The geometry of toric syzygies

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$\Bbbk = \overline{\Bbbk}$

Takeaway: min. free res. $\rightsquigarrow \mathbb{P}^n$

virtual res. $\rightsquigarrow$ sm proj TV X

I. Background

$$\vec{x} = (x_0, x_1, \dots, x_m) \in \Bbbk^{m+1}$$

$$\Bbbk^\times \times \Bbbk^{m+1} \longrightarrow \Bbbk^{m+1}$$

$$(t, \vec{x}) \longmapsto (tx_0, tx_1, \dots, tx_m) =: t \cdot \vec{x}$$

$\vec{x} \in \Bbbk^{m+1} \setminus \{\vec{0}\}$: $\quad [\vec{x}] := \{ t \cdot \vec{x} \mid t \in \Bbbk^\times \}$

$$\Rightarrow \mathbb{P}^m = \left\{ [\vec{x}] \mid \vec{x} \in \Bbbk^{m+1} \setminus \{\vec{0}\} \right\} \twoheadleftarrow \Bbbk^{m+1} \setminus \{\vec{0}\}$$

$$[\vec{x}] \longmapsfrom \vec{x}$$

Note: $[\vec{x}] = [\vec{y}] \iff \exists\, t \in \Bbbk^\times$ s.t. $\vec{x} = t \cdot \vec{y}$.

$\alpha \in \mathbb{N}^{m+1}$: $\quad \vec{x}^{\alpha} := x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ & $|\alpha| := \sum_{i=0}^{m} \alpha_i$

$f(\vec{x}) \in \Bbbk[x_0, x_1, \dots, x_m] =: S \qquad [\vec{z}] \in \mathbb{P}^m$

$f(\vec{y}) = 0 \quad \forall \vec{y} \in [\vec{z}] \iff f(\vec{x})$ is homogeneous

$X \subseteq \mathbb{P}^m$: $\quad I(X) := \langle f(\vec{x}) \mid f(\vec{y}) = 0 \;\; \forall \vec{y} \in [\vec{z}] \in X \rangle$

(homogeneous) ideal of X

Ex: $\mathbb{P}^2 \supseteq X = \{ [1:0:0], [0:1:0], [0:0:1], [1:1:1], [1:1:0] \}$

$$\Rightarrow I(X) = \langle \underline{x_0x_2 - x_1x_2},\; x_1^2x_2 - x_1x_2^2,\; \underline{x_0^2x_1 - x_0x_1^2} \rangle$$

$$-x_0x_1(\quad - \quad) + x_2(\quad - \quad)$$

II. Enter Hilbert

$S = \Bbbk[x_0, x_1, \dots, x_m] \qquad \deg(x_i) = 1$

$M = \bigoplus_{d \in \mathbb{Z}} M_d$ f.g. graded S-module

$HF_M(d) = \dim_{\Bbbk} M_d$

[Hilbert]: $\exists\, P_M(z) \in \mathbb{Q}[z]$ s.t. $\forall\, d >> 0$,

$$P_M(d) = HF_M(d).$$

Def: An acyclic graded free complex

$$F: \; F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} F_2 \leftarrow \cdots$$

is a minimal free resolution of M if:

$$H_0 F = M, \quad F = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{ij}}, \; \& \; \partial(F_i) \subseteq \langle \underline{x} \rangle F_{i-1}$$

Ex: (5 pts. in $\mathbb{P}^2$): $M = S/I(X)$

$$S \leftarrow \begin{matrix} S(-2) \\ \oplus \\ S(-3)^2 \end{matrix} \longleftarrow S(-4)^2 \leftarrow 0$$

$$m+d-j \geq 0 \implies \dim_{\Bbbk} S_{d-j} = \frac{(m+d-j)(m-1+d-j)\cdots(1+d-j)}{m!}$$

III. Classical Results

Hilbert's Syzygy Thm: $X \subseteq \mathbb{P}^m \implies$

$S/I(X)$ has a min. free res. of length $\leq m$.

