Boundedness questions for polynomials in many variables

Daniel Erman (Wisconsin)
Steven Sam (UCSD)
Andrew Snowden (Michigan)

May 14, 2020: Zoom

Hilbert's Landmark Theorems (1890s)

Throughout: $\mathbf{k}$ is a field and $S = \mathbf{k}[x_1, \ldots, x_n]$. All ideals, modules, etc. will be homogeneous.

- **Hilbert Basis Theorem**: S is noetherian (i.e. every ideal in S is finitely generated).
- **Hilbert Syzygy Theorem**: every module has a free resolution

$$F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_n \leftarrow 0$$

  of length $\leq n$.

These yield a huge array of finiteness results in modern algebra.

Question

Are there interesting analogues of these results as $n \to \infty$?

Projective Dimension

Any (graded) S-module M has a free resolution:

$$F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} F_2 \leftarrow \cdots$$

Each F_i is free; $\operatorname{coker} \phi_1 = M$; and $\operatorname{image} \partial_{i+1} = \ker \partial_i$ for all $i \geq 1$.

Definition

The projective dimension of M, denoted $\operatorname{pdim}(M)$ is

$$\min \left\{ \begin{array}{c} p \text{ where M has a free resolution} \\ \text{of the form } F_0 \leftarrow \cdots \leftarrow F_p \leftarrow 0 \end{array} \right\} \quad (\text{or } \infty).$$

Theorem (Hilbert Syzygy Theorem, 1890)

For any S-module M, $\operatorname{pdim}(M) \leq n$.

Example

$S/(f)$ has free resolution $S \xleftarrow{f} S$; so $\operatorname{pdim}(S/(f)) = 1$.

What happens to Hilbert's Theorems as $n \to \infty$?

At first pass, nothing good seems to happen! Over $\mathbf{k}[x_1, x_2, \dots]$:

- **Basis Theorem fails**: $(x_1, x_2, \dots)$ requires infinitely many generators.
- **Syzygy Theorem fails**: there are ideals of arbitrarily large (or even ∞) projective dimension. E.g. $\operatorname{pdim}((x_1, x_2, \dots, x_r)) = r - 1$.

But what if we narrow the question, e.g.:

Question

Let I be an ideal generated by 5 cubic polynomials in 10^{10} variables. Hilbert's Syzygy Theorem says $\operatorname{pdim}(I) \leq 10^{10}$. Can we do better?

Goal: to describe some recent frameworks where analogues of Hilbert's results hold, even as $n \to \infty$.

Stillman's Conjecture

Stillman's Conjecture (Proven by Ananyan–Hochster 2015)

Let $f_1, \ldots, f_r$ be polynomials of degree $\leq d$. One can bound the projective dimension of $(f_1, \ldots, f_r)$ solely in terms of r and d.

This is a version of Hilbert's Syzygy Theorem as $n \to \infty$. Note:

Example

$\operatorname{pdim}(x_1, \ldots, x_r) = r$. (So bound must involve r.)

Example (Beder, McCullough, Nunez-Betancourt, Seceleanu, Snapp, Stone, 2011)

For any d, one can find polynomials f_1, f_2, f_3 of degree d where $\operatorname{pdim}((f_1, f_2, f_3)) \geq \sqrt{d}^{\sqrt{d}-1}$. (So bound must involve d.)

A notion of complexity for polynomials

The **strength** of a homogeneous polynomial f is the minimal s for which we can write $f = \sum_{i=0}^{s} g_i h_i$ with g_i and h_i of positive degree, or ∞ if no such decomposition exists.

The **collective strength** of $f_1, \ldots, f_r$ is the minimal strength of a homogeneous **k**-linear combination of the f_i.

Example

The strength $x_1 x_2 + x_3 x_4 + x_5 x_6$ is 2.

Example

A polynomial has strength ∞ if and only if it is a nonzero linear form.

This is a measure of complexity which can only go to ∞ if $n \to \infty$.

Example (Banks-Bruce, in progress)

A general quartic in 100 variables has strength 90. (Related: a general quartic hypersurface $V(Q) \subseteq \mathbb{P}^{99}$ contains a $\mathbb{P}^9$.)

