References:

[KV] Kapulkin, Voevodsky, A cubical approach to straightening '18

[KLW] Kapulkin, Lindsey, Wong, A co-reflection of cubical sets into simplicial sets w/ applications, NYJM '19

[DKLS] Doherty, Kapulkin, Lindsey, Sattler, Cubical models of $(\infty,1)$-categories

$$\overbrace{[KV] \longrightarrow [KLW]}^{\text{today}} \longrightarrow \overset{\text{Wednesday}}{[DKLS]}$$

Plan:
1. Remarks on $\square \underset{\text{full}}{\subseteq} \mathrm{Cat}$ [KV]
2. Homotopy coherent nerve & straightening [KV] agnostic about $\square$
3. Straightening over the point [KLW] specific choice of $\square$

Prop: fdcd $\Rightarrow$ full

Proof: Have products $\square([1]^m, [1]^n) \cong \square([1]^m, [1])^n$

Induction w.r.t. m.

$f : [1]^{m+1} \longrightarrow [1]$ $\qquad f_t(x) := f(x,t)$, $t \in \{0,1\}$

Claim: $f(x,t) = \max(f_0(x), \min(f_1(x), t))$

$\square$

Fact: $\square$ is a test category
Maltsiniotis, La theorie d'homotopie de Grothendieck
Thm: A full subcat of Cat closed under products and not containing the empty cat is a test cat.
+ extra cond. e.g. $\cdot \to \cdot$.

$$\square \hookrightarrow \mathrm{Cat} \xrightarrow{N} \mathrm{sSet}$$

$J_{jef}|\square$ $\qquad$ J_{jef} jointly epi families

Thm: $\mathrm{Sh}(\square, J_{jef}|\square) \simeq \mathrm{sSet}$.

Proof: $\mathrm{Sh}(\square, J_{jef}|\square) \xleftarrow{\sim} \mathrm{Sh}(\mathrm{sSet}, J_{jef}) \xleftarrow{\sim} \mathrm{sSet}$

$\Uparrow$ Comparison Lemma

every repr. $\Delta^n \in \mathrm{sSet}$ can be covered by sth from $\square$

$$(\Delta^1)^n \twoheadrightarrow \Delta^n$$

$\circlearrowleft S_n$ $\qquad$ $\square$

Have: $\mathrm{sSet} \underset{\text{sheafify}}{\overset{\perp}{\rightleftarrows}} \mathrm{cSet}$ = triangulation
$T \dashv U$

Another take: $\mathrm{Split}(\square)$ = fin. lattices w/ poset maps

Ex: A fin. poset P is a retract of some $[1]^n$ iff it is a lattice

$$cSet \simeq Set^{Split(\square)^{op}}$$

Inclusion $\Delta \xhookrightarrow{i} Split(\square)$

induces

$$\begin{tikzcd} sSet \arrow[r, bend left=40, "i_!"] \arrow[r, bend right=40, "i_*"'] \arrow[r, phantom, bend left=20, "\perp"] \arrow[r, phantom, bend right=20, "\perp"] & cSet \arrow[l, "i^*" description] \end{tikzcd}$$

is the triangulation

End of Part 1.

Until further notice: $\square$ arbitrary (fd + ?)

<u>Recall</u>: $(cSet, \otimes, \square^0)$ monoidal cat

$$\square^m \otimes \square^n = \square^{m+n}$$

<u>Rmk</u>: w/ diagonals $\otimes = \times$

Consider $cCat$ = category of cubical cats

<u>Goal</u>: Construct homotopy coherent nerve

$$N_\square : cCat \longrightarrow sSet$$

$$(N_\square \mathfrak{C})_n = cCat(\mathfrak{C}[n], \mathfrak{C})$$

What's $\mathfrak{C}[n]$?

- objects $0, 1, \ldots, n$
- maps $\mathfrak{C}[n](i,j) = \square^{j-i-1}$
  $= \{I \subseteq \{i, \ldots, j\} \mid i, j \in I\}$
- composition "union of subsets"

$$\mathfrak{C}[n](i,j) \otimes \mathfrak{C}[n](j,k) \longrightarrow \mathfrak{C}[n](i,k)$$

$$\square^{k-i-2} \cong \square^{j-i-1} \otimes \square^{k-j-1} \xrightarrow{\partial_{j-i,1}} \square^{k-i-1}$$

- identity $\quad \square^{-1} := \square^{0}$

Ex:

$\mathfrak{C}[0]$: $\overset{0}{\bullet}$

$\mathfrak{C}[1]$: $\overset{0}{\bullet} \longrightarrow \overset{1}{\bullet}$

$\mathfrak{C}[2]$:

