Commutative Algebra with $\mathfrak{S}_n$-invariant monomial ideals

$\Bbbk$ = a field

$S = \Bbbk[e_1, \dots, e_n]$

$e^{\underline{x}} = e_1^{x_1} \cdots e_n^{x_n}$ monomial

$\underline{x}$ = exponent vector

Fix subgroup $G \subseteq GL_n(\Bbbk)$ and let it act on S by linear change of coords.

$\boxed{\text{Problem 1}}$ Classify G-invariant ideals $I \subseteq S$.

$\boxed{\text{Problem 2}}$ Given any such I, describe basic invariants:

- Castelnuovo-Mumford regularity
- projective dimension
- Cohen-Macaulay property
- Betti numbers
- ⋮

<u>Today</u> $G = (k^\times)^n \rtimes \mathfrak{S}_n$ semi-direct product

· $(k^\times)^n$ <u>torus</u> acting by coordinate rescaling

· <u>$\mathfrak{S}_n$ symmetric group</u> acting by coordinate permutations

<u>Concretely:</u> G generalized permutation matrices

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 4 & 0 & 0 \end{pmatrix}$$

(exactly one non-zero entry in each row/column)

$\boxed{\text{Rmk}}$ G-invariant ideals $\Longleftrightarrow$ $\mathfrak{S}_n$-invariant monomial ideals

Basic Example $1 \le p \le n$, let

$$J_p = \langle e_{i_1} \cdots e_{i_p} \mid 1 \le i_1 < \dots < i_p \le n \rangle$$

Know a lot about J_p:

- $J_p = \sqrt{J_p}$ (only such G-ideals)
- Cuts out

$$Z_{p-1} = V(J_p) = \text{union of } (p-1)\text{-diml. coordinate planes}$$

- Betti numbers using Hochster's formula

- Cohen-Macaulay, linear resolution

$$\operatorname{reg}(J_p) = p, \quad \operatorname{pdim}(J_p) = n-p$$

- **Boochur**:

$$J_p = \text{maximal minors} \begin{bmatrix} \times e_1 & \times e_2 & \cdots & \times e_n \\ & & & \\ \times e_1 & \times e_2 & \cdots & \times e_n \end{bmatrix} \quad p \times n$$

$\times$ = random coeffs.

minimal resolution = Eagon-Northcott complex

- **Galetto '16**: describes $\operatorname{Tor}^S_\bullet(J_p, k)$ as σ_n-representations.

- **Mustaţă '00**

  Yanagawa '00

$$\operatorname{Ext}^i_S(J_p, S) \longleftrightarrow \operatorname{Tor}(J_{n-p+1}, k)$$

Alexander dual

Classification For $\underline{x} \in \mathbb{Z}^n_{\geq 0}$ and let

$$I_{\underline{x}} = \langle \sigma(e^{\underline{x}}) : \sigma \in \mathfrak{S}_n \rangle$$ "principal G-ideals"

Bayer–Sturmfels '98: permutohedron ideals, when x_i distinct
minimal cellular resolution.

Kumar–Kumar '13: Betti numbers for general $\underline{x}$.

$\boxed{\text{WLOG}}$ $x_1 \geq x_2 \geq \dots \geq x_n$, i.e. $\underline{x}$ is a partition,
picture via Young diagram: $(4,2,2,1) \longleftrightarrow$ 

Exercise: $I_{\underline{x}} \subseteq I_{\underline{y}} \iff \underline{x} \geq \underline{y} \overset{\text{def}}{\iff} x_i \geq y_i$

Notation: $\mathcal{P}_n = \{\text{partitions with} \leq n \text{ parts}\}$, with order

Def. $\mathcal{X} \subseteq \mathcal{P}_n$ subset, let

$$I_{\mathcal{X}} = \sum_{\underline{x} \in \mathcal{X}} I_{\underline{x}}$$

Exercise (solution to P1)

ⓐ Every G-ideal is $I_{\mathcal{X}}$.

