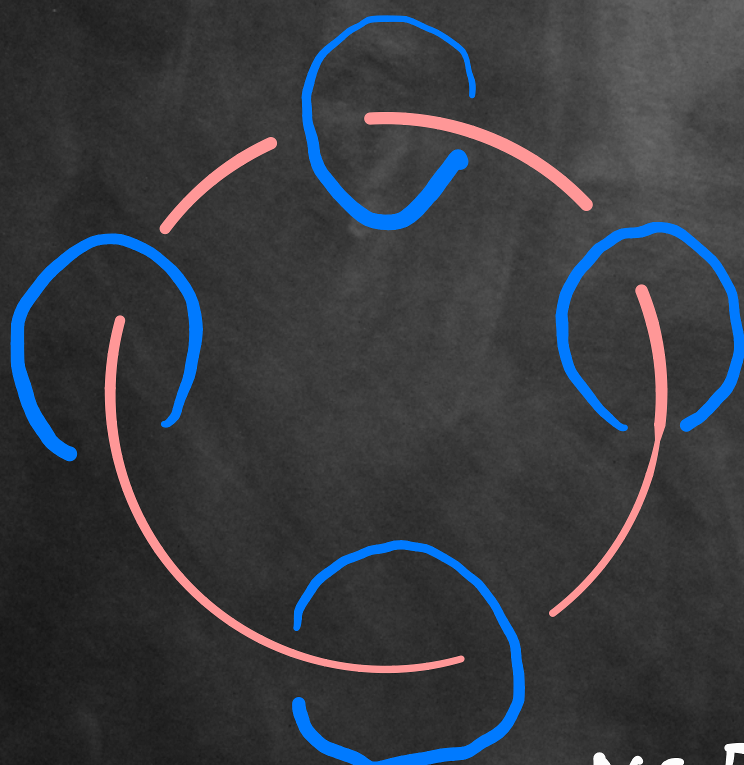
Representations of Motion Groups

joint with:

- Gustafson, Kimball, Zhang
- Bullivant, Kimball, Martin
- Damiani, Martin
- Kadar, Martin, Wang



NSF, MSRI, Simons Foundation: Thanks!

Motivation.

- 2D picture: B_n reps are key!
  - Anyon statistics in 2D Topol. phases.

  - Density of unitary reps $\longleftrightarrow$ Universal Top. Quantum Computation.
  - Link/3-manifold invariants via Trace.
  - (2+1)TQFTs: genus 0 part.
- 3D picture: Motion Groups.
  - loop excitations in 3D TPMs?

  - Any useful Q.C. Models?
  - Interesting invariants?
  - (3+1)TQFT "Canary in the coal mine"

Motion Groups.

Dahm 1962
Goldsmith 1981, 1982

M oriented
$\cup|$
N oriented, cpt

A <u>motion</u> of N in M is an ambient isotopy $f_t(x)$ of N in M st. $f_0(x) = id_M$ & $f_1(N) = N$ as an oriented submanifold.
(N.b. $f_1|_N \neq id_N$ in gen.)
If $f_t(N) = N \ \forall t$, f is <u>stationary</u>. $f \simeq f'$ if $f^{-1} \circ f' \sim$ a stationary motion.

$\mathcal{M}(M,N)$: motions$/\simeq$

Heuristic Defn

More details

- $H_c(M,N)$ homeos (or diffeos) of M w/

  cpt support

  - ∂M fixed ptwise
  - N fixed setwise
  - orientation on M & N preserved.
- Motions are paths f in $H_c(M)$ st
  - $f_0 = id_M$
  - $f_1 \in H_c(M,N)$

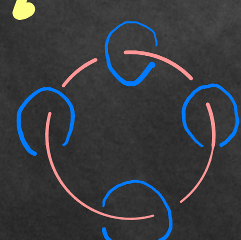
  $$\boxed{\mathcal{M}(M,N) = \pi_1(H_c(M), H_c(M,N); id_M)}$$

- $\mathcal{H}^+(M,N) := \pi_0(H_c(M,N), id_M)$ orientation preserving
- $\mathcal{M}(M,N) \xrightarrow{\partial} \mathcal{H}^+(M,N)$ $\quad \partial([f]) = [f_1]$.
  - ∂ is iso if $M = \mathbb{R}^3$
  - ∂ surj. if $M = S^3$, $\ker(\partial) \cong \mathbb{Z}_2$.