Hilbert-Burch Thm: If $X \subseteq \mathbb{P}^m$ is s.t. $S/I(X)$ has a m.f.r. of form $S \leftarrow F_1 \xleftarrow{M} F_2 \leftarrow 0$, then $I(X) = \langle (\text{maximal minors of } M)\cdot(\text{NZD})\rangle$.

Regularity: M: f.g. grd. S-mod, $p \in \mathbb{Z}$

Def: M is p-regular if, $\forall q \in \mathbb{Z}$,

$$[H^q_{\langle x \rangle}(M)]_r = 0 \text{ for } r = p-q+1,$$

& $$[H^0_{\langle x \rangle}(M)]_r = 0 \quad \forall r \geqslant p+1$$

Castelnuovo-Mumford Regularity & Linearity:

$M_{\geqslant p}$ has m.f.r: $S(-p)^{\beta_0} \leftarrow S(-p-1)^{\beta_1} \leftarrow S(-p-2)^{\beta_2} \leftarrow \cdots$

$\iff$ M is p-regular.

Ⓑ

I. Background:

$a_1, a_2, \ldots, a_N \in \mathbb{Z}^r$: $(\Bbbk^\times)^r \times \Bbbk^N \longrightarrow \Bbbk^N$

$$(\vec{t}, \vec{x}) \longmapsto \overset{\vec{t}\cdot\vec{x}}{\overset{\parallel}{(\vec{t}^{a_1}x_1, \vec{t}^{a_2}x_2, \ldots, \vec{t}^{a_N}x_N)}}$$

$S = \Bbbk[x_1, x_2, \ldots, x_N]$ $\quad \deg(x_i) = a_i \in \mathbb{Z}^r$

$\cup\!|$

$B \rightsquigarrow \mathrm{Var}(B) = \{\vec{x} \in \Bbbk^N \mid f(\vec{x}) = 0 \quad \forall \vec{x} \in B\}$

$\vec{x} \in \mathbb{k}^N \setminus \mathrm{Var}(B)$: $[\vec{x}] := \{\vec{t} \cdot \vec{x} \mid \vec{t} \in (\mathbb{k}^\times)^r\}$

<u>Def:</u> <u>toric</u> <u>variety</u> $X := \{ [\vec{x}] \mid \vec{x} \in \mathbb{k}^N \setminus \mathrm{Var}(B)\}$

$$= \left(\mathbb{k}^N \setminus \mathrm{Var}(B)\right) /\!\!/ (\mathbb{k}^\times)^r$$

Assume: X sm. proj.

<u>Ex:</u> $\overset{x_0\,x_1}{\mathbb{P}^1} \times \overset{y_0\,y_1\,y_2}{\mathbb{P}^2}$

$$(\mathbb{k}^\times)^2 \times \mathbb{k}^5 \longrightarrow \mathbb{k}^5$$

$$(\vec{t}, \vec{x}) \longmapsto (t_1x_0, t_1x_1, t_2y_0, t_2y_1, t_2y_2)$$

$S = \mathbb{k}[x_0, x_1, y_0, y_1, y_2]$ has $\deg(x_i) = \binom{1}{0} \in \mathbb{Z}^2$

$\cup\!\!\mid$ & $\deg(y_j) = \binom{0}{1} \in \mathbb{Z}^2$

$B = \langle x_0, x_1 \rangle \cap \langle y_0, y_1, y_2 \rangle$

II. Virtual Resolution

M B-saturated f.g. $\mathbb{Z}^r$-grd. S-module.