ⓑ $I_{\mathcal{X}} = I_{\mathcal{Y}} \iff \mathcal{X}, \mathcal{Y}$ contain same minimal partitions

$\boxed{\text{Caution}}$ Conclusions may fail when $|k| < \infty$: either

ⓘ k = infinite

ⓘⓘ G = group scheme

Running Examples: $n=3$, $\mathcal{X} = \{$  ,  $\} = \{(211), (42)\}$

$$I_{\mathcal{X}} = \langle e_1^2 e_2 e_3,\ e_1 e_2^2 e_3,\ e_1 e_2 e_3^2,\ e_1^4 e_2^2,\ e_1^4 e_3^2,\ e_2^4 e_1^2,\ e_2^4 e_3^2,\ e_3^4 e_1^2,\ e_3^4 e_2^2 \rangle$$

Betti table (M2):

	0	1	$\boxed{2}$ ← projective dimension
4	3	3	1
5	–	–	–
6	6	6	–
regularity → $\boxed{7}$	–	3	3

<u>Exercise</u> $\sqrt{I_{\mathcal{X}}} = J_p$, where $p =$ least number of parts of a partition in $\mathcal{X}$

$$\dim\left(S/I_{\mathcal{X}}\right) = \dim V(I_{\mathcal{X}}) = p-1.$$

[In example $\mathcal{X} = \{(211), (421\}$, $p=2$

$V(I_{\mathcal{X}}) =$  dim 1]

$\boxed{\text{Def}}$ Say $I_{\mathcal{X}}$ is Cohen-Macaulay if

$$\operatorname{pdim}\left(S/I_{\mathcal{X}}\right) = 1 + \operatorname{pdim}(I_{\mathcal{X}}) = n - \dim V(I_{\mathcal{X}}) = \operatorname{codim} V(I_{\mathcal{X}})$$

test: $1+2 \neq 3-1$ $I_{\mathcal{X}}$ NOT C-M.

Strategy to analyze $\left\{\begin{array}{l}\text{regularity}\\ \text{proj. dim}\\ \text{C-M property}\end{array}\right]$ independent on char(k)

is to compute

$$\mathrm{Ext}_S^{\bullet}(I,S) \text{ and use}$$

$$\mathrm{pdim}(I) = \max\{j \mid \mathrm{Ext}^j(I,S) \neq 0\}$$

$$\mathrm{reg}(I) = \max\{-j-r \mid \mathrm{Ext}^j(I,S)_r \neq 0\}$$

$\boxed{\text{Rmk}}$ Betti numbers are hard to compute. Examples of Murai show they may depend on char(k).

To compute Ext, break it up into pieces, by constructing a filtration (finite)

$$S = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_\varkappa$$

with

$$\mathrm{Ext}^{\cdot}_S(S/I_\varkappa, S) \cong \bigoplus_t \underbrace{\mathrm{Ext}(I_t/I_{t+1}, S)}_{\text{manageable}}$$

as graded vector spaces

Index filtration by $\mathcal{Z}(\varkappa)$ consisting of pairs $(\underline{z}, \ell)$

$\underline{z} \in P_n$ (combinatorial)

$\ell \geq 0$ (geometric) $\left(\ell = \dim \frac{I_t}{I_{t+1}}\right)$

I_t/I_{t+1} Cohen-Macaulay

know reg, pdim

Theorem If $I_{\mathcal{X}} \neq S$ then

$$\operatorname{reg}(I_{\mathcal{X}}) = \max\{ |\underline{z}| + \ell + 1 \mid (\underline{z}, \ell) \in \mathcal{Z}(\mathcal{X}) \}$$

$$\operatorname{pdim}(I_{\mathcal{X}}) = \max\{ n - 1 - \ell \mid (\underline{z}, \ell) \in \mathcal{Z}(\mathcal{X}) \}$$

$\boxed{\text{Example}}$ $n = 3$, $\mathcal{X} = \{ (211), (42) \}$

$(\underline{z}, \ell)$	$(\emptyset, 1)$	( , 1)	( , 0)	( , 0)	( , 0)	( , 0)
$\lvert\underline{z}\rvert + \ell + 1$	2	4	4	5	6	$\boxed{7}$
$n - 1 - \ell$	1	1	2	2	2	$\boxed{2}$