Remarks & Subtleties

- M cpt? $\partial M \neq \phi$? orientation, diffeo?
- Focus on M^2, M^3: $\boxed{S^3, D^3, \mathbb{R}^3}$ or $\boxed{S^2, D^2, \mathbb{R}^2}$
- $\mathcal{M}(M, N) \longrightarrow \mathrm{Aut}(\pi_1(M \setminus N))$ or $\mathrm{Out}(\pi_1(M \setminus N))$

  can be useful for getting a presentation.

- Few presentations available.
  - $\bigsqcup_i \bigcirc$, $\left(\bigsqcup_i\right.$  $\left.\right) \bigsqcup \left(\bigsqcup_j \bigcirc\right)$ [Damiani-Kamada]
  -  [Bellingeri-Bodin]
  - Torus Links [Goldsmith, Qiu-Wang]

Example I. The Braid Group

$$B_n := \mathcal{M}(D^2, P)$$
$$|P| = n.$$

Motions of points in a disk.

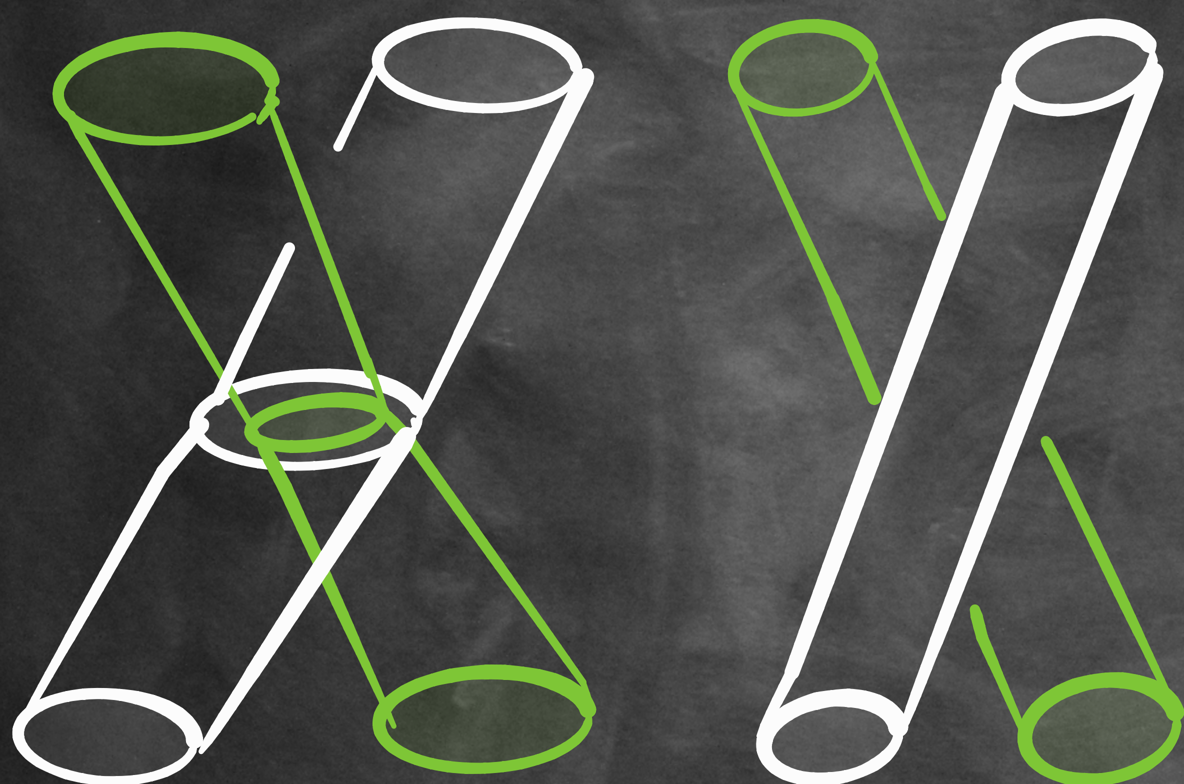
Surface Braid gps generally…

- Close to Hurwitz' 1891 formulation…
- Artin's presentation:

(B1) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$
(B2) $\sigma_i\sigma_j = \sigma_j\sigma_i$ *for* $|i-j|>1$,

Example II (McCool, Fenn-Rourke-Rimanyi, ...)