<u>Def:</u> A $\mathbb{Z}^r$-grd. free complex

$$F : \quad F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots$$

is a <u>virtual</u> <u>resolution</u> of M if

$$H_0F : B^\infty = M \quad \& \quad H_iF : B^\infty = 0 \quad \forall i > 0.$$

$(\Longleftrightarrow \widetilde{F}$ is a locally free res. of $\widetilde{M}.)$

<u>Ex:</u> $Z = \{([1:0],[1:0]),([0:1],[0:1])\} \subseteq \underline{\overset{x_0\,x_1}{\mathbb{P}^1}} \times \overset{y_0\,y_1}{\mathbb{P}^1}$

$S/I(Z)$ has m.f.r:

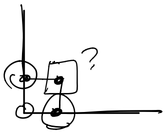
m.f.r of $S/I(Z) \cap \langle x \rangle$

$S(-2$

$$\begin{tikzcd} S & \begin{matrix} \oplus \\ S\binom{-1}{-1}^2 \\ \oplus \\ S\binom{0}{-2} \end{matrix} \arrow[l] & \begin{matrix} S\binom{-2}{-1}^2 \\ \oplus \\ S\binom{-1}{-2}^2 \end{matrix} \arrow[l] & S\binom{-2}{-2} \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

mfr of $S/I(Z) \cap \langle y \rangle$.

$= \text{vResPair}\left(S/I(Z), \binom{0}{1}+\binom{1}{1}\right)$



$$\star \quad S \leftarrow S\binom{-1}{-1}^2 \leftarrow S\binom{-2}{-2} \leftarrow 0 \quad \text{mfr of } S/I(Z) \cap B$$

Ex: $\overset{\text{curve}}{Y} \subseteq \mathbb{P}^1 \times \mathbb{P}^2$, mfr of $S/I(Y)$:

$$\begin{tikzcd} S & \begin{matrix} S(-3,-1) \\ \oplus \\ S(-2,-2) \\ \oplus \\ S(-2,-3)^2 \\ \oplus \\ S(-1,-5)^3 \\ \oplus \\ S(0,-8) \end{matrix} \arrow[l] & \begin{matrix} S(-3,-3)^3 \\ \oplus \\ S(-2,-5)^6 \\ \oplus \\ S(-1,-7) \\ \oplus \\ S(-1,-8)^2 \end{matrix} \arrow[l] \arrow[l, "\overset{4\times 3}{\varphi}"'] & \begin{matrix} S(-3,-5)^3 \\ \oplus \\ S(-2,-7)^2 \\ \oplus \\ S(-2,-8) \end{matrix} \arrow[l] & S(-3,-7) \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

$\sim$ $\text{vResPair}\left(S/I(Y), \binom{4}{1}+\binom{1}{2}\right)$

III. Virtual Results $\qquad \mathbb{P}^{\vec{n}} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \dots \times \mathbb{P}^{n_r}$

v. HST.
[B-Erman-Smith]: For every $Y \subseteq \mathbb{P}^{\vec{n}}$,
$S/I(Y)$ has a v. res. of length $\leq |\vec{n}| = n_1 + n_2 + \dots + n_r$

v. HBT
[B-Erman-Smith]: $X = \underline{\mathbb{P}^1 \times \mathbb{P}^1} \supseteq Y$ coll of points
Then $S/I(Y)$ has a v. res. of form

$$S \xleftarrow{J} F_1 \xleftarrow{M} F_2 \leftarrow 0$$

w/ $J = \langle \text{minors of } M \rangle$.

Cor: vHBT also holds for X toric surface $\supseteq$ Y general pts.

[Duarte-Seceleanu]: use v.res. to compute residual resultants over $\mathbb{P}^1 \times \mathbb{P}^1$.

$\mathbb{P}^{\vec{n}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ $\quad S = \mathrm{Cox}(\mathbb{P}^{\vec{n}})$

M B-saturated f.g. $\mathbb{Z}^r$-grd S-mod., $\vec{d} \in \mathbb{Z}^r$

[Maclagan-Smith]: M is $\vec{d}$-regular if

$$\left[H^i_B(M)\right]_{\vec{p}} = 0 \quad \forall i \geq 1, \ \vec{p} \in \bigcup_{\substack{\vec{q} \in \mathbb{N}^r \\ |\vec{q}| = i-1}} \left(\vec{d} - \vec{q} + \mathbb{N}^r\right)$$

$\Delta_i \subseteq \mathbb{Z}^r$ denotes the set of twists of summands in the i^{th} step of mfr of S/B.

