The equivariant uniform Kan fibration model of cubical homotopy type theory
(w/ Steve Awodey, Evan Cavallo, Thierry Coquand, + Christian Sattler)

Goal: build a constructive cubical sets model of HoTT + a Quillen model structure that is at least classically equivalent to spaces

Bonus: the Quillen equivalence is via a nice functor, eg "triangulation"

Bonus: use a cube category that permits "inductive constructions"

team	cubes	equivalent to spaces?
Bezem-Coquand-Huber	Symmetric { faces, degeneracies, symmetries	No! $Q = I^2/_{swap} \not\simeq *$ (Buchholtz)
Awodey, Angiuli-Brunerie-Coquand-Favonia-Harper-Licata	Cartesian { faces, degeneracies, symmetries, diagonals	No! $Q = I^2/_{swap} \not\simeq *$ (Sattler)
Cohen-Coquand-Huber-Mörtberg	DeMorgan { faces, degeneracies, symmetries, diagonals, connections, reversals	No! $Q = I/_{reversal} \not\simeq *$ (Buchholtz)
	Dedekind { faces, degeneracies, symmetries, diagonals, connections	???

Idea to remove counterexamples: make $* \longrightarrow Q$ a trivial cofibration by adding an equivariance condition to the definition of the fibrations

eg $Q = I^2/_{\text{swap}}$

$$\begin{tikzcd}
* \arrow[r, equal] \arrow[d] & * \arrow[r, equal] \arrow[d, "{(0,0)}"'] & * \arrow[r, "x"] \arrow[d, "{(0,0)}"'] & X \arrow[d, two heads] \\
I^2 \arrow[r, shift left, "id"] \arrow[r, shift right, "swap"'] & I^2 \arrow[r, two heads, "\pi"'] \arrow[urr, dashed, "{j(x,y\pi)}"] & Q \arrow[r, "y"'] \arrow[ur, dashed, "{j(x,y)}"'] & Y
\end{tikzcd}$$

$$\begin{aligned} j(x, y\pi)\cdot \text{swap} &= j(x, y\pi \cdot \text{swap}) \\ &= j(x, y\pi) \end{aligned}$$

Theorem (ACCRS) Equivariant uniform Kan fibrations on cartesian cubical sets satisfy the above desiderata.

Plan: I. the EZ category of cartesian cubes, II. equivariant uniform Kan fibrations

III. Quillen equivalence to spaces

I. the cartesian cube category $\square \simeq \mathrm{Fin}^{op}_{0\neq 1}$ ⇜ finite bipointed sets

objects

$* = I^0 \leftrightsquigarrow \{0,1\}$

$I = I^1 \leftrightsquigarrow \{0,x,1\}$

$I^2 = I \times I \leftrightsquigarrow \{0,x,y,1\}$

$I^3 = I \times I \times I \leftrightsquigarrow \{0,x,y,z,1\}$

$\vdots \qquad\qquad \vdots$

maps include

faces $I^{n-1} \hookrightarrow I^n \leftrightsquigarrow$ - - - -

degeneracies $I^n \twoheadrightarrow I^{n-1} \leftrightsquigarrow$ - - - -

symmetries $I^{n+1} \xrightarrow{\cong} I^{n+1} \leftrightsquigarrow$ - - - -

diagonals $I^n \longrightarrow I^{n+1} \leftrightsquigarrow$ - - -

$n \geq 1$

(exercise)

Proposition $\mathbb{\square}$ w/ $\mathrm{ob}\,\square \xrightarrow{\dim} \mathbb{N}$ is an Eilenberg–Zilber category meaning:

(1) $\forall \left\{ \begin{array}{l} \text{isos} \\ \text{non-inv monos} \\ \text{non-inv splitepis} \end{array} \right\}$ the dim of the domain is $\left\{ \begin{array}{c} = \\ < \\ > \end{array} \right\}$ the dim of the codomain

(2) $I^n \xrightarrow{\forall \alpha} I^m$

$\exists$ splitepi $\twoheadrightarrow I^k \rightarrowtail$ mono

unique up to iso (in Σ_k)