Ananyan–Hochster Principle

Ananyan–Hochster Principle

If the collective strength of $f_1, \dots, f_r$ is sufficiently large (relative to d, r) then $f_1, \dots, f_r$ will behave approximately like independent variables.

The principle takes a fact about independent variables and predicts a corresponding statement for polynomials of high strength:

Fact about linear forms:		Predicted statement:
$x_1, \dots, x_r$ form a regular sequence	$\overset{AH\ Principle}{\Longrightarrow}$	Any $f_1, \dots, f_r$ of high enough strength form a reg. seq

The predicted statement could then be proven or disproven. (In the above case, it is a theorem of Ananyan–Hochster.)

Other instances (due to Ananyan–Hochster): algebraic independence; defining a prime ideal; defining a smooth variety ...

Results in the spirt of the Ananyan-Hochster Principle

Let $\mathbf{k} = \overline{\mathbf{k}}$. Let $f_1, \dots, f_r$ be polynomials of degree $\leq d$ in n variables. If the collective strength is sufficiently large then:

1. $f_1, \dots, f_r$ form a regular sequence (Ananyan-Hochster).
2. $V(f_1, \dots, f_r)$ is smooth in high codimension (Ananyan-Hochster, Kazhdan–Ziegler).
3. $V(f_1, \dots, f_r)$ has trivial Picard and Chow groups (Grothendieck, Paranjape, Esnault-Levine-Viehweg, ...).
4. $V(f_1)$ is unirational (Harris-Mazur-Pandharipande, Chen).
5. If $\mathbf{k} = \mathbb{Q}$, $V(f_1)$ satisfies the Hasse principle (Birch).

Question

Can we better describe when the Ananyan-Hochster Principle will/won't apply to a specific property?

Non-example: $(x_1, \dots, x_r)$ define a regular ring; but this can never happen for polynomials of higher degree.

Proof of Stillman's Conjecture I: setup

Lemma

Let $I \subseteq S = \mathbf{k}[x_1, x_2, \ldots]$. If the generators of I belong to a subalgebra $\mathbf{k}[g_1, \ldots, g_t]$ where $g_1, \ldots, g_t$ are a regular sequence, then $\operatorname{pdim}(I) \leq t$.

- Let $R = \mathbf{k}[z_1, \ldots, z_t]$. Let $\phi : R \to S$ given by $z_i \mapsto g_i$. This is faithfully flat since the g_i form a regular sequence.
- Since the generators of I lie in $\mathbf{k}[g_1, \ldots, g_t]$, we can find an ideal $J \subseteq R$ where $\phi(J) = I$.
- By flatness: $\operatorname{pdim}(J) = \operatorname{pdim}(\phi(J))$.

Proof of Stillman's Conjecture II: induction argument

Sample case: Start with $I = (f_1, f_2, f_3, f_4)$ with $\deg(f_i) = 3$.

At each step, we either have a regular sequence and are done, or we can replace one polynomial with lower degree polynomials.

- If the collective strength of f_1, f_2, f_3, f_4 is high enough, then they form a regular sequence and $\mathrm{pdim}(I) \leq 4$.
- If not, we can rewrite $f_4 = \sum_{i=0}^{N} g_i h_i$. Then:
  - If the collective of strength of $f_1, f_2, f_3, g_0, \ldots, g_N, h_0, \ldots, h_N$ is high enough then they form a regular sequence and $\mathrm{pdim}(I) \leq 2n + 5$.
  - If not, we can rewrite $f_3 = \sum_{i=0}^{N'} g'_i h'_i$. Then:
    - ★ If the new set of f, g, h, g', h' are a regular sequence, then $\mathrm{pdim}(I) \leq 2N + 2N' + 6 \ldots$
    - ★ If not, we can rewrite ...
  - If not, we can rewrite $h_N = \sum_{i=0}^{N''} g''_i h''_i$. Then: ...

Eventually this process terminates, yielding:

Proof of Stillman's Conjecture III: small subalgebras

Theorem (Ananyan-Hochster's Small Subalgebra Theorem)

Let $I \subseteq \mathbf{k}[x_1, x_2, \ldots]$ be an ideal generated by $\leq r$ polynomials of degree $\leq d$. There exists $s = s(r, d)$ (not depending on n) such that the generators of I lie in a subalgebra generated by a regular sequence of length $\leq s$.