$$LB_n = \mathcal{M}(B^3, S^1 \sqcup S^1 \sqcup \cdots \sqcup S^1) \cong B_n * S_n / \langle L0, L1, L2 \rangle$$



σ_i $\quad$ s_i

The **braid relations***:*

(B1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
(B2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ *for* $|i-j| > 1$,

the **symmetric group relations***:*

(S1) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
(S2) $s_i s_j = s_j s_i$ *for* $|i-j| > 1$,
(S3) $s_i^2 = 1$

and the **mixed relations***:*

(L0) $\sigma_i s_j = s_j \sigma_i$ *for* $|i-j| > 1$
(L1) $s_i s_{i+1} \sigma_i = \sigma_{i+1} s_i s_{i+1}$
(L2) $\sigma_i \sigma_{i+1} s_i = s_{i+1} \sigma_i \sigma_{i+1}$

or $\sigma_{i+1} \sigma_i s_{i+1} = s_i \sigma_{i+1} \sigma_i$

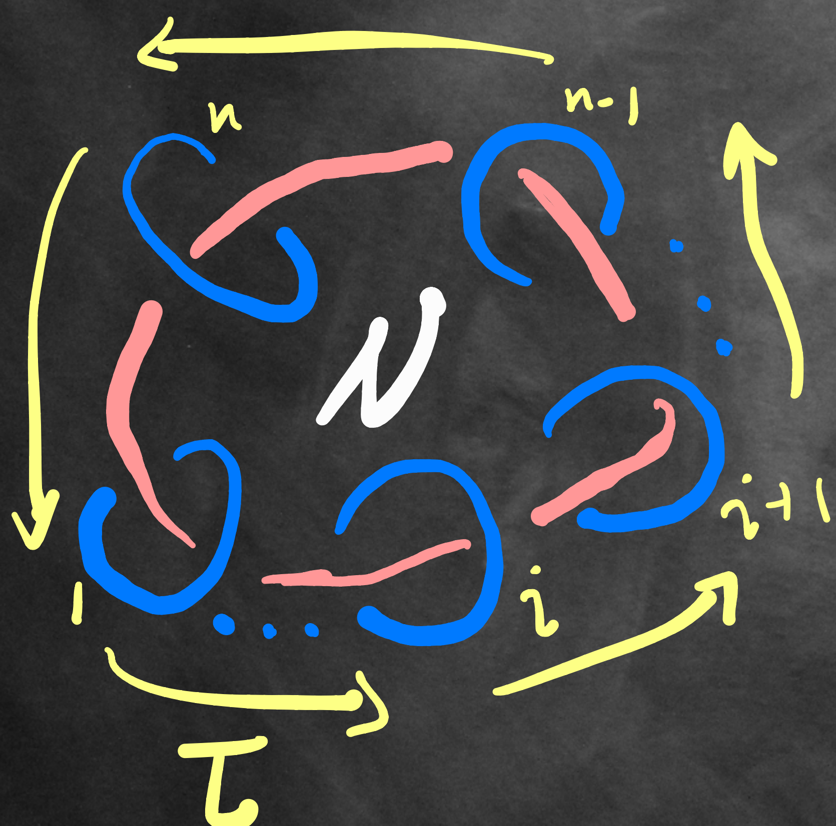
But not both!

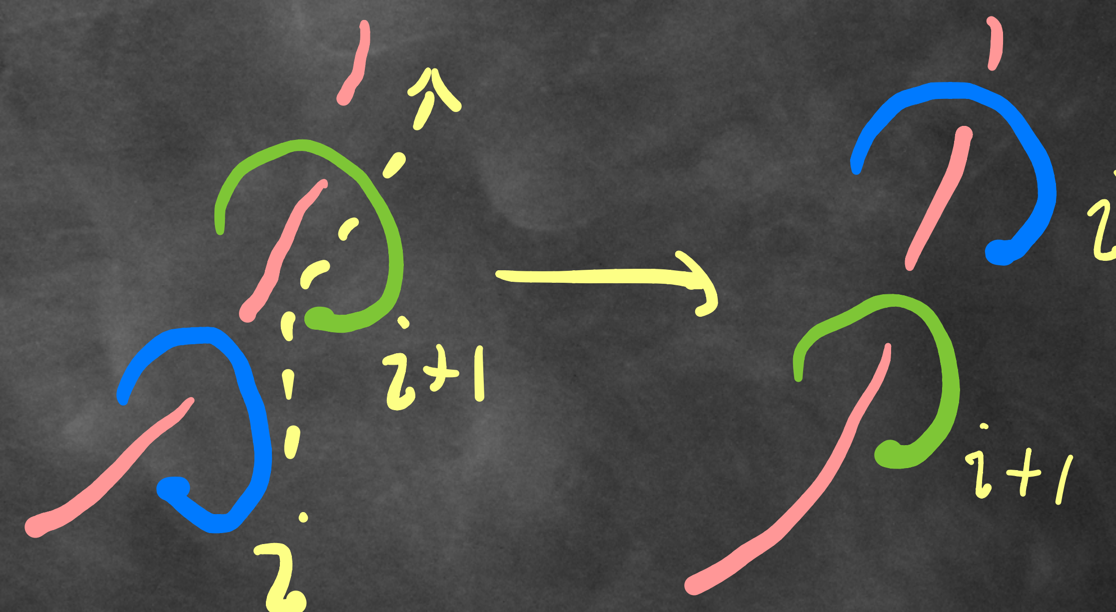
Example III (Bellingeri-Bodin)