[Bayer-Sturmfels]: $\Delta_0 = \{\vec{0}\}$ & $\forall i \geq 1$,

$$\Delta_i = \left\{ -\vec{a} \in \mathbb{Z}^r \;\middle|\; \vec{0} \leq \vec{a} - \vec{1} \leq \vec{n}, \ |\vec{a}| = r + i - 1 \right\}$$

Regularity & Linearity:

[B-Erman-Smith]: $\vec{d} \in \mathbb{Z}^r$, over $\mathbb{P}^{\vec{n}}$, M as above

Then M is $\vec{d}$-regular $\Longleftrightarrow$ $M(\vec{d})$ has a v.res. of the form

$$F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots \leftarrow F_{|\vec{n}|} \leftarrow 0$$

& $\forall\, 0 \leq i \leq |\vec{n}|$:

degree of each generator of F_i is in $\Delta_i + \mathbb{N}^r$.

[B-Ermen-Smith]: $Y \subseteq \mathbb{P}^{\vec{n}} = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$ w/

gens. of $I = I(Y)$ in degrees $\vec{d}_1, \vec{d}_2, \dots, \vec{d}_s$

with $[S/I]_{\vec{d}_i} \xrightarrow{\sim} H^0(Y, \mathcal{O}_Y(\vec{d}_i))$ $\forall\ 1 \leq i \leq s$. }

If one of:

(i) $\operatorname{codim}(Y) = 2$ & $\exists d$ s.t. S/I is $\vec{d}$-regular & vResPair$(S/I, \vec{d})$ has length 2

(ii) — " — 3 ———— " ———— has length 3 & is self-dual.

(iii) vResPair$(S/I, \vec{d})$ is a Koszul complex,

then:

the embedded deformation theory of $Y \subseteq \mathbb{P}^{\vec{n}}$ is unobstructed.

What makes a complex ~~exact~~? a virtual res?

[Buchsbaum - Eisenbud]: [Loper]

A grd. free chain complex

$$F: \quad F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} F_2 \leftarrow \cdots \xleftarrow{\varphi_t} F_t \leftarrow 0$$

over $S = \operatorname{Cox}(\cancel{\mathbb{P}^n})$ is ~~exact~~ virtual
X sm. proj. TV

$\iff$ both of these hold:

(i) $\operatorname{rank}(\varphi_i) + \operatorname{rank}(\varphi_{i+1}) = \operatorname{rank}(F_i)$

& (ii) $\operatorname{depth}(\underset{\substack{\uparrow \\ \text{rank-size} \\ \text{minors}}}{I}(\varphi_i) : B^\infty) \geq i$.

$\boxed{\text{Constructions}}$ M B-saturated f.g. grd. S-mod. $S = \operatorname{Cox}(X)$

⓪ Naïve: insert components of B into $\mathrm{im}(\varphi_i)$ & resolve tail

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① Resolution of diagonal (Pf. v. HST)

$$X = \mathbb{P}^{\vec{n}} \qquad K_i = \bigoplus_{\vec{0} \leq \vec{a} \leq \vec{n}} \mathcal{O}_X(\vec{a}) \boxtimes \Big(\underbrace{\Lambda^{\vec{a}} \Omega_X}_{\boxtimes_{i=1}^{r} \Omega_{\mathbb{P}^{n_i}}(a_0)} \Big)(\vec{a})$$

$$\begin{tikzcd} & X \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & X \end{tikzcd}$$

For $\vec{d} >> \vec{0}$,

$$\mathbb{R}\pi_{1*}\left(\pi_2^* \underbrace{\tilde{M}(\vec{d})}_{} \underset{X \times X}{\otimes} K_\bullet \right)$$

[B-Erman-Smith]:

