Constructing Cubes from Semicubes

Tuesday, May 5, 2020 3:27 AM

<u>Notation</u>: faces $\partial^n_{i,\varepsilon}$; degeneracies σ^n_i; connections $\gamma^n_{i,\varepsilon}$;
symmetries τ^{n}_{j}; reversals ρ^n_i; diagonals $\delta^n_{i,k}$

cube categories $\Box_a$ $\quad a \subseteq \{\partial, \sigma, \gamma_0, \gamma_1, \tau, \rho, \delta\}$

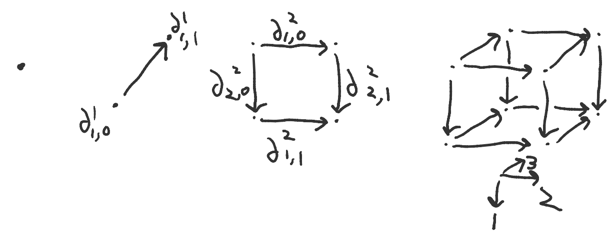
$\hat{C} := Set^{C^{op}}$

$$\boxed{\text{Theorem: Each } \hat{\Box}_a \text{ is the category of algebras for a monad on } \hat{\Box}_\partial}$$

Prelude on $\Box_\partial$

- $\Box_\partial$ is the "free" monoidal category generated by $I = \Box^0 \underset{\partial^1_{1,1}}{\overset{\partial^1_{1,0}}{\rightrightarrows}} \Box^1$

$$\Box^0 \underset{\partial^1_{1,1}}{\overset{\partial^1_{1,0}}{\rightrightarrows}} \Box^1 \underset{\partial^2_{i=1,2;\,\varepsilon=0,1}}{\overset{}{\rightrightarrows\!\rightrightarrows}} \Box^2 \rightrightarrows\!\rightrightarrows\!\rightrightarrows \Box^3 \cdots$$



$$\partial^n_{i,\varepsilon} = \underset{(1)}{id_{\Box^1}} \otimes \underset{\cdots}{\cdots} \otimes \underset{(i-1)}{id_{\Box^1}} \otimes \underset{(i)}{\partial^1_{1,\varepsilon}} \otimes \underset{(i+1)}{id_{\Box^1}} \otimes \underset{\cdots}{\cdots} \otimes \underset{(n)}{id_{\Box^1}}$$

A Monad Adding Degeneracies

— For $\partial : \Box^m \to \Box^n$ in $\Box_\partial$ let $A_\partial = \{\text{identity components of } \partial\} \subseteq \{1, \ldots, n\}$

eg $\partial = \underset{(1)}{id_{\Box^1}} \otimes \underset{(2)}{\partial^1_{1,0}} \otimes \underset{(3)}{id_{\Box^1}} \otimes \underset{(4)}{\partial^1_{1,1}} \otimes \underset{(5)}{\partial^1_{1,0}} \otimes \underset{(6)}{id_{\Box^1}} : \Box^3 \to \Box^6$ here $A_\partial = \{1, 3, 6\}$

Note $|A_\partial| = m$, so $A_\partial \cong \{1, \ldots, m\}$

— For $A \subseteq \{1,\dots,n\}$ and $\partial : \square^m \to \square^n$ in $\square_\partial$ define

$$\partial_A : \square^{|A\cap A_\partial|} \to \square^{|A|} \text{ by restricting to the } A\text{-components}$$

In the example above, if $A=\{1,2,4\}$, $\underset{(1)}{\partial_A = \mathrm{id}_{\square^1}} \otimes \underset{(2)}{\partial^1_{1,0}} \otimes \underset{(4)}{\partial^1_{1,1}} : \square^1 \to \square^3$

We now have
$$\begin{array}{ccccc} B_{A,\partial} := A\cap A_\partial & \subseteq & A \\ \cap & & \cap \\ \{1,\dots,m\} \cong A_\partial & \subseteq & \{1,\dots,n\} \end{array}$$
and $\partial_A : \square^{|B_{A,\partial}|} \to \square^{|A|}$

— Consider a semicubical set X in $\widehat{\square}_\partial$

$$X_0 \overset{d^1_{1,0}}{\underset{d^1_{1,1}}{\leftleftarrows}} X_1 \leftleftarrows\!\leftleftarrows X_2 \;\underset{d^n_{i,\varepsilon}}{\leftleftarrows\!\leftleftarrows\!\leftleftarrows}\; X_3 \;\cdots$$

