Subadditivity of Syzygies and Related Problems

Outline:
I. Notation
II. Constructions
  a. Idealization
  b. Bourbaki ideals
III. Subadditivity
IV. General bounds on maximal graded shifts of ideals
V. Open Questions

K a field

$S = K[x_1, \dots, x_n]$ std. graded

($\deg(x_i) = 1 \quad \forall i$)

$S_i =$ K-v.s. of homog. deg i polys

So $S = \bigoplus_{i \geq 0} S_i$ as K-v.s.

Let $M = \bigoplus_i M_i$ be a f.g. graded S-module

Denote by $M(j)$ the graded S-module with $M(j)_i = M_{i+j}$

(e.g. $S(-j)$ is a rank one free

S-module with gen in degree j)

M has a (finite) minimal graded free S-resolution

$$F_\bullet : F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} \cdots \leftarrow F_p \leftarrow 0$$

i.e. $H_0(F_\bullet) \cong M$

$H_i(F_\bullet) = 0 \quad \forall i > 0$

$F_i = \bigoplus_j S(-j)^{\beta_{ij}(M)}$

and $\mathrm{Im}(\partial_i) \subseteq (x_1, \dots, x_n) F_{i-1}$.

(unique up to iso $\Rightarrow \beta_{ij}(M)$ are invariants of M)

$\mathrm{pd}(M) = \max\{i \mid \beta_{ij}(M) \neq 0\}$

$\mathrm{reg}(M) = \max\{j \mid \beta_{i,i+j}(M) \neq 0\}$

Betti Table of M:

$$
\begin{array}{c|cccccc}
 & 0 & 1 & \cdots & i & \cdots & pd(M) \\
\hline
0 & \beta_{00} & \beta_{11} & & \vdots & & \\
1 & \beta_{01} & \beta_{12} & & \vdots & & \\
\vdots & & & & \vdots & & \\
j & \cdots & \cdots & \cdots & \beta_{i,i+j} & & \\
\vdots & & & & & & \\
reg(M) & & & & & &
\end{array}
\;\to 0
$$

$$\downarrow 0$$

Max / Min graded shifts

$$\bar{t}_i(M) = \max\{j \mid \beta_{ij}(M) \neq 0\} = \max\{j \mid Tor_i^S(M,k)_j \neq 0\}$$

$$\underline{t}_i(M) = \min\{\quad \text{''} \quad\} = \quad \text{''}$$

<u>Note:</u> $reg(M) = \max\limits_{0 \leq i \leq pd(M)} \{\bar{t}_i(M) - i\}$.

Q1: What sequences of $\bar{t}_i(M)$ are possible?

A1: Almost anything.

A graded S-module is <u>pure</u> if
$\bar{t}_i(M) = \underline{t}_i(M) \quad \forall i$

Thm: (Eisenbud - Fløystad - Weyman '11
Eisenbud - Schreyer '09
Berkesch - Erman - Kummini - Sam '13
Fløystad '15)

$\forall$ sequence of integers $\underline{d}: d_0 < d_1 < \cdots < d_c$
$\exists$ a pure CM S-module M with

$$\bar{t}_i(M) = \underline{t}_i(M) = d_i$$

(M tends to have many generators in these constructions)

Q1': What sequences $\bar{t}_i(S/I)$ are possible?

Clearly **not** every increasing sequence of integers is possible.

$\boxed{\text{Example}}$ Take $\underline{d} = (0,1,3,4)$

Suppose $\bar{t}_i(S/I) = d_i \ \forall i$

$\bar{t}_1(S/I) = 1 \Rightarrow I$ is gen by lin. forms

$\Rightarrow I$ is " " a reg. seq. of lin. forms

$\Rightarrow S/I$ is resolved by a Koszul complex

$\Rightarrow \bar{t}_i(S/I) = i \ \forall i$

On the other hand, $\exists$ a pure CM module M with max shifts $(0,1,3,4)$

Check: $M = \operatorname{coker}\begin{pmatrix} w & x & y & z \\ w & 2x & 3y & 4z \end{pmatrix}$ works.