(3) split epis have absolute pushouts

$$\begin{tikzcd}
I^{\langle \vec{x}, \vec{y}, \vec{z} \rangle} \arrow[r, two heads] \arrow[d] & I^{\langle \vec{y}, \vec{z} \rangle} \arrow[l, dashed, bend right, "\vec{x} = \vec{0}"'] \arrow[d] \\
I^{\langle \vec{x}, \vec{y} \rangle} \arrow[u, dashed, bend left, "\vec{z} = \vec{0}"] \arrow[r] & I^{\langle \vec{y} \rangle} \arrow[u, dashed, bend right, "\vec{z} = \vec{0}"'] \arrow[l, dashed, bend left, "\vec{x} = \vec{0}"]
\end{tikzcd} \in \widehat{\mathbb{\square}}$$

II. equivariance

warmup: two (cofibrantly generated) weak factorization systems on $\mathrm{Set}^{\Sigma_n^{op}}$ = right Σ_n-sets

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Sigma_n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

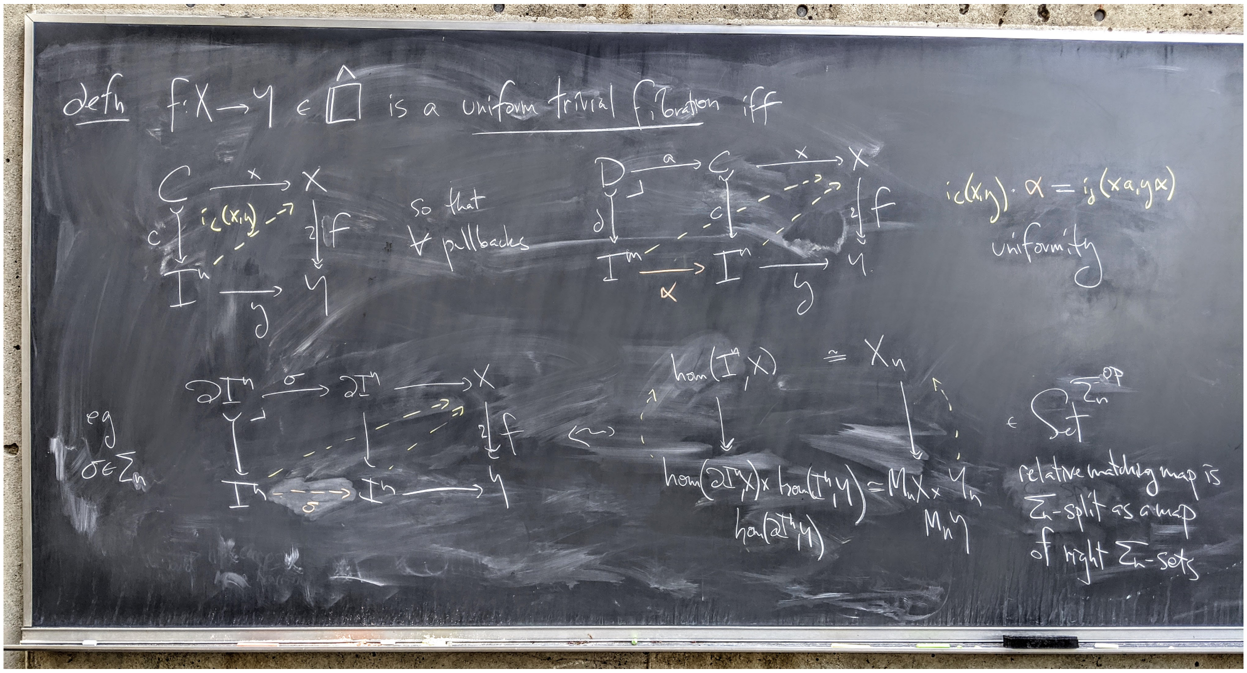
iff f is epi

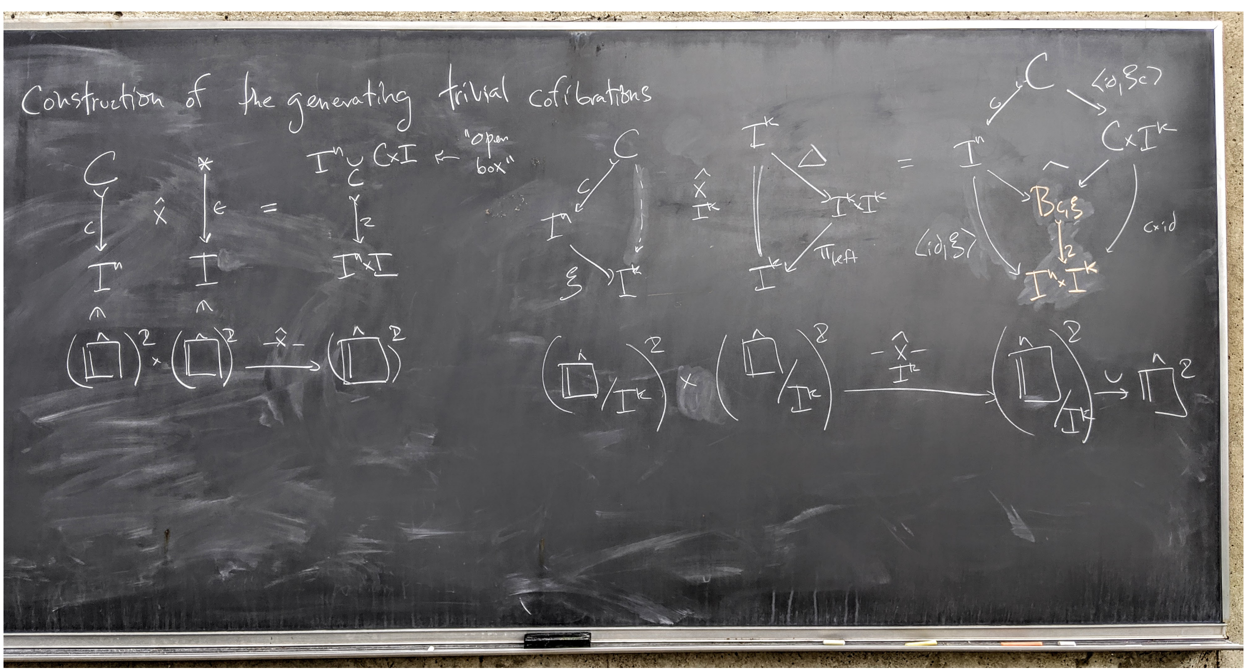
$$\forall \sigma \in \Sigma_n \quad \begin{tikzcd}
\emptyset \arrow[r, equal] \arrow[d] & \emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Sigma_n \arrow[r, "\sigma"] & \Sigma_n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

iff f is Σ_n-split epi

$$\begin{tikzcd}
\hom(\Sigma_n, X) \cong X \arrow[d] \\
\hom(\emptyset, X) \times_{\hom(\emptyset, Y)} \hom(\Sigma_n, Y) \cong Y \arrow[u, dashed, bend left]