— For $A \subseteq \{1,\dots,n\}$, let $X_A = X_{|A|}$

— Define $(T_\sigma X)_n = \coprod_{A\subseteq\{1,\dots,n\}} X_A$

For $\partial : \square^m \to \square^n$ in $\square_\partial$ define $d : \coprod_{A\subseteq\{1,\dots,n\}} X_A \to \coprod_{B\subseteq\{1,\dots,m\}} X_B$

by restricting to $d_A : \quad X_A \to X_{B_{A,\partial}}$

Ex: The representable $X = \square^2$ has

$X_0 = \{1,2,3,4\}$ $X_1 = \{a,b,c,d\}$, $X_2 = \{\alpha\}$

$$\begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ {\scriptstyle b}\downarrow & \alpha & \downarrow{\scriptstyle c} \\ 3 & \xrightarrow[d]{} & 4 \end{array}$$

$TX_0 = \{1,2,3,4\}$ $TX_1 = \{a,b,c,d\} \sqcup \{1,2,3,4\}$ $TX_2 = \{\alpha\} \sqcup \{a_1,b_1,c_1,d_1\} \sqcup \{a_2,b_2,c_2,d_2\} \sqcup \{1,2,3,4\}$

where in TX_2

$$\begin{array}{ccc} 2 & \overset{2}{=} & 2 \\ 2\| & 2 & \|2 \\ 2 & \underset{2}{=} & 2 \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ 1\| & a_1 & \|2 \\ 1 & \xrightarrow[a]{} & 2 \end{array} \qquad \begin{array}{ccc} 1 & \overset{1}{=} & 1 \\ {\scriptstyle a}\downarrow & a_2 & \downarrow{\scriptstyle a} \\ 2 & \underset{2}{=} & 2 \end{array}$$

— Define the unit $X \to T_\sigma X$ by

$$X_n = X_{\{1,\dots,n\}} \hookrightarrow \coprod_{A\subseteq\{1,\dots,n\}} X_A = T_\sigma X_n$$

— Multiplication amounts to $B \subseteq \{1,\dots,m\} \cong A \subseteq \{1,\dots,n\}$

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$X_A \xrightarrow{s_A} X_n$ for all $n, A \subseteq \{1, \dots, n\}$ such that

(1) $X_{\{1,\dots,n\}} \xrightarrow{s_{\{1,\dots,n\}}} X_n$ is the identity

(2) $X_B \xrightarrow{s_B} X_m \cong X_A \xrightarrow{s_A} X_n$ agrees with $X_B \xrightarrow{s_B} X_n$

(3)
$$\begin{tikzcd} & X_n \arrow[dr, "d"] & \\ X_A \arrow[ur, "s_A"] \arrow[dr, "d_A"'] & & X_m \\ & X_{B_{A,\partial}} \arrow[ur, "s_{B_{A,\partial}}"'] & \end{tikzcd}$$
commutes

write s_i^n for $s_{\{1,\dots,\hat{i},\dots,n\}} : X_{n-1} \to X_n$

(2) shows $s_i^n s_j^{n-1} = s_{j+1}^n s_i^{n-1} \; (i \leq j) = s_{\{1,\dots,\hat{i},\dots,\widehat{j+1},\dots,n\}}$

(3) shows $d_{i,\varepsilon}^n s_j^{n-1} = \begin{cases} s_{j-1}^n d_{i,\varepsilon}^{n-1} & i < j \\ s_j^n d_{i-1}^{n-1} & i > j \\ \mathrm{id}_{X_{n-1}} & i = j \end{cases}$

so $\{s_i^n\}$ extend X to a functor $\square_{\partial\sigma}^{op} \to \mathrm{Set}$

— For any X in $\hat{\square}_{\partial\sigma}$, the underlying semicubical set uX in $\hat{\square}_\partial$ has a canonical T_σ-algebra structure

— The full subcategory of T_σ-Alg spanned by $\{T_\sigma \square^n\}$ is isomorphic to $\square_{\partial\sigma}$

$\square^1 \xrightarrow{s_1^1(*)} T_\sigma \square^0$ $\quad$ $\begin{tikzcd} \hat{\square}_\partial \arrow[r, bend left, "T_\sigma"] \arrow[r, phantom, "\perp"] & T_\sigma\text{-Alg} \arrow[l, bend left, "u"] \end{tikzcd}$ $\quad$ $T_\sigma \square^1 \xrightarrow{\sigma} T_\sigma \square^0$

