Symbolic powers, stable containments, and degree bounds
Fellowship of the ring, 28/05/2020

<u>Question</u> What is the smallest degree of a homogeneous $f \in k[x_0, \dots, x_N]$ vanishing to order n on a variety $X \subsetneq \mathbb{P}^n_k$?
(k algebraically closed/perfect)

$$X \subsetneq \mathbb{P}^N \rightsquigarrow I = I(X) \subseteq k[x_0, \dots, x_N]$$
homogeneous radical ideal
of all f vanishing along X

$\alpha(I)$ = minimal degree of a homogeneous $0 \neq f \in I$

What are the polynomials that vanish to order n along X?
I radical ideal in R (a any excellent regular ring)
$I = P_1 \cap \dots \cap P_k$ $\quad P_i$ prime
The n-th symbolic power of I is
$$I^{(n)} = \bigcap_{P \in \mathrm{Min}(R/I)} \left(I^n R_P \cap R \right) = \bigcap_i \left(I^n R_{P_i} \cap R \right)$$
= minimal components in a primary decomposition of I^n
$= \{ r \in R : sr \in I^n, \text{ for some } s \notin \bigcup_i P_i \}$

$X = \{p_1, \dots, p_s\}$ points in $\mathbb{A}^N_k$ or $\mathbb{P}^N_k$. then

$$I(X)^{(n)} = I(p_1)^n \cap \dots \cap I(p_s)^n$$

<u>theorem</u> (Zariski – Nagata) $k = \bar{k}$, I radical in $R = k[\underline{x}]$

$$I^{(n)} = \bigcap_{\underline{\alpha} \in V(I)} (x_1 - \alpha_1, \dots, x_N - \alpha_N)^n$$

= elements that vanish to order n along X

<u>Facts</u>: 1) $I^{(1)} = I^1$

2) $I^{(n+1)} \subseteq I^{(n)}$

3) $I^n \subseteq I^{(n)}$

4) If $I = $ (regular sequence), $I^{(n)} = I^n$ for all n.

<u>Warning</u> In general, $I^{(n)} \neq I^n$, and finding $I^{(n)}$ is hard.

5) $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$

Can form the <u>symbolic Rees algebra</u> of I

$$\bigoplus I^{(n)} t^n \subseteq R[t]$$

<u>Warning</u> Not always finitely generated!

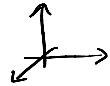
so $I^{(n)}$ could have unexpected elements for infinitely many n.

<u>Examples</u>

1) $P = \ker(k[x,y,z] \to k[t^a, t^b, t^c])$ prime of height 2

$\bigoplus P^{(n)}$ can be infinitely generated, or generated in degrees $\leq 1, 2, 3, 4, \dots$

2) $I = (xy, xz, yz) \subseteq k[x,y,z]$
$= (x,y) \cap (y,z), (x,z)$



$$I^{(3)} \subseteq I^2 \subsetneq I^{(2)} = (x,y)^2 \cap (y,z)^2 \cap (x,z)^2 \ni xyz$$

$$\alpha(I^2) = 4 \quad \text{but} \quad \alpha(xyz) = 3$$

3) X 3×3 generic matrix
$R = k[X]$
$I = I_2(X)$ = ideal of 2×2 minors of X
$f = \det(X) \in I^{(2)}$, but $f \notin I^2$

How do we find lower bounds for $\alpha(I^{(n)})$?

$$I^{(a)} I^{(b)} \subseteq I^{(a+b)} \qquad \text{for all } a, b$$

$$\Downarrow$$

$$\alpha(I^{(a)}) + \alpha(I^{(b)}) \geq \alpha(I^{(a+b)}) \qquad \text{for all } a, b$$

$\therefore$ $\alpha(I^{(-)})$ is subadditive

$\Downarrow$ Fekete's lemma

$$\lim_{n \to \infty} \frac{\alpha(I^{(n)})}{n} = \inf_n \frac{\alpha(I^{(n)})}{n} \quad (\geq 0)$$