\end{tikzcd} \in \mathrm{Set}$$

$$\begin{tikzcd}
\hom(\Sigma_n, X) \arrow[d] \\
\hom(\emptyset, X) \times_{\hom(\emptyset, Y)} \hom(\Sigma_n, Y) \arrow[u, dashed, bend left]
\end{tikzcd} \in \mathrm{Set}^{\Sigma_n^{op}}$$


defn f: X → Y ∈ □̂ is a uniform trivial fibration iff
so that ∀ pullbacks
i_c(x,y) · α = i_d(xa, yα)
uniformity
eg σ ∈ Σ_n
hom(I^n, X) ≅ X_n
hom(∂I^n, X) × hom(I^n, Y) = M_nX × Y_n
hom(∂I^n, Y) M_nY
∈ Set^(Σ_n^op)
relative matching map is Σ_n-split as a map of right Σ_n-sets


Construction of the generating trivial cofibrations
I^n ∪ C×I ← "open box"
⟨id, ξc⟩
C×I^k
B_{c,ξ}
⟨id, ξ⟩
c×id
π_left
I^n × I^k

Uniformity and equivariance square

$$\left(\begin{tikzcd} D \arrow[r, "a"] \arrow[d, tail, "\partial"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, tail, "c"] \\ I^m \arrow[r, "\alpha"'] & I^n \arrow[dr, "\xi"] \\ & & I^k \end{tikzcd}\right) \hat{\times}_{I^k} \left(\begin{tikzcd} I^k \arrow[r, equal] \arrow[d, "\Delta"'] & I^k \arrow[d, "\langle id, \sigma\rangle"] \\ I^k \times I^k \arrow[r, "id\times\sigma"'] & I^k \times I^k \arrow[dr, "\pi_{left}"] \\ & & I^k \end{tikzcd}\right)$$

$$\left(\hat{\square}_{/I^k}\right)^2 \times \left(\square_{/I^k}\right)^2 \xrightarrow{-\times_{I^k}-} \left(\hat{\square}_{/I^k}\right)^2 \xrightarrow{U} \hat{\square}^2$$