$(T_\sigma \square^0)_n = (\square^0)_0 = \{*\}$

Formalism

Formalism

— The data specifying T_σ was

- The sets $\mathcal{A}_n = \{A \subseteq \{1,\dots,n\}\}$ for all n
- For each $\partial : \square^m \to \square^n$, the function $\mathcal{A}_n \to \mathcal{A}_m : A \mapsto B_{A,\partial}$
- The assignment $A \mapsto \square^{|A|}$ and $(\partial, A) \mapsto \partial_A : \square^{|B_{A,\partial}|} \to \square^{|A|}$
- The "unit" $\{1,\dots,n\} \in \mathcal{A}_n$ and "multiplication" $B \subseteq \{1,\dots,m\} \cong A \subseteq \{1,\dots,n\}$

— More concisely, we have

- $\mathcal{A} : \square_\partial^{op} \to \mathrm{set}$
- $\mathcal{F} : \mathrm{el}\mathcal{A} \to \square_\partial$
- $e : * \to \mathcal{A}$ with $\square_\partial \cong \mathrm{el}* \xrightarrow{e} \mathrm{el}\mathcal{A} \xrightarrow{\mathcal{F}} \square_\partial$ the identity
- (multiplication data)

— Given this data, define $TX_n = \coprod_{A \in \mathcal{A}_n} X_{\mathcal{F}A}$

— For T_σ, $(\mathcal{A}, \mathcal{F})$ are "monoidally generated" by

$\mathcal{A}_0 = \{e_0\}$ $\quad$ $\mathcal{A}_1 = \{e_1, \sigma\}$

so $\mathcal{A}_n$ contains $\underbrace{e_1 \otimes \sigma \cdots \sigma \otimes \sigma \otimes e_1}_{n\text{-components}}$

$$\underline{\mathrm{el}\mathcal{A}} \qquad\qquad\qquad \underline{\square_\partial}$$

$$\begin{tikzcd}
e_0 \arrow[r, shift left, "\partial'_{1,0}"] \arrow[r, shift right, "\partial'_{1,1}"'] \arrow[dr, "\partial'_{1,0}"] \arrow[d, "\partial'_{1,1}"'] & e_1 & & \square^0 \arrow[r, shift left, "\partial'_{1,0}"] \arrow[r, shift right, "\partial'_{1,1}"'] \arrow[dr, no head] & \square^1 \\
{} & \sigma & & & \square^0
\end{tikzcd}
\qquad \xrightarrow{\mathcal{F}}$$

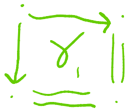
($\mathcal{A}$ is the free Day-convolution-monoid generated by the pointed semicubical set with $A_1 = \{e_1, \sigma\}$, $A_n = \{e_n\}$, which determines $\mathcal{F}$)

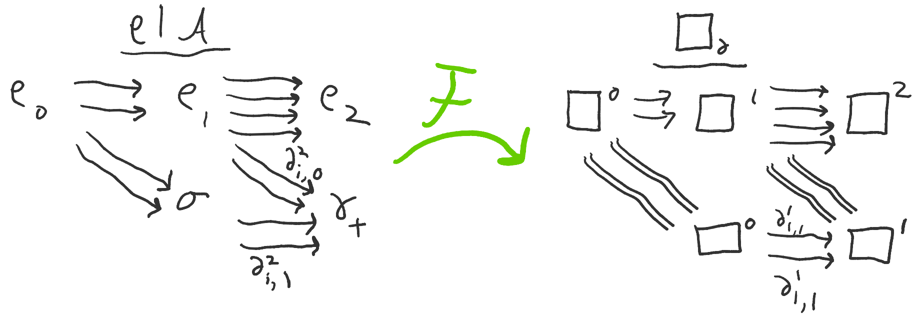
More Examples

connections:

— Let $\mathcal{A}$ be monoidally generated from

$\mathcal{A}_0 = \{e_0\}$ $\mathcal{A}_1 = \{e_1, \sigma\}$ $\mathcal{A}_2 = \{e_2, \gamma_1\}$





Generated operations include $\sigma_i^n, \gamma_{i,1}^n \in \mathcal{A}_n \xrightarrow{\mathcal{F}} \square^{n-1}$