(Fekete's lemma does not prevent $-\infty$, but our setting does)

Waldschmidt Constant $\hat{\alpha}(I) = \lim\limits_{n\to\infty} \dfrac{\alpha(I^{(n)})}{n}$

Notes: • $\alpha(I^{(n)}) \geq n\,\hat{\alpha}(I)$ for all n

• lower bounds for $\hat{\alpha}(I)$ $\Leftrightarrow$ lower bounds for all $\alpha(I^{(n)})$

Philosophy For lower bounds on $\alpha(I^{(n)})$ $\forall n$, enough to study $n \gg 0$

other idea $I^{(n)} \subseteq J \Rightarrow \alpha(I^{(n)}) \geq \alpha(J)$

Goal: Find a good J

Containment Problem When is $I^{(a)} \subseteq I^b$?

$$\begin{array}{c} I^{(a)} \\ \cup\!| \\ I^a \end{array}$$

(cofinal: Schenzel
lin. equiv: Swanson)

Setup R regular, excellent
I radical $I = P_1 \cap \dots \cap P_k$ P_i primes
h = big height of I = $\max\{\operatorname{ht} P_i\}$

theorem (Ein–Lazarsfeld–Smith 2001, Hochster–Huneke 2002, Ma–Schwede 2018)

$$I^{(hn)} \subseteq I^n \text{ for all } n \geq 1$$

$$\Rightarrow \quad I^{((\dim R)n)} \subseteq I^n \text{ for all } n \geq 1$$

Uniform Symbolic Topologies Problem R complete local domain
Is there a constant c (depending only on R) such that
$P^{(cn)} \subseteq P^n$ for all $n \geq 1$ and all primes P?
(cf. Huneke–Katz(–Validashti), R. Walker, Carvajal-Rojas–Smolkin)

Consequence $\alpha(I^{(hn)}) \geq n\,\alpha(I)$

$$\Downarrow$$

$$\hat{\alpha}(I) \geq \frac{\alpha(I)}{h} \quad (\text{Waldschmidt}_{1977},\ \text{Skoda}_{1977})$$

Example $I = (xy, xz, yz) \to h = 2$, so $I^{(2n)} \subseteq I^n$ for all n

eg, $I^{(4)} \subseteq I^2$ But actually, $I^{(3)} \subseteq I^2$

Question (Huneke, 2000) P prime of height 2 in a RLR. Is $P^{(3)} \subseteq P^2$?

Conjecture (Harbourne, 2008) $I^{(hn-h+1)} \subseteq I^n$ for all $n \geq 1$

Remark (see Hochster–Huneke) In char $p>0$, $n = q = p^e$

$$I^{(hq)} \subseteq I^{[q]} = (f^q \mid f \in I) \subseteq I^q$$

Proof

- Enough to show $I^{(hq)} R_Q \subseteq I^{[q]} R_Q$ for $Q \in \mathrm{Ass}(R/I^{[q]})$
- By a theorem of Kunz, $\mathrm{Ass}(R/I^{[q]}) = \mathrm{Ass}(R/I) = \{P_1, \ldots, P_k\}$
- $I R_{P_i} = P_i R_{P_i}$ maximal ideal in RLR of dim $\leq h$

ETS: $(x_1, \ldots, x_h)^{hq-h+1} \subseteq (x_1^q, \ldots, x_h^q)$

gen by $x_1^{a_1} \cdots x_h^{a_b}$, $a_1 + \cdots + a_h \geq hq - h + 1 \Rightarrow \exists a_i \geq q$

$\therefore$ $I^{(hq-h+1)} \subseteq I^{[q]}$ for all $q = p^e$

Counterexample (Dumnicki – Szemberg – Tutaj-Gasińska, Harbourne-Seceleanu)
2013 2015

$\exists$ radical ideal I, $h=2$ in $k[x,y,z]$, char $k \neq 2$
n^3+3 points in $\mathbb{P}^2$

$$I^{(3)} \not\subseteq I^2$$

But actually, $I^{(2n-1)} \subseteq I^n$ for all $n \geq 3$

Harbourne's Conjecture is satisfied by:

- General points in $\mathbb{P}^2$ (Bocci – Harbourne) and $\mathbb{P}^3$ (Dumnicki)
- squarefree monomial ideals
  $\cap$
- In char $p>0$, if R/I is F-pure (G-Huneke)
  equal char 0, if R/I is of dense F-pure type

strongly F-regular
- R/I Veronese $\cong k$[all monomials of deg d, v vars
- $I = I_t(X^{gen}_{n\times m})$, $R = k[X]$
- R/I ring of invariants of linearly reductive group

(G-Huneke) If R/I is strongly F-regular and $h \geq 2$, can replace h by $h-1$

$$h=2 \implies I^{(n)} = I^n \text{ for all } n \geq 1$$

(G-Ma-Schwede: a version of this result over Gorenstein rings)

Stable Harbourne Conjecture $\quad I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$.

Question Does it suffice to show $I^{(hk-h+1)} \subseteq I^k$ for some k?

Remark If that's sufficient, then Stable Harbourne holds in char k.

Theorem (G) If $I^{(hk-h)} \subseteq I^k$ for some k, then
$$I^{(hn-h)} \subseteq I^n \text{ for all } n \gg 0$$

Question Given c, is $I^{(hn-c)} \subseteq I^n$ for $n \gg 0$?

Resurgence (Bocci–Harbourne)

$$\rho(I) = \sup\left\{\frac{a}{b} : I^{(a)} \not\subseteq I^b\right\}$$

$$1 \leq \rho(I) \leq h$$

Remark If $\rho(I) < h$, stable Harbourne holds.

Why? $\quad \dfrac{hn-c}{n} > \rho(I) \quad \Longrightarrow \quad I^{(hn-c)} \subseteq I^n$

$$\Updownarrow$$

$$n > \frac{c}{h-\rho(I)} \neq 0$$

We say I has expected resurgence if $\rho(I) < h$.

theorem (G-Huneke-Mukundan)

(R, m) RLR or polynomial ring / k with I homogeneous

Assume $I^{(n)} = I^n : m^\infty$.

(eg, I defining a finite set of points in $\mathbb{P}^N$)

If $I^{(hn-h+1)} \subseteq m I^n$, then $\rho(I) < h$.

Applications (Assume $I^{(n)} = I^n : m^\infty$)

1) $R = k[x_1, \dots, x_d]$, char $k = 0$, I generated degree $< h$.

2) P defining ideal of (t^a, t^b, t^c) in char $\neq 3$

$P^{(3)} \subseteq m P^2$ (Knödel - Schenzel - Zonsarow)

3) R/I Gorenstein, char p (or char 0 and $\oplus I^{(n)}$ noetherian)

Back to degree bounds

$$\hat{\alpha}(I) = \lim_{n \to \infty} \frac{\alpha(I^{(n)})}{n} = \inf_n \frac{\alpha(I^{(n)})}{n}$$

ELS - HH - MS $\Rightarrow \hat{\alpha}(I) \geq \dfrac{\alpha(I)}{h}$ (Waldschmidt, Skoda)

so for s points in $\mathbb{P}^N \longrightarrow \hat{\alpha}(I) \geq \dfrac{\alpha(I)}{N}$

Theorem (G. Chudnovsky, 1977, Esnault–Viehweg, 1983)
$N=2$

I defining s points in $\mathbb{P}^N$. Then $\hat{\alpha}(I) \geq \dfrac{\alpha(I)+1}{N}$.