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] \arrow[rr, phantom, shift left=3, "\Uparrow"] & & 2 \end{tikzcd}$$

$$\{0,1,2\} \supseteq \{0,2\}$$

$\mathfrak{C}[3]$: $\quad 0, 1, 2, 3 \quad$ obj

$\mathfrak{C}[3](0,3)$

$$\begin{tikzcd} 03 \arrow[r] \arrow[d] & 013 \arrow[d] \\ 023 \arrow[r] & 0123 \end{tikzcd}$$

Want: extend $\mathfrak{C}$ to a cosimp obj:

$$[3] \xrightarrow{\sigma_1} [2]$$

$3 \mapsto 2$

$2 \mapsto 1$

$1 \mapsto 1$

$0 \mapsto 0$

$$\mathfrak{C}[3] \xrightarrow{\mathfrak{C}\sigma_1} \mathfrak{C}[2]$$

max

Mapping space from 0 to 3

$$\begin{tikzcd} & 03 \arrow[dl] \arrow[dr] & & & \boxed{02} \arrow[d] \\ 013 \arrow[dr] & & 023 \arrow[dl] & & \boxed{012} \\ & 0123 & & & \end{tikzcd}$$

$\boxed{03} \mapsto \boxed{02}$

$\boxed{013 \to 0123 \leftarrow 023} \mapsto \boxed{012}$

<u>Conclusion</u>: need max connections to define h.c.n.

Homotopy coherent nerve:

$$\begin{tikzcd}
\Delta \arrow[r, "\mathfrak{C}"] \arrow[d, hook] & c\mathcal{C}at \arrow[dl, bend left, "N_\square"] \\
sSet \arrow[ur, "\mathfrak{C}"] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

<u>Obs:</u> $N_\Delta : sCat \to sSet$

$$\begin{tikzcd}
s\mathcal{C}at \arrow[r, "U_\bullet"] \arrow[rr, bend right, "N_\Delta"', "\cong"] & c\mathcal{C}at \arrow[r, "N_\square"] & sSet
\end{tikzcd}$$

<u>Thm:</u> If $\mathcal{C}$ is locally Kan, then $N_\square \mathcal{C}$ is a quasicat.

locally Kan = $\mathcal{C}(x,y)$ is Kan for all x,y

i.e.
$$\begin{tikzcd}
\sqcap^n_{i,\varepsilon} \arrow[r] \arrow[d, hook] & \mathcal{C}(x,y) \\
\square^n \arrow[ur, dashed] &
\end{tikzcd}$$

<u>Proof:</u>
$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & N_\square \mathcal{C} \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}
\iff
\begin{tikzcd}
\mathfrak{C}\Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & \mathcal{C} \\
\mathfrak{C}[n] \arrow[ur, dashed] &
\end{tikzcd}$$

$$\Longleftarrow \quad
\begin{tikzcd}
\sqcap^n_{i,0} \arrow[r] \arrow[d, hook] & \mathcal{C}(f(0), f(1)) \\
\square^n \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

$\square$

<u>Obs:</u> Only need half of Kan cond.

<u>Ex:</u> Half Kan $\Longrightarrow$ all Kan

fillers for all $\sqcap^n_{i,0}$ (or $\sqcap^n_{i,1}$) $\Longrightarrow$ fillers for both

Ex: what are some locally Kan cubical cats?

Kan = cubical cat of universal Kan cpx

X universal Kan cpx if $\forall K$ X^K Kan

Rmk: If $\square$ Reedy: univ Kan = Kan

Ex: $X \in sSet$ Kan cpx, then UX univ Kan

(Un)Straightening

$$Q_S : sSet/S \rightleftarrows cSet^{\mathfrak{C}[S]} \simeq N_\square cSet : \int_S^S$$

We will focus on $\int_S$.

Simple problem: given $f: \Delta^n \to N_\square cSet$ construct its set of sections

$n = 0$

$$\frac{f: \Delta^0 \to N_\square cSet}{\square^0 \to X_0 \in cSet}$$

$$\begin{array}{c} X_0 \\ \cup \\ x_0 \end{array}$$

$Sect(f) = (X_0)_0$

$n = 1$

$$\frac{f: \Delta^1 \to N_\square cSet}{X_0 \xrightarrow{f_{01}} X_1}$$

$$\begin{tikzcd} & X_0 \arrow[r, "f_{01}"] & X_1 \\ \square^0 \arrow[ur] \arrow[urr, bend right, "x_0"'] & & f_{01}(x_0) \\ & & x_1 \arrow[u, "p_{01}"'] \end{tikzcd}$$