<u>defn</u> $f: X \to Y \in \hat{\square}$ is a <u>uniform equivariant fibration</u> iff

$$\forall \begin{tikzcd} C \arrow[d, tail, "c"] \\ I^n \arrow[dr, "\xi"'] \\ & I^k \end{tikzcd} \quad \begin{tikzcd} B_{c,\xi} \arrow[r, "\langle x, w\rangle"] \arrow[d, tail] & X \arrow[d, "f"] \\ I^n \times I^k \arrow[r, "y"'] \arrow[ur, dashed, "j_{c,\xi}(x,w,y)"] & Y \end{tikzcd}$$

so that $\forall$

$$\begin{tikzcd} B_{\partial,\xi\alpha} \arrow[r, "\langle \alpha, \alpha\times\sigma\rangle"] \arrow[d] & B_{c,\xi\sigma} \arrow[r, "\langle x, w\rangle"] \arrow[d] & X \arrow[d, "f"] \\ I^m \times I^k \arrow[r, "\alpha\times\sigma"'] \arrow[urr, dashed] & I^n \times I^k \arrow[r, "y"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

$$j_{c,\sigma\xi}(x,w,y)\cdot(\alpha\times\sigma) = j_{\partial,\xi\alpha}(x\alpha, w(\alpha\times\sigma), y(\alpha\times\sigma))$$

III. equivalence to spaces (Sattler)

$$\begin{tikzcd} \Delta \arrow[r, hook, "i"] & \square_{Dede}^{\natural} & \\ & \square_{Dede} \arrow[u, hook] & \square_{Cart} = \square \arrow[l, "j"'] \end{tikzcd}$$

$\leftarrow$ idempotent completion

$$\begin{tikzcd} \hat{\Delta} \arrow[r, bend left, "i_!"] \arrow[r, phantom, shift left=2, "\perp"] & \hat{\square}_{Dede} \arrow[l, "i^*"'] \arrow[r, "j^*"] \arrow[l, bend left, "i_*"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \hat{\square}_{Cart} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

NB: $T := i^* j_!$ is triangulation

Step 1: Show i^* and $j_!$ are left Quillen

($\Rightarrow T := i^* j_!$ is left Quillen)

Step 2: Show j^* and $i_!$ are left Quillen

($\Rightarrow L := j^* i_!$ is left Quillen)

Cor: $i^*, j^*, i_!, j_!$ preserve weak equivalences

$$\begin{tikzcd} \hat{\Delta} \arrow[r, "i_!"] \arrow[r, phantom, shift right=1, "\perp"] \arrow[rr, bend left, "L"] & \hat{\square}_{Dede} \arrow[l, shift left=2, "i^*"] \arrow[r, "j^*"] \arrow[r, phantom, shift right=1, "\perp"] & \hat{\square}_{Cart} \arrow[l, shift left=2, "j_!"] \arrow[ll, bend left, "T"] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Ho}\,\hat{\Delta} \arrow[r, "i_!"] \arrow[r, phantom, shift right=1, "\perp"] \arrow[rr, bend left, "L"] & \mathrm{Ho}\,\hat{\square}_{Dede} \arrow[l, shift left=2, "i^*"] \arrow[r, "j^*"] \arrow[r, phantom, shift right=1, "\perp"] & \mathrm{Ho}\,\hat{\square}_{Cart} \arrow[l, shift left=2, "j_!"] \arrow[ll, bend left, "T"] \end{tikzcd}$$

Step 3: L and T are inverse equivalences

$$\mathrm{Ho}\,\hat{\Delta} \simeq \mathrm{Ho}\,\hat{\square}_{Cart}$$

$$\mathrm{id}_{\mathrm{Ho}\hat{\Delta}} \xrightarrow[\cong]{\eta} i^* i_! \xleftarrow[\sim]{i^* \varepsilon i_!} i^* j_! j^* i_! = TL$$

$$\mathrm{id}_{\mathrm{Ho}\hat{\square}_{Cart}} \xrightarrow[\sim]{\eta} j^* j_! \xleftarrow{j^* \varepsilon j_!} j^* i_! i^* j_! = LT$$