$NB_n : \mathcal{M}(S^3, N)$

(B1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
(B2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ *for* $|i - j| \neq 1 \pmod{n}$,
(N1) $\tau \sigma_i \tau^{-1} = \sigma_{i+1}$ *for* $1 \leq i \leq n$
(N2) $\tau^{2n} = 1$

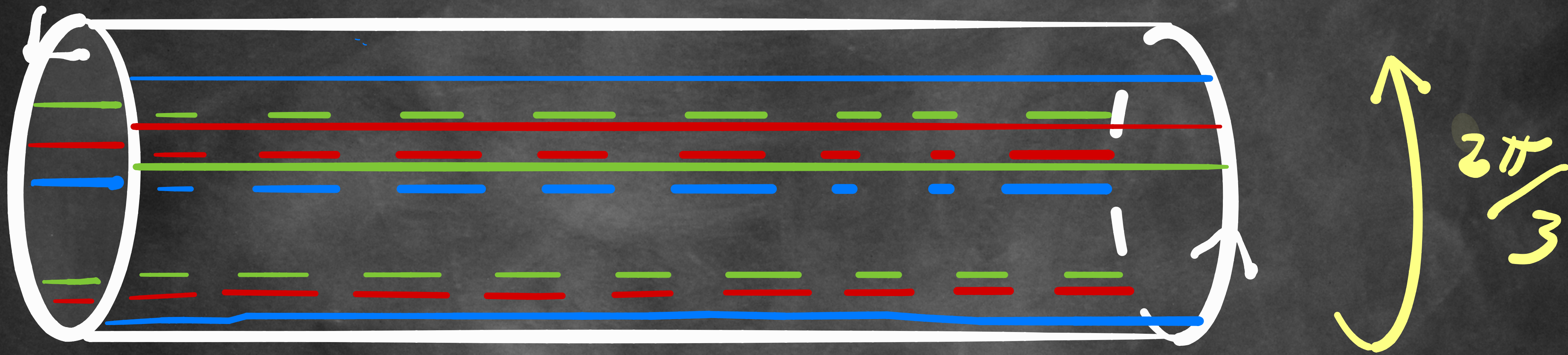
Here indices are taken modulo n, with $\sigma_{n+1} := \sigma_1$ *and* $\sigma_0 := \sigma_n$.





Example IV (Goldsmith, Qiu–Wang)

Torus Links $TL(np, nq)$



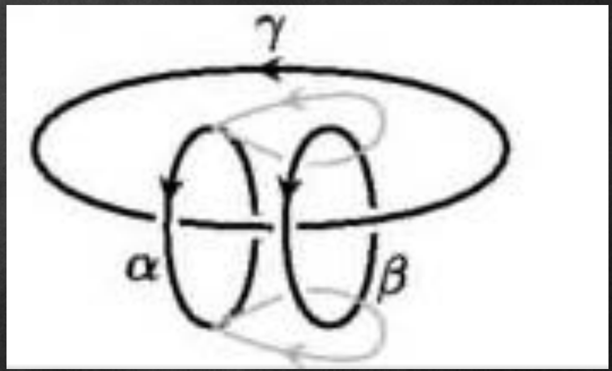
σ_i interchange ith component & $(i+1)$st.

r_i rotate ith component by $2\pi/p$

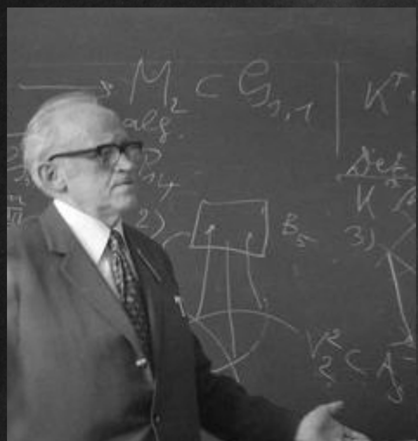
σ_i satisfy Braid rels...