$Sect(f) = \{ (x_0, x_1, p_{01}) \}$

$n=2$ $\qquad$ $f : \Delta^2 \longrightarrow N_{\square}\mathrm{cSet}$

$$\begin{tikzcd} & X_1 \arrow[dr, bend left, "f_{12}"] & \\ X_0 \arrow[ur, bend left, "f_{01}"] \arrow[rr, bend right, "f_{02}"'] & \arrow[phantom, "\Uparrow \alpha_{012}"] & X_2 \end{tikzcd}$$

x_0 $\qquad$ $\begin{array}{c} f_{01}(x_0) \\ \uparrow p_{01} \\ x_1 \end{array}$

$$\begin{tikzcd} & f_{12}f_{01}(x_0) \arrow[dr, "\alpha_{012}"] & \\ f_{12}(x_1) \arrow[ur, "f_{12}(p_{01})"] & \Theta_{012} & f_{02}(x_0) \\ & x_2 \arrow[ul, "p_{12}"] \arrow[ur, "p_{02}"'] & \end{tikzcd}$$

$$\mathrm{Sect}(f) = \left\{ (x_0, x_1, x_2, p_{01}, p_{12}, p_{02}, \Theta_{012}) \right\}$$

Let's make n arb: For $f : \Delta^n \to N_{\square}\mathrm{cSet}$ define $\mathrm{Sect}(f) = \left\{ g : \Delta^{1+n} \to N_{\square}\mathrm{cSet} \mid g(0) = \square^0,\ g|_{\Delta^{\{1,\dots,1+n\}}} = f \right\}$

$$\left(\int_S f\right)_n = \left\{ \left(s : \Delta^n \to S,\ g \in \mathrm{Sect}(fs)\right) \right\}$$

$f : S \longrightarrow N_{\square}\mathrm{cSet}$

<u>Thm</u>: $\int_S : N_{\square}\mathrm{cSet}^S \longrightarrow \mathrm{sSet}/S$ admits a left adjoint.

<u>Proof</u>: Construct straightening.
Given $p : X \longrightarrow S$:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "p"'] & X^{\triangleleft} = \{0\} * X \arrow[d] \\ S \arrow[r] & P \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

$Q_S p : \mathfrak{C}S \longrightarrow \mathrm{cSet}$
$(Q_S p)_s = \mathfrak{C}P(0,s)$

$\square$

<u>Rmk</u>: This recovers all notions of straightening

(relative to a map $\phi : \mathfrak{C}S \to \mathcal{C}$, simplicial)

Straightening over the point:

$S = \Delta^0$

$$Q : sSet \underset{\int}{\overset{Q}{\rightleftarrows}} cSet : \int \qquad (Q \dashv \int)$$

Intuition: Q is the $(\infty,1)$-inverse of $\int$

Construction of Q:

$$\begin{tikzcd}
\Delta \arrow[r, "Q"] \arrow[d, hook] & cSet \arrow[dl, bend left, "\int"] \\
sSet \arrow[ur, "Q"] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

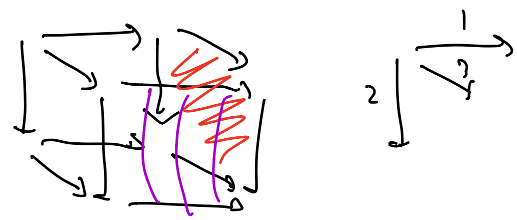
Q^n is constructed as a quotient of $\square^n$

$$\begin{tikzcd}
\bigcup\limits_{0<i<n} \square^{i-1} \otimes \square^{n-i} \arrow[r, "{[\partial_{i,1}]}"] \arrow[d] & \square^n \arrow[d] \\
\bigcup\limits_{0<i<n} \square^{i-1} \arrow[r] & Q^n \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Ex: $Q^0 = \square^0$

$Q^1 = \square^1$

$$Q^2 = \begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d, equal] \\
\bullet \arrow[r] & \bullet
\end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "1"] \arrow[d, "2"'] & {} \\ {} & \end{tikzcd}$$

$Q^3 =$ 

$\vdots$

Q defines a cosimplicial object in cSet

Faces: $\partial_{n,1}, \partial_{n,0}, \ldots, \partial_{1,0}$

Degeneracies: $\sigma_n, \gamma_{n-1}, \ldots, \gamma_1$

Moral: Need max-connections to write down Q.

For the next 5 minutes:

$\square$ = faces + degen + conn.

Thm: $\varepsilon : Q\int X \to X$ is a monomorphism

Intuition: $\int$ picks out cubes that look like simplices

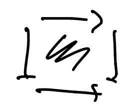
Thm: $\eta : K \to \int QK$ is an iso

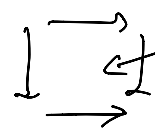
$\therefore$ Q is a coreflective embedding of sSet into cSet

Thm: Any cofibrantly generated model str. on sSet in which $\{\text{cofibs}\} \subseteq \{\text{monos}\}$ can be transferred along $\int : \text{cSet} \to \text{sSet}$ to give a Quillen equivalent model str.

Comments: Kan-Quillen, Joyal are ex.

Problem: not all obj are cofib.

Q^n  $= \square^2$ not cofibrant!

$Q \int \square^2 =$  empty inside