M has a v.res. F w/

$$F_i = \bigoplus_{\substack{0 \leq \vec{u} \leq \vec{n} \\ |\vec{u}| = i}} S(-\vec{u}) \otimes_{\mathbb{k}} \underbrace{H^q\left(\mathbb{P}^{\vec{n}}, \Omega^{\vec{u}}_{\mathbb{P}^{\vec{n}}} \otimes \tilde{M}(\vec{u} + \vec{d})\right)}_{}$$

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② vResPair: $\mathbb{P}^{\vec{n}}$, $\vec{d} \in \mathbb{Z}^r$

vResPair$(M, \vec{d} + \vec{n})$ is subcomplex of m.f.r. of M w/ summands generated in deg $\leq \vec{d} + \vec{n}$.

[B-Erman-Smith]: If M is $\vec{d}$-regular, then vResPair$(M, \vec{d} + \vec{n})$ is v.res. of M.

Idea:

$$0 \to \mathrm{vResPair}(M, \vec{d}+\vec{n}) \to \overset{\text{m.f.r.}}{F_\bullet} \to \underset{\text{irrel. hom.}}{\boxed{E}} \to 0$$

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③ Monomial Ideals

[Yang]: X sm. proj. TV, $S = \mathrm{Cox}(X)$
$I \neq S$ B-saturated monomial ideal
$\Longrightarrow$ $\exists$ v.res. of S/I of length $\leq \dim(X)$.

[Kenshur-Lin-McNally-Xu-Yu]: $X = \mathbb{P}^{\vec{n}}$
Δ: simplicial complex, pure, balanced (each facet touchs each $\mathbb{P}^{n_i}$)
$\Longrightarrow$ S/I_Δ has v.res. of length $= \mathrm{codim}(S/I_\Delta)$

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④ Mapping Cone Construction

any X

F: mfr of M of length t & $\mathrm{Ext}^t_S(M,S) : B^\infty = 0$

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & F_0^* \arrow[r] \arrow[d] & F_1^* \arrow[r] \arrow[d] & \cdots \arrow[r] & F_{t-2}^* \arrow[r] \arrow[d, "\alpha^*_{t-2}"] & F_{t-1}^* \arrow[r, "\varphi_t^*"] \arrow[d, equal, "\alpha^*_{t-1}"] & F_t^* \arrow[r] \arrow[d, equal, "\alpha^*_t"] & 0 \\
0 \arrow[r] & G_{-1}^* \arrow[r] & G_0^* \arrow[r] & G_1^* \arrow[r] & \cdots \arrow[r] & G_{t-2}^* \arrow[r] & G_{t-1}^* \arrow[r, "\underline{\varphi_t^*}"'] & G_t^* \arrow[r] & 0
\end{tikzcd}$$

if this is 0

$\Longrightarrow$ $\mathrm{cone}(\alpha)$ is a vres of M:

$$\begin{tikzcd}
\begin{matrix} F_0 \\ \oplus \\ G_{-1} \end{matrix} & \begin{matrix} F_1 \\ \oplus \\ G_0 \end{matrix} \arrow[l] & \cdots \arrow[l] & \begin{matrix} F_{t-2} \\ \oplus \\ G_{t-3} \end{matrix} \arrow[l] & G_{t-2} \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

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⑤ Points:

[B-Erman-Smith]: $Z \subseteq \mathbb{P}^{\vec{n}}$ zero-dim'l scheme w/ $I = I(Z)$

Then $\underline{\vec{a}} \in \mathbb{N}^r$ w/ $\underline{\underline{a_r = 0}}$ s.t.

m.f.r. of $S/(I \cap B^{\vec{a}})$ has length $= |\vec{n}|$

[Gao-Li-Loper-Mattoo]: points in $\mathbb{P}^1 \times \mathbb{P}^1$

VCI?

Open Problems:

$$\left.\begin{array}{l} \cdot\ \text{vGor.} \\ \cdot\ \text{VCM} \\ \cdot\ \text{VCI} \end{array}\right\} \text{defns, tests}$$

*vHST