Remarks on Reps. of Motion Gps

- Explicit reps from (3+1) TQFT are not easy.
- In many cases, $B_n \subseteq \mathcal{M}(M^3, L)$
  so $\boxed{\text{extend}}$!
- Might hope for invariants of surfaces in $\mathbb{R}^4$ via Markov traces, generalized.
- Physics applications! Levin-Wang PRL 2014



Representations of B_n.



- 1936 $\boxed{\text{Burau}}$: $\sigma_i \longmapsto \begin{bmatrix} I & & \\ & \begin{matrix} 1-t & t \\ 1 & 0 \end{matrix} & \\ & & I \end{bmatrix}$

- 1980s Yang-Baxter eqn. $\boxed{\text{local}}$ reps

$R_1 = R \otimes I$, $R_2 = I \otimes R$, $R_1 R_2 R_1 = R_2 R_1 R_2$

- 1990s Jones, Birman-Wenzl, others...

towers of f.d. $\boxed{\text{quotients}}$ of $\mathbb{k}[B_n]$:

$TL_n(t)$, $\mathcal{H}_n = \mathbb{C}(q)[B_n] / \langle (\sigma_i + 1)(\sigma_i - q) \rangle$

$BMW_n(r, q)$

Representations of B_n (cont)

- 1980s Drinfeld quasi triangular, quasi-Hopf algs.

- 1990s $\boxed{\text{Braided fusion categories}}$ $\mathcal{C} \ni X$

$$\varphi : \mathbb{C}[B_n] \longrightarrow \mathrm{End}(X^{\otimes n}), \quad \sigma_i \longmapsto I_X^{i-1} \otimes c_{X,X} \otimes I_X^{n-i-1}$$

B_n acts on $\bigoplus_{Y \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}(Y, X^{\otimes n})$.

- 1989 Jones-Goldschmidt: $\boxed{\text{Metaplectic}}$ reps

$$\mathcal{A}_n = \mathbb{C}\langle u_i : u_i u_{i+1} = q^2 u_{i+1} u_i,\ u_i^\ell = q^\ell = 1,\ [u_i, u_j] = 1\ |i-j|>1 \rangle$$

$\sigma_i \longmapsto \frac{1}{\sqrt{\ell}} \sum_j q^{j^2} u_i^j$ defines a rep into $\mathcal{A}_n$.

Representations of B_n (even more!)

Gustafson
Kimball
R
Zhang

Inspired by metaplectic reps.

Fix finite gp G and a bihom. $\alpha: G\times G \to \mathbb{Z}_m$ & $q^m = 1$. Iterated twisted tensor power

$$\mathcal{A}_n(G,\alpha) = \underset{\{g_1\}}{\mathbb{C}[G]} \otimes_\alpha \cdots \otimes_\alpha \underset{\{g_i\}}{\mathbb{C}[G]} \otimes_\alpha \cdots \underset{\{g_{n-1}\}}{\mathbb{C}[G]}$$

$$g_i h_j = \begin{cases} h_j g_i, & |i-j| > 1, \\ q^{\pm\alpha(g,h)} h_{i\pm 1} g_i, & j = i \pm 1, \\ (gh)_i, & j = i, \end{cases}$$

Look for

$$r_i = \sum_{g\in G} f(g) g_i$$

satisfying B_n rels, possibly in

$$\mathcal{A}_n(G,\alpha)/\mathcal{I}$$

- Metaplectic: $G = \mathbb{Z}_\ell$
- $G = Q_8 \to$ Quaternionic B_n reps.
- $G = \mathbb{Z}_p \times \mathbb{Z}_p$ factors...

Categorical connections

- **Reutter**: s.s. (3+1)TQFT Z. M^4 spinless then $Z(M^4)$ depends on classical stuff (π_1, χ, σ, ...).
- **Liang Chang**: $\mathcal{B}_n$ reps from $X, Y \in \mathcal{C}$ BFC if $X \otimes Y = \bigoplus_i Z_i$, Z_i bosons/fermions. In fact, uses 2-dim. braiding on $(X \otimes Y)^{\otimes n}$.