This implies Stillman's Theorem by previous slides. It also explicitly connects Stillman's Conjecture to Hilbert's Syzygy Theorem.

Example (Two Quadrics)

For independent quadrics q_1, q_2 (in any number of variables):

- If q_1, q_2 is a regular sequence then they lie in $\mathbf{k}[q_1, q_2]$.
- If q_1, q_2 are not a regular sequence, then they must be reducible quadrics with a common factor. So $q_1 = \ell_1\ell_2$ and $q_2 = \ell_1\ell_3$ with ℓ_i linear. In this case, the subalgebra is $\mathbf{k}[\ell_1, \ell_2, \ell_3]$.

Limit Ananyan–Hochster Principle

Limit Ananyan–Hochster Principle

If $f_1, \ldots, f_r$ have infinite collective strength, then they should behave exactly like independent variables.

Let $\mathbf{k}[\![x_1, x_2, \ldots]\!]$ be the graded ring where

$$\mathbf{k}[\![x_1, x_2, \ldots]\!]_d = \left\{ \begin{matrix} \text{arbitrary } \mathbf{k}\text{-linear combinations of} \\ \text{degree } d \text{ monomials in } x_1, x_2, \ldots \end{matrix} \right\}$$

For example $\sum_{i=1}^{\infty} x_i^2$ is a degree two element.

- This is an inverse limit of the polynomial rings as $n \to \infty$.
- It contains new elements of strength ∞ like $\sum_{i=1}^{\infty} x_i^2$.
- Non-noetherian: even $\mathbf{k}[\![x_1, x_2, \ldots]\!]_1$ has an uncountable basis.
- Had appeared in Snellman's work on universal Gröbner bases.

Big polynomial rings

Theorem (E-Sam-Snowden, 2018)

The limit ring $\mathbf{k}[\![x_1, x_2, \ldots]\!]$ is isomorphic to a polynomial ring $\mathbf{k}[\mathcal{Z}]$ where $\mathcal{Z}$ is any maximal set of collective strength ∞.

$\mathcal{Z}$ contains uncountably many elements of degree d for each $d \geq 1$.

- Example: the power sums $\{\sum_{i=1}^{\infty} x_i^d\}_d$ have collective strength ∞.
- The theorem verifies the limit Ananyan–Hochster principle.
- New possibilities for Stillman's Conjecture, etc.
- One can define "universal polynomials" like $\sum c_{ij} x_i x_j$ to study universal Gröbner bases. (See work of Draisma, Laśon, Leykin).
- Corollary: Finitely generated ideals in $\mathbf{k}[\![x_1, x_2, \ldots]\!]$ have finite projective dimension.

Proof over $\mathbb{C}$:

By definition of $\mathcal{Z}$, we get a surjection:

$$\Phi : \mathbb{C}[\mathcal{Z}] \to \mathbb{C}[\![x_1, x_2, \dots]\!]$$

where $z_i \mapsto g_i$. Choose a kernel element of minimal degree. WLOG it only involves $z_1, \dots, z_r$. By considering this as a polynomial in z_r with coefficients in $\mathbb{C}[z_1, \dots, z_{r-1}]$, we get a nontrivial relation:

$$a_r g_r^n = \sum_{i=0}^{n-1} a_i g_r^i \qquad \text{in} \quad \mathbb{C}[\![x_1, x_2, \dots]\!]$$

where $a_i \in \mathbb{C}[g_1, \dots, g_{r-1}]$.

Now for the trick: take an appropriate partial derivative of both sides to get a relation of lower degree, yielding a contradiction.

(Subtlety: check that the new relation is nontrivial. Easy in characteristic 0. In characteristic p, use Hasse deriviatives.)

The ultraproduct ring

Let **S** be the graded ultraproduct of polynomial rings $\mathbf{k}[x_1, \dots, x_{n_i}]$ for $i \in \mathbb{N}$, where $n_i \to \infty$ as $i \to \infty$. A degree d element of **S** is a collection (f_i) of degree d polynomials in $\mathbf{k}[x_1, \dots, x_{n_i}]$.