Some basics:

1. Minimality $\Rightarrow \underline{t}_{i-1}(M) < \underline{t}_i(M) \;\forall i$

2. $\bar{t}_{i-1}(M) < \bar{t}_i(M)$ for $i \le \operatorname{codim}(M)$

   (Hint: Try dualizing)

   but $\bar{t}_{i-1}(M) \ge \bar{t}_i(M)$ is possible $\forall i > \operatorname{codim}(M)$

Silly Example: $S = K[x,y,z]$

$$M = S/(x^3, y^3) \oplus S/(x,y,z)$$

Then $\bar{t}_i(M) = (0, 3, \underline{\underline{6}}, \underline{\underline{3}})$

$\operatorname{codim}(M) = 2$

Betti Table:

	0	1	2	3
0	1+1	3	3	1
1	-	-	-	-
2	-	2	-	-
3	-	-	-	-
4	-	-	1	-

Q2: How do we construct ideals with similar behavior?

Option 1: **Idealization** (trivial extensions)

Fix an S-module M gen in degree 0 by $g_1, \dots, g_m$. The std graded ring $S \ltimes M$ is the graded abelian gp $S \oplus M(-1)$ with multiplication

$$(s,m)\cdot(s',m') = (ss', sm' + s'm)$$

i.e. M becomes an ideal in $S \ltimes M$ with $M^2 = 0$.

Presentation of $S \ltimes M$:

$T := S[y_1, \dots, y_m]$ where $m = \mu(M)$

$$I = \underbrace{(y_1, \dots, y_m)^2}_{A} + \underbrace{\left(\sum_i c_i y_i \;\middle|\; \sum_i c_i g_i = 0 \right)}_{B}$$

$$S \ltimes M \cong T/I$$

SES:

$$0 \longrightarrow \frac{I}{A} \xrightarrow{i} \frac{T}{A} \longrightarrow \frac{T}{I} \longrightarrow 0$$

$$\begin{array}{ccccc} \uparrow & & \uparrow & & \| \\ K_\bullet(y) \otimes_T \tilde{F}_\bullet & \xrightarrow{\tilde{i}} & \text{linear res} & & \uparrow \end{array}$$

cone$(\tilde{i})$ is a (minimal) free res of T/I

$\boxed{\tilde{F}_\bullet}$ = res of $\mathrm{Syz}_1(M) \otimes_S T$

Aside: Rees-Like Algebra of I is

$$S[It, t^2] \cong S \oplus I \oplus S \oplus I \oplus \cdots$$

$$S[It, t^2]/(t^2) \cong S \ltimes I(-1)$$

t^2 nzd on $S[It, t^2]$

Get graded Betti #'s of defining ideal of $S[It, t^2]$ as above

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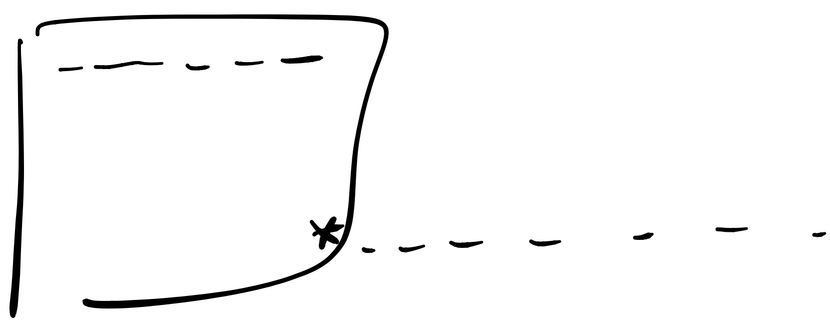
Thm (Ullery '13): Assume $\bar{t}_i(M) < \bar{t}_{i+1}(M)$ $\forall i$. $p = \mathrm{pd}(M)$, $m = \mu(M)$.

$$\bar{t}_i(T/I) = \begin{cases} \bar{t}_{i+1}(M) + 1 & 1 \le i \le p-1 \\ \bar{t}_p(M) + (i-p+1) & p \le i \le p+m-1 \end{cases}$$

"anything"

long linear tail

(Easy to check $\operatorname{codim}(I) = m$).