$\Rightarrow$ $\mathcal{C}[X, Y]$ is weakly integral. So finite image, probably.

Categorical construction: Dijkgraaf-Witten Theory

Qiu-Wang provide evidence that

(Conj) reps of motion gps from $\boxed{DW_G^{3+1}}$
are determined by those from $\boxed{DW_G^{2+1}}$.

Eg. Torus Links with labels pure fluxes,
(& mapping class gps of closed mfds).

o How general is this?

Loop BFCs?

non-symmetric braiding

Very Naive guess: $X \in \mathcal{C}^\vee$ BFC. $c_{X,X} \in \mathrm{End}(X^{\otimes 2})$.

A symmetric braiding $s_{X,X} \in \mathrm{End}(X^{\otimes 2})$ also?

Unusual by results of Nikshych.

Loop Braided Vector spaces? [Kádár, Martin, R, Wang]

(R, S, V) $R, S \in \mathrm{End}(V^{\otimes 2})$ solns to YBE

s.t. $LB_n \to \mathrm{Aut}(V^{\otimes n})$ $\sigma_i \longmapsto R_i$, $s_i \longmapsto S_i$.

Th^m: If R is of group type (i.e, YD-module)

$S(v \otimes w) = w \otimes v$ works! (Essentially DW).

Otherwise, may be not.

Extensions?

B_n rep	IB_n	NB_n
Burau	✓	✓
Lawrence-Krammer	✗ stringent conditions	✓
local	✓ group type	✓
Arb. completely reducible rep	✗ many counterexs.	✓ central extn..
B_3 irrep. dim ≤ 5	✓	✓
Gaussian, Quaternionic	probably not	✓
Categorical constructions	$(X\otimes Y)^{\otimes n}$ if $X\otimes Y = \oplus$ bosons/fermions	$Y\otimes X^{\otimes n}$ if $c_{Y,X}\circ c_{X,Y} = id$.

Finite dim'l quotients? [Damiani-Martin-R]

$\mathbb{C}[LB_n] \big/ \langle(\sigma_i - 1)(\sigma_i + t)\rangle$ not f.d. 

However, extended Burau is a rep.

$$\sigma_i \longmapsto M_i = \begin{bmatrix} I & & \\ & \begin{matrix} 1-t & t \\ 1 & 0 \end{matrix} & \\ & & I \end{bmatrix}, \quad s_i \longmapsto \sigma_i \Big|_{t=1} = P_i.$$

These satisfy: $(M_i - 1)(P_i + 1) = (P_i - 1)(M_i + t) = 0$.

Set $LH_n := LB_n \big/ \langle (\sigma_i - 1)(\sigma_i + 1), (\sigma_i - 1)(s_i + 1), (s_i - 1)(\sigma_i + t) \rangle$

Finite dim'l!

Properties of LH_n (Loop Hecke algebra)

- **Not** semisimple!
- Admits a local rep: $R = \begin{bmatrix} 1 & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & -t \end{bmatrix}$

Look!

$S = R\big|_{t=1}$.

$$\sigma_i \longrightarrow I_2^{i-1} \otimes R \otimes I_2^{n-i-1}$$

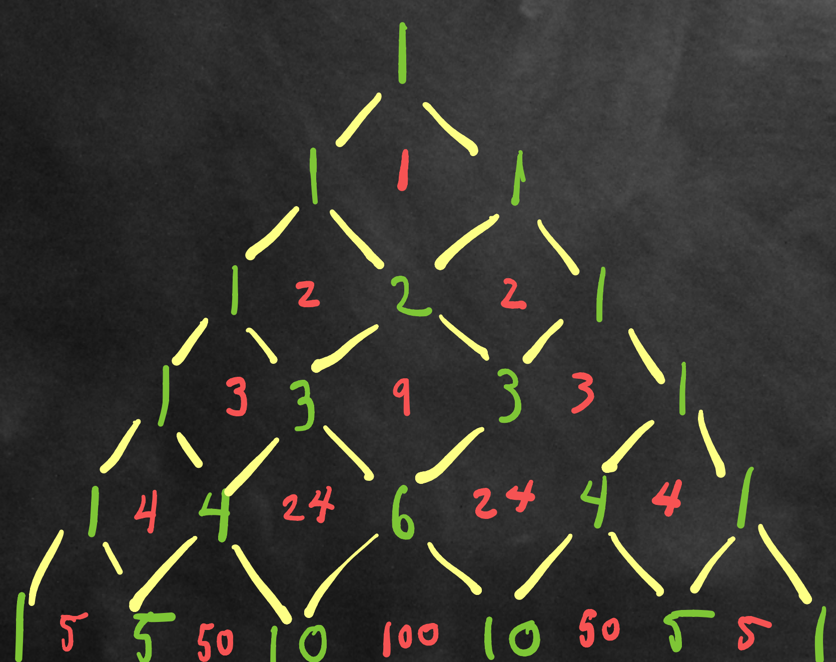
$$s_i \longrightarrow I_2^{i-1} \otimes S \otimes I_2^{n-i-1}$$