<u>Example</u> $n=3$, $\mathfrak{X} = \{ \ydiagram{1,1,1}, \ydiagram{2,2} \}$

$$I_{\mathfrak{X}} = \langle e_1 e_2 e_3, e_1^2 e_2^2, e_1^2 e_3^2, e_2^2 e_3^2 \rangle$$

$$= \langle e_1, e_2 \rangle^2 \cap \langle e_1, e_3 \rangle^2 \cap \langle e_2, e_3 \rangle^2$$

$$= J_2^{(2)} \quad \text{symbolic square}$$

<u>Filtration:</u> $S \supseteq J_2 \supseteq J_2^{(2)}$ $\qquad S/J_2$ ✓

$$\frac{J_2}{J_2^{(2)}} = \frac{\langle e_1 e_2, e_1 e_3, e_2 e_3 \rangle}{\langle e_1 e_2 e_3, \ldots \rangle} = e_1 e_2 \frac{S}{\langle e_1, e_2, e_3 \rangle} \oplus e_1 e_3 \frac{S}{\langle e_1, e_3, e_2 \rangle} \oplus e_2 e_3 \frac{S}{\langle e_2, e_3, e_1 \rangle}$$

twist

$$\frac{S}{\langle e_1, e_2, e_3 \rangle} \cong \frac{k[e_1, e_2]}{\langle e_1, e_2 \rangle}$$

$$\Lambda \subseteq \{1, \ldots n\}$$

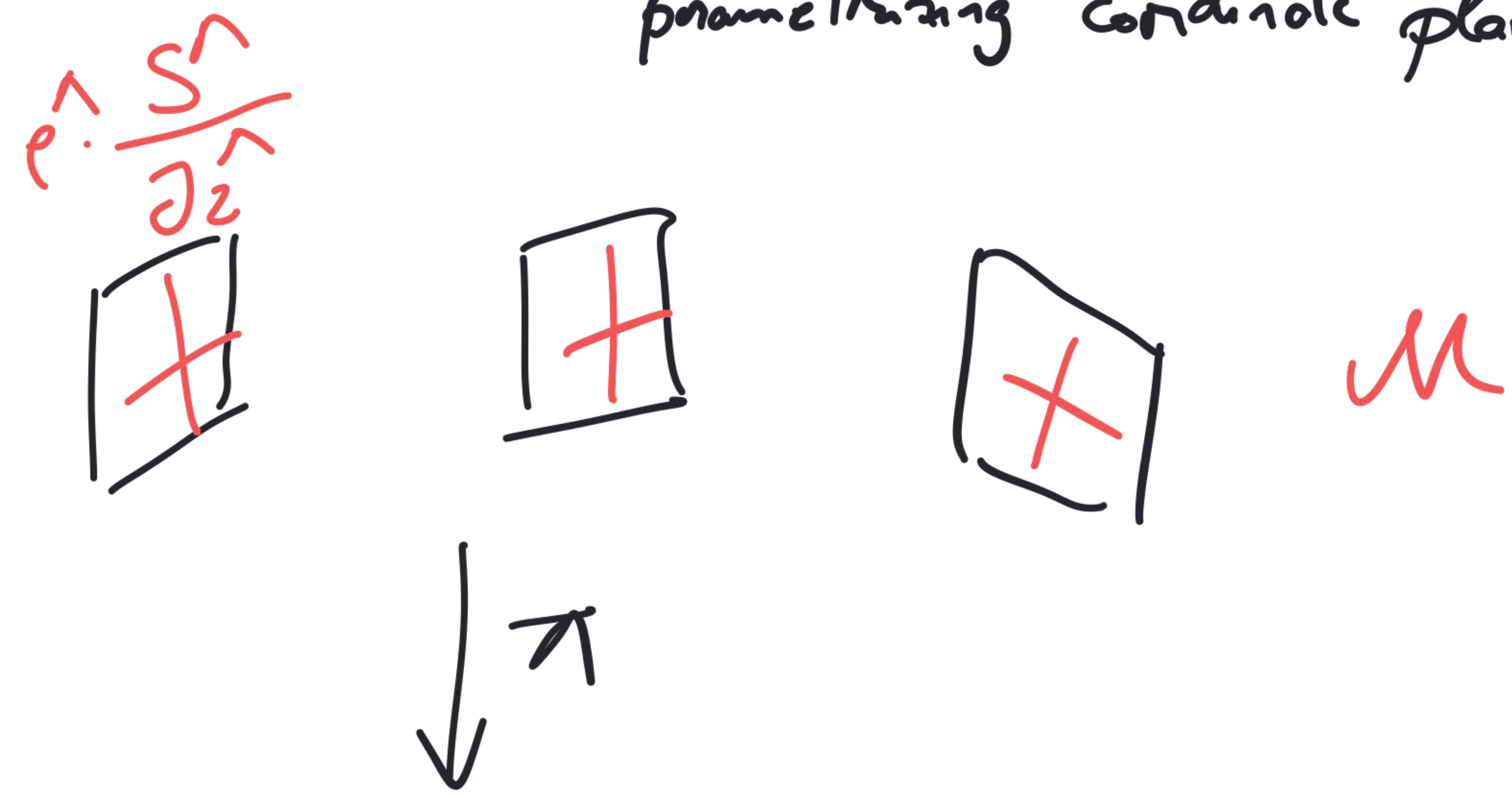
$$S^\Lambda = \frac{S}{\langle e_i \mid i \notin \Lambda \rangle} \supseteq J_p^\Lambda$$