Theorem (E-Sam-Snowden, 2018)

The ultraproduct ring **S** *is isomorphic to a polynomial ring. Variables correspond to sequences* (f_i) *where the strength of* f_i *is unbounded.*

Advantage of the ultraproduct limit is that we can work with *arbitrary* sequences of polynomials (f_i). Makes it easier to pass properties from the infinite strength case to the finite strength.

A Hilbert Basis Theorem as $n \to \infty$?

Topological Version: descending chains of Zariski closed subsets in $\mathbb{A}^n$ stabilize.

Fails for $\mathbb{A}^\infty$. But something similar is true if we study infinite dimensional affine spaces with large group actions.

Example

Let $X_1 = \mathrm{Spec}(\mathbf{k}[c_1, c_2, \ldots])$. X_1 corresponds to $\mathbf{k}[\![x_1, x_2, \ldots]\!]_1$ via $(c_1, c_2, \ldots) \mapsto \sum c_i x_i$, and this gives X_1 a GL_∞-action. Any descending chain of GL_∞-invariant, closed subsets of X_1 stabilizes.

Why? The only GL_∞-invariant closed subsets are X_1 and the origin.

Example

Let $X_2 = \{\mathbf{k}[\![x_1, x_2, \ldots]\!]_2\}$ with the induced GL_∞-action. Any descending chain of GL_∞-invariant, closed subsets of X_2 stabilizes.

Why? The only GL_∞-invariant closed subsets are the loci of quadrics of rank $\leq r$ for some r.

A Hilbert Basis Theorem as $n \to \infty$?

Sam–Snowden asked whether this held in a far more general setting. Draisma proved that it did.

Our setup: Let $X_d = \{\mathbf{k}[\![x_1, x_2, \dots]\!]_d\}$, endowed with the corresponding GL_∞-action. Let X_d^r be the product of r copies of X_d.

Theorem (Draisma)

X_d^r is GL_∞-noetherian, that is, any descending chain of GL_∞-invariant closed subsets of X_d^r stabilizes.[a]

[a] Draisma's actual result applies to other Schur functors, etc.

Many related results: S_∞-actions, *FI*-modules, twisted commutative algebras, stronger results in cases of quadrics and cubics, ...

Stillman's Conjecture and more via GL_∞-noetherianity

Write $X_{d,r} = \mathrm{Spec}(\mathbf{k}[c_{i,\alpha}])$. We can define universal "polynomials":

$$F_i = \sum_\alpha c_{i,\alpha} x^\alpha \in \mathbf{k}[c_{i,\alpha}][\![x_1, x_2, \ldots]\!].$$

We have a universal family $V(F_1, \ldots, F_r) \to X_{d,r}$.

- Over the generic point, $\mathbf{k}(c_{i,\alpha})[\![x_1, x_2, \ldots]\!]$ is a polynomial ring.
- This implies "generic flatness" for the universal family.
- We can thus build a (potentially infinite) flattening stratification for the universal family. We may assume each strata is GL_∞-invariant.
- Draisma's result then implies that the stratification is finite.

This provides an alternate proof of Stillman's Conjecture. Moreover:

Theorem (E-Sam-Snowden, 2018)

Any ideal invariant which is semicontinuous in flat families and stable under adjoining a variable can be bounded in terms of d and r.

Is there a “dual” of Stillman’s Conjecture?

Context 1: Koszul duality/BGG correspondence:

$$\left\{\begin{array}{c}\text{modules over}\\ \text{polynomial ring}\\ \mathrm{Sym}(V)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{modules over}\\ \text{exterior algebra}\\ \bigwedge(V^*)\end{array}\right\}$$

Is there a “dual” of Stillman’s Conjecture?

Context 1: Koszul duality/BGG correspondence:

$$\left\{\begin{array}{c}\text{modules over}\\ \text{polynomial ring}\\ \mathrm{Sym}(V)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{modules over}\\ \text{exterior algebra}\\ \bigwedge(V^*)\end{array}\right\}$$

Theorem (McCullough, 2019)

There is no “good” analogue of Ananyan and Hochster’s results for modules over exterior algebras.

Is there a "dual" of Stillman's Conjecture?

Context 1: Koszul duality/BGG correspondence:

$$\left\{\begin{array}{c}\text{modules over}\\ \text{polynomial ring}\\ \mathrm{Sym}(V)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{modules over}\\ \text{exterior algebra}\\ \bigwedge(V^*)\end{array}\right\}$$

Theorem (McCullough, 2019)

There is no "good" analogue of Ananyan and Hochster's results for modules over exterior algebras.