Not unitary!

Loop Burau-Rittenberg rep.

Structure of LBR_n algebra $:= \langle R_i, S_i \rangle$.

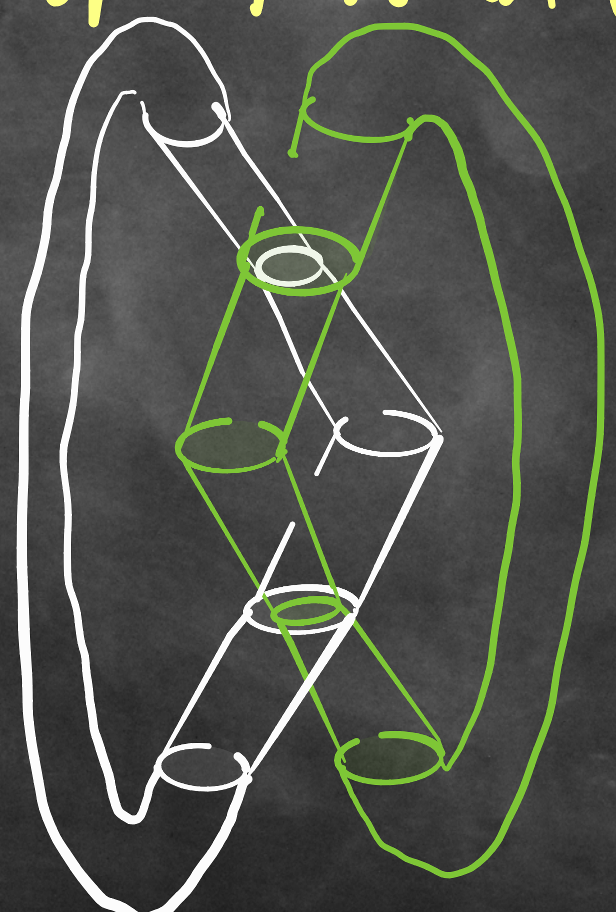
$$\overbrace{\begin{bmatrix} \mathbb{A}_1 & \mathbb{A}_{1,2} & \cdots & \\ & \ddots & \ddots & \mathbb{A}_{n-1,n} \\ & & \ddots & \mathbb{A}_n \end{bmatrix}}^{2^n}$$



- $\dim(\mathbb{A}_i) = \binom{n}{i}^2$ (simple)
- $\dim(\mathbb{A}_{i,i+1}) = \binom{n}{i}\binom{n}{i+1}$
- Jacobson Radical $\bigoplus_i \mathbb{A}_{i,i+1}$
- $LBR_n / J(LBR_n) \cong \bigoplus_i \mathbb{A}_i$
- Bratteli diagram : Pascal's Δ

Outlook / Future directions

- Non-s.s. is a feature, not a failing?
- A more robust categorical approach is needed.
- Other f.d. quotients of Motion Group algebras?
- Topological invariants (of surfaces in M^4)?



Markov Trace on LH_n?

- $Tr(ab) = Tr(ba)$ $\quad a, b \in LH_n$
- $Tr(a\sigma_n) = Tr(a)\alpha$ $\quad$ Then adjust by
- $Tr(a s_n) = Tr(a)\beta$ $\quad$ writhe...
- $Tr(1) = 1$

Relations $\Rightarrow$ $(\alpha - t\beta)(\alpha - \beta) = 0$

$$\underset{(-t,\,-1)}{\Downarrow} \qquad \underset{(1,1)}{\Downarrow}$$

Leads to trivial invariant...