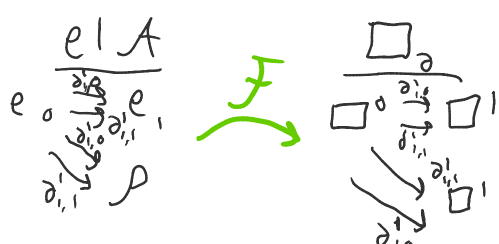
$\sigma_i^n = e_{i-1} \otimes \sigma \otimes e_{n-i}$, $\gamma_{i,1}^n = e_{i-1} \otimes \gamma_1 \otimes e_{n-i}$

but do not include "$\gamma^3_{1,1}\gamma^2_{1,1} = \gamma^3_{2,1}\gamma^2_{1,1}$"

- Need to add in "composites" which correspond to composition of degeneracy / connection maps in $\square_{\partial\sigma\gamma_1}$
- The category of pairs $(\mathcal{A}, \mathcal{F})$ has two monoidal structures, one based on Day convolution and the other corresponding to composition of the functors T.
  We want $(\mathcal{A}, \mathcal{F})$ to be a monoid in both.

Reversals — Let $(\mathcal{A}, \mathcal{F})$ be generated by

$\mathcal{A}_0 = \{e_0\}$ $\mathcal{A}_1 = \{e_1, \rho\}$



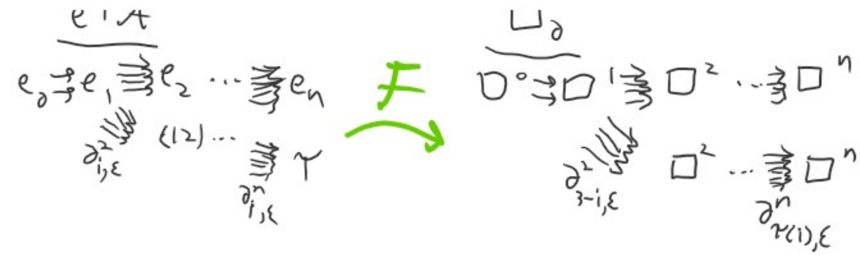
So $\mathcal{A}_n \cong \mathcal{A}_1^n$

Symmetries — Let $(\mathcal{A}, \mathcal{F})$ be given by

$\mathcal{A}_n = \Sigma_n$

$\underline{el\,\mathcal{A}}$: $e_0 \rightrightarrows e_1 \rightrightarrows e_2 \cdots \rightrightarrows e_n$ $\xrightarrow{\mathcal{F}}$ $\underline{\square_\partial}$: $\square^0 \rightrightarrows \square^1 \rightrightarrows \square^2 \cdots \rightrightarrows \square^n$

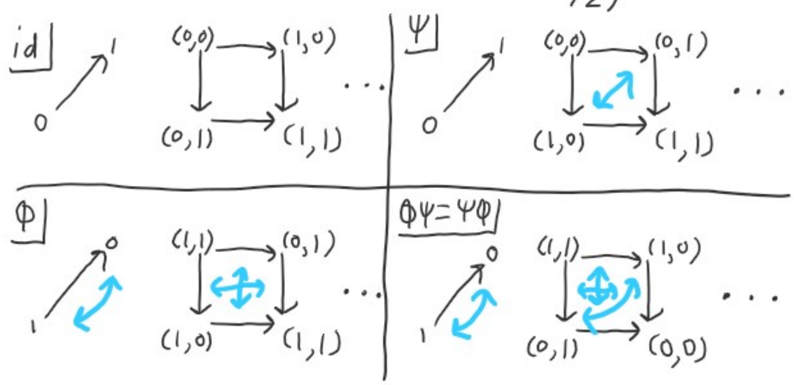
$\mathcal{A}_n = \mathcal{E}_n$



Diagonals

$(\mathcal{A}, \mathcal{F})$ contains $\delta_k \in \mathcal{A}_k$ with $\mathcal{F}\delta_k = \square^{2k} \ldots$

— There are exactly 4 ($\mathbb{Z}/_2 \times \mathbb{Z}/_2$) automorphisms of $\square_\partial$



These extra operations added to semicubical sets seem related to these symmetries:

	Φ	Ψ
"degeneracies"	σ	γ_0, γ_1
"symmetries"	ρ	τ
"extra faces"		δ

Φ has no fixed points, unless we add in composites  ..