$$\| \quad \bigoplus_{|\Lambda|=2} e^\Lambda \cdot \frac{S^\Lambda}{J_2^\Lambda} \; ✓$$

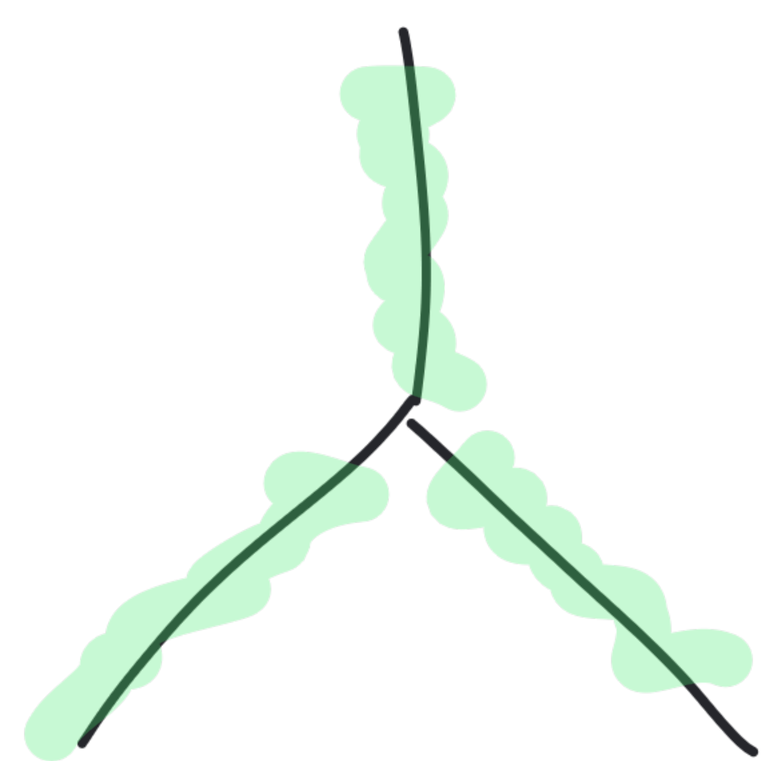
understand

Geometrically: each $\Lambda \longleftrightarrow$ point in discrete Grassmannian parametrizing coordinate planes

$\mathcal{V}$ = universal bundle



Ext calculation can be done upstairs.



$$\frac{J_2}{J_2^{(2)}} = \pi_* \mathcal{M}$$

S/J_2 , $J_2/J_2^{(2)}$: Cohen-Macaulay dim 1 ...

<u>S.e.s:</u>

$$0 \to \frac{J_2}{J_2^{(2)}} \longrightarrow \frac{S}{J_2^{(2)}} \longrightarrow \frac{S}{J_2} \longrightarrow 0$$

<u>l.e.s</u>

$$\longrightarrow \mathrm{Ext}^j\left(\frac{S}{J_2}, S\right) \longrightarrow \boxed{\mathrm{Ext}^j\left(\frac{S}{J_2^{(2)}}, S\right)} \longrightarrow \mathrm{Ext}_S^j\left(\frac{J_2}{J_2^{(2)}}, S\right) \longrightarrow$$

$\longrightarrow$

$\neq 0$ only for $j = 2$.