Conjecture (G. Chudnovsky, 1977) I defining s points in $\mathbb{P}^N$

$$\hat{\alpha}(I) \geq \frac{\alpha(I)+N-1}{N}$$

Conjecture (Harbourne–Huneke, 2013)

I radical of big height h in R regular

- $I^{(hn)} \subseteq m^{(h-1)n} I^n$ for all $n \geq 1$

(• $I^{(hn-h+1)} \subseteq m^{(n-1)(h-1)} I^n$ for all $n \geq 1$)

$$\Downarrow$$

$$\frac{\alpha(I^{(hn)})}{hn} \geq \frac{n\alpha(I)+(h-1)n}{nh}$$

$$\Downarrow$$

$$\hat{\alpha}(I) \geq \frac{\alpha(I)+h-1}{h}$$

Note only need stable containments

<u>theorem</u> (Bisui – G – Hà – Nguyen)
char $k=0$, $N \geq 3$
I defining s general points in $\mathbb{P}^N$
If $s \geq 4^N$ (or $s \geq 2^N$ and $N \geq 9$), then

$$I^{(Nr)} \subseteq \mathfrak{m}^{r(N-1)} I^r \quad \text{for all } r \gg 0$$

$$\Rightarrow \hat{\alpha}(I) \geq \frac{\alpha(I)+N-1}{N}$$

<u>theorem</u> (Fouli-Mantero-Xie 2018, Dumnicki – Tutaj-Gasińska 2017)
I defining s ($\geq 2^N$) <u>very general</u> points in $\mathbb{P}^N$, char $k=0$, $\overline{k}=k$
then $I^{(Nr)} \subseteq \mathfrak{m}^{r(N-1)} I^r$ for all $r \geq 1$

$$\Rightarrow \hat{\alpha}(I) \geq \frac{\alpha(I)+N-1}{N}$$

(Malara, Szemberg, Szpond, Chang-Jow) Demailly's conjecture for sufficiently large sets of <u>very general</u> points in $\mathbb{P}^N$

A property holds for <u>general</u> points if it holds for all $X \in \mathcal{U}$, $\mathcal{U}$ some open dense set in the Hilbert scheme of sets of s points in $\mathbb{P}^N$

A property holds for <u>very general</u> points if it holds on $\bigcap_{n=1}^{\infty} \mathcal{U}_n$, $\mathcal{U}_n \neq \emptyset$ open

Roadmap very general vs general

Step 1 Consider s generic points in $\mathbb{P}^N$: $1 \le i \le s$

$$(z_{i0} : \cdots : z_{iN}) \in \mathbb{P}^N_{k(\text{all } z_{ij})}$$

Show that for s generic points,

$$I^{(Nr)} \subseteq m^{r(N-1)} I^r \quad \text{for all } r \geq 1$$

Step 2 Specialize. For each r, get open dense set $\mathcal{U}_r$ where $I^{(Nr)} \subseteq m^{r(N-1)} I^r$ for all $V(I) = X \in \mathcal{U}_r$

Step 3 take $\bigcap_{r=1}^{\infty} \mathcal{U}_r$.

Remark $\pi =$ specialization map. For each fixed m,

$$\left(\pi(I^{(m)})\right) = (\pi(I))^{(m)} \quad \text{on an open dense set.}$$

General (not very)

Step 1 Show that for s generic points, there exists c such that

$$I^{(Nc-N)} \subseteq m^{c(N-1)} I^c$$

Step 2 Specialize. Get open dense set $\mathcal{U}$ where

$$I^{(Nc-N)} \subseteq m^{c(N-1)} I^c \quad \text{for all } V(I) = X \in \mathcal{U}.$$

Step 3 Apply the theorem:

Theorem (Bisui – G – Hà – Nguyễn)
I any radical ideal of big height h in a regular ring
If $I^{(hc-h)} \subseteq m^{c(h-1)} I^c$ for some c, then
$I^{(hn-h)} \subseteq m^{n(h-1)} I$ for all $n \gg 0$ $\square$

Theorem (Bisui – G – Hà – Nguyễn)
char $k=0$, $N \geq 3$, I defining s general points in $\mathbb{P}^N$. Then

$$\hat{\alpha}(I) \geq \frac{\alpha(I)+N-2}{N}$$

Corollary I defining s general points in $\mathbb{P}^N_k$, $N \geq 2$, char $k=0$
then $\rho(I) < N$, and thus for all $r \geq 1$, $I^{(Nr-C)} \subseteq I^r$ for $r \gg 0$.