Context 2: Boij-Söderberg theory: (Eisenbud-Schreyer, 2006)

$$\left\{\begin{array}{c}\text{free resolutions of modules over}\\ \text{polynomial ring } \mathrm{Sym}(V)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{cohomology of}\\ \text{sheaves on } \mathbb{P}(V^*)\end{array}\right\}$$

Sheaf Cohomology Tables

Question

Are there finiteness bounds for sheaf cohomology similar to the results of Ananyan and Hochster?

Let $\mathcal{E}$ be a coherent sheaf on $\mathbb{P}^n$. Define $\gamma_{i,j}(\mathcal{E}) := \dim_{\mathbf{k}} H^i(\mathbb{P}^n, \mathcal{E}(j))$.

$$\gamma(\mathcal{E}) = \begin{pmatrix} \cdots & \gamma_{n,0} & \gamma_{n,1} & \gamma_{n,2} & \cdots \\ \cdots & \gamma_{n-1,1} & \gamma_{n-1,2} & \gamma_{n-1,3} & \cdots \\ & \vdots & \vdots & \vdots & \cdots \\ \cdots & \gamma_{0,n} & \gamma_{0,n+1} & \gamma_{0,n+2} & \cdots \end{pmatrix}$$

Example

$$\gamma(\mathcal{O}_{\mathbb{P}^2}) = \begin{pmatrix} \cdots & 6 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 3 & 6 & 10 & \cdots \end{pmatrix}$$

Main Theorem: Finiteness of Cohomology Tables

Theorem (E-Sam-Snowden, 2019)

Let $\mathbf{b}$ *and* $\mathbf{b}'$ *be column vectors and fix any* k*. Even as* $n \to \infty$*, there are only finitely many cohomology tables* $\gamma(\mathcal{E})$ *whose* k *and* $(k+1)$*'st columns are* $\mathbf{b}$ *and* $\mathbf{b}'$*.*

$$\gamma(\mathcal{E}) = \begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \gamma_{k,0} & \gamma_{k,1} & \gamma_{k,2} & \cdots \\ \cdots & \gamma_{k-1,1} & \gamma_{k-1,2} & \gamma_{k-1,3} & \cdots \\ & \vdots & \vdots & \vdots & \ldots \\ \cdots & \gamma_{0,k} & \gamma_{0,k+1} & \gamma_{0,k+2} & \cdots \end{pmatrix}$$

Note: this parallels an alternate version of Stillman's Conjecture, which is phrased in terms of the Betti table of the ideal.

Small Projective Spaces

Parallel of Ananyan-Hochster's Small Subalgebra Theorem:

Theorem (E-Sam-Snowden, 2019)

Fix $\mathbf{b}$ *and* $\mathbf{b}'$ *and* k. *Then there exists* $c = c(\mathbf{b}, \mathbf{b}')$ *with the following property: any sheaf* $\mathcal{E}$ *where the* k *and* $(k+1)$*'st columns of* $\gamma(\mathcal{E})$ *are* $\mathbf{b}$ *and* $\mathbf{b}'$ *is the pushforward of a sheaf on* $\mathbb{P}^c$ *via an embedding* $\mathbb{P}^c \to \mathbb{P}^n$.

- While c depends only on $\mathbf{b}$ and $\mathbf{b}'$, but $\mathbb{P}^c \to \mathbb{P}^n$ depends on $\mathcal{E}$.
- Cohomology is stable under proper maps, so this implies the boundedness result on cohomology tables.

Conclusion: What next?

These results open the door to many more questions:

- Effective questions: (Work of McCullough, Mantero-McCullough, Ananyan–Hochster, and many more)
- GL_∞-noetherianity beyond topological statements? (Connected to FI-modules, twisted commutative algebra, and more)
- Geometric applications? (Applications to Hartshorne's Conjecture, Fano varieties, $\ldots$)
- How strength relates to other notions of complexity: (Bik–Draisma–Eggermont, Kazhdan–Ziegler)
- Multigraded questions (Nobody yet?)
- How to compute strength in actual examples (Nobody? Chen?)