Similar filtrations for general $I_{\mathcal{X}}$

indexed $\boxed{Z(\mathcal{X})}$

<u>Example</u>: $\mathcal{X} = \{$ ,  $\}$

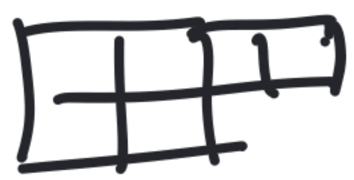
$Z(\mathcal{X}) = \{ (\varnothing, 1),$ (, $1) \}$

$$\frac{S}{\partial_2} \qquad\qquad \frac{\partial_2}{\partial_2^{(2)}}$$

To establish which $(\underline{z}, \ell)$ in $\mathcal{Z}(\mathcal{X})$, need:

$\boxed{\text{truncation}}$ if $\underline{x} \in \mathcal{P}_n$, $c \geq 0$

$\underline{x}(c)$ = first c columns of $\underline{x}$

 $(2) =$ 

$\boxed{\text{transpose}}$ $\underline{x}'$ of $\underline{x}$, rows $\longleftrightarrow$ columns

 $' =$ 

Take any $\underline{z} \in \mathcal{P}_n$ (fixed $\mathcal{X}$)

define $c = z_1$ and

$$\ell = \min \left\{ x'_{c+1} - 1 \;\middle|\; \begin{array}{c} \underline{x} \in \mathcal{X} \\ \underline{x}(c) \leq \underline{z} \end{array} \right\}$$

if $\nearrow$ nonempty and

$\ell \geq 0$

then

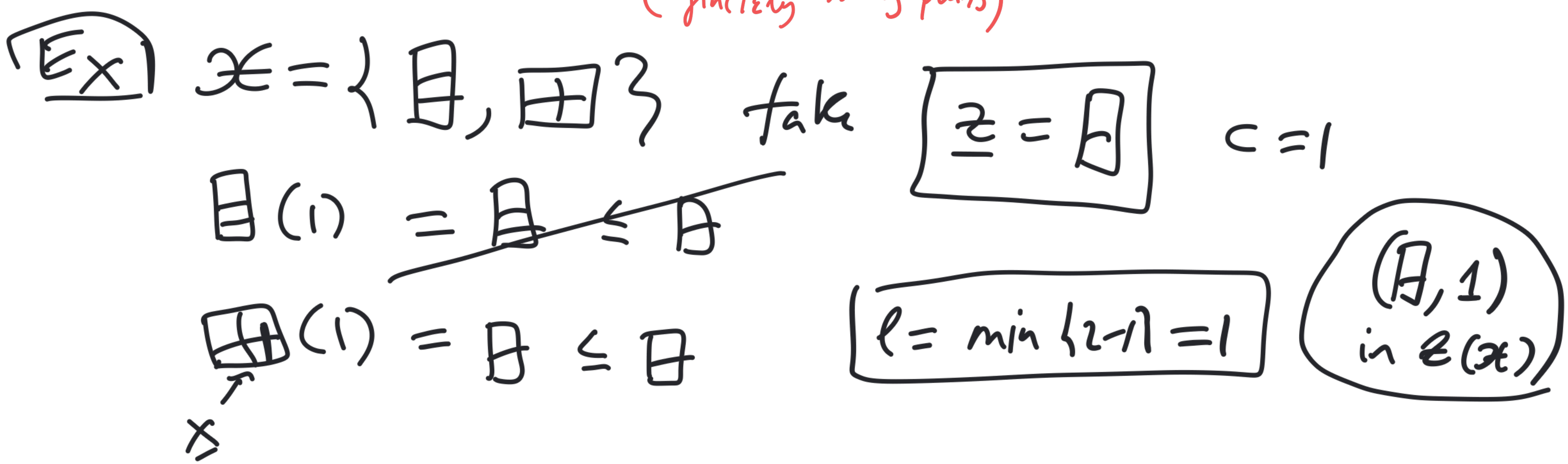
put $(\underline{z}, \ell)$ in $\mathcal{Z}(\mathcal{X})$

(finitely many pairs)

Exercise

Check $\mathcal{Z}(\mathcal{X})$ when

$\mathcal{X} = \{(211), (42)\}$



$\boxed{\text{Rmk}}$ Since we allow non-radical ideals, the corresponding schemes $V(I_{\mathcal{X}})$ may have embedded components.

$\boxed{\text{Theorem}}$ Let $\mathcal{X} \subseteq \mathcal{P}_n$ be a set of incomparable partitions.

Let $p = \dim V(I_{\mathcal{X}}) + 1$ (so that $\sqrt{I_{\mathcal{X}}} = J_p$)

TFAE:

1. $I_{\mathcal{X}}$ is Cohen-Macaulay
2. no embedded components ($I_{\mathcal{X}}$ unmixed)

] geometric

3. $(*)$ Every $\underline{x} \in \mathcal{X}$ satisfy $x_1 = \ldots = x_p$
4. Every $(\underline{x}, \ell) \in Z(\mathcal{X})$ satisfies $\ell = p-1$.

] combinatorial

5. $\operatorname{Ext}(S/I_{\mathcal{X}}, S) \hookrightarrow H_{I_{\mathcal{X}}}(S)$ injective

Eisenbud – Mustaţă – Stillman '00

$S/I_{\mathcal{X}}$ cohomologically full $\leftarrow$ Dao – De Stefani – Ma '18

Examples

$\mathcal{X} = \{$  $\}$ $\boxed{p=2}$

NOT C-M

$\mathcal{X} = \{$  $\}$

✓ C-M

Que How to remove embedded cpts?

Answer Saturation

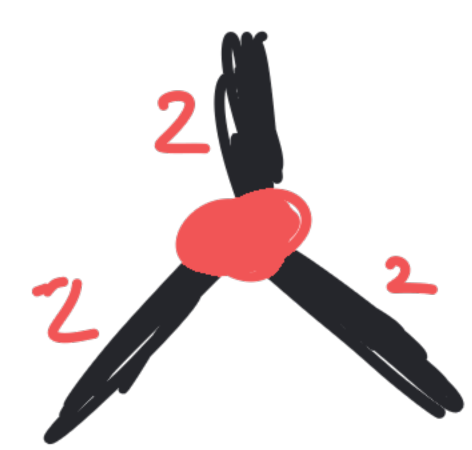
$$I_{\mathfrak{X}} : J_i^{\infty} \qquad \text{eliminates cpts. in } V(J_i)$$

$$\|$$

$$I_{\mathfrak{X}^{:i}} \qquad \mathfrak{X}^{:i} = \{ \text{remove columns of size} \le i \text{ from each } \lambda \in \mathfrak{X} \}$$

Ex: $\mathfrak{X} = \{$  $\}$

$\mathfrak{X}^{:1} = \{$  $\} \leftrightarrow J_2^{(2)}$

$V(I_{\mathfrak{X}})$ 

Moreover:

$$\mathcal{Z}(x^{:i}) = \{ (\underline{z}, \ell) \in \mathcal{Z}(x) \mid \ell \geq i \}$$

<u>Example</u> $x = \{(211), (42)\}$

$$\mathcal{Z}(x) = \{ (\emptyset, 1), (\ydiagram{1,1}, 1), \cancel{(\ydiagram{1,1}, 0)}, \ldots \cancel{(\ydiagram{3,3}, 0)} \}$$

$$\mathcal{Z}(x^{:1}) = \{ (\emptyset, 1) (\ydiagram{1,1}, 1) \}$$

Furthermore:

- $S/I_{\mathfrak{X}}$ is sequentially C-M (Stanley)

$$S \supseteq \dots \supseteq I_{\mathfrak{X}:2} \supseteq I_{\mathfrak{X}:1} \supseteq I_{\mathfrak{X}}$$

- characterise $I_{\mathfrak{X}}$ have linear resolution

Answer: Symmetric shifted

Biermann - De Alba - Galetto - Murai - Nagel - O'Keefe - Römer - Seceleanu '19

- $\operatorname{reg}(I_{\underline{x}}^{d}) = \underline{\underline{a}}d + \underline{b}$
- $\operatorname{reg}(I_{\mathfrak{X},n})$ $I_{\mathfrak{X},n} \subseteq k[e_1, \dots e_n]$

linear in n