Option 2: Bourbaki Ideals

Bourbaki: Chapter VII

Thm: (Kumashiro '19, Herzog-Kumashiro-Stamate '20)
$|K| = \infty$. Let M be a f.g. graded torsionfree S-module of $\operatorname{rank}(M) = r$, gen in degree ≤ 0.
Then $\exists$ a graded Bourbaki sequence

$$0 \to S^{r-1} \to M \to I(m) \to 0$$

where $m \in \mathbb{Z}$, I is a graded ideal.

(Note: S need only be graded, Noeth, normal domain.)

Moreover, if M has no free summand, $\operatorname{codim}(I) = 2$. (requires S to be a UFD)

<u>Example</u>: Take $S = K[x,y,z]$

$$M = S/(x^3, y^3)(3) \oplus S/(x,y,z)(1)$$

Then $\mathrm{Syz}_1(M)$ is torsion free, rank $=2$ and <u>gen in degree 0</u>. So there is a Bourbaki sequence:

$$0 \to \underline{\underline{S}} \to \mathrm{Syz}_1(M) \to \underline{\underline{I(-4)}} \to 0$$

Betti table of S/I:

	0	1	2	3
0	1	-	-	-
1	-	-	-	-
2	-	-	-	-
3	-	4	3	1
4	-	-	-	-
5	-	-	1	-

codim$(I) = 2$

Part II: Bounds

$I \subseteq S$ is said to satisfy the **subadditivity condition** if

$$\bar{t}_a(S/I) + \bar{t}_b(S/I) \geq \bar{t}_{a+b}(S/I)$$

$$\forall a, b.$$

Remarks:

① Not true $\forall I$, not even CM ideals.

② True for graded, complete intersections.

③ Open for: Koszul, monomial, toric.

in particular if $R = S/I$ is Koszul:

- $\bar{t}_i(R) \leq 2i$ (Backelin, Kempf)
- $\bar{t}_{i+1}(R) \leq \bar{t}_i(R) + \underset{= \bar{t}_1(R)}{2}$

and $\cdot\ \bar{t}_a(R) + \bar{t}_b(R) \underline{\underline{+1}} \geq \bar{t}_{a+b}(R)$

$\boxed{\text{Correction:}}$ Requires R to also be if $\underline{\underline{\mathrm{char}(K) = 0}}$

(-M. Thanks to Srikanth for pointing this out.)

(Avramov – Conca – Iyengar)

(4) Failure of subadditivity is a witness of gen of the Koszul homology algebra $H_*(\underline{x}; S/I)$ in degree $a+b$.

Thm (-, Seceleanu '20)

$\forall\ s \geq 3\ \exists$ a graded, quadratic, Artinian Gorenstein ideal $I \subseteq S$ s.t.

(1) $\bar{t}_1(S/I) = 2$

(2) $\bar{t}_2(S/I) = s$

In particular, subadditivity can fail for quadratic Gorenstein ideals.

Use:

Thm (Mastroeni - Schenck - Stillman '19)

Let $I \subseteq S$ be a graded, quadratic, Artinian ideal. $R = S/I$.

ω_R = canonical module of R

$= \mathrm{Ext}^n_S(S/I, S)(-n)$

Assume R is <u>level</u>, i.e. ω_R is gen in 1 degree.

Set: $r = \mathrm{reg}(S/I)$, $m = \mathrm{type}(S/I) = \mu(\omega_R)$

① $G = R \ltimes \omega_R(-r-1)$ is Gorenstein, std. graded, and

$$\cong \frac{S[y_1, \dots, y_m]}{\left(I \cdot S[y] + (y_1, \dots, y_m)^2 + \left(\sum c_i y_i \mid \sum c_i w_i = 0 \right) \right)}$$

($w_1, \dots, w_m$ for S-gens of ω_R).

(2) In particular, if ω_R is linearly presented (<u>super level</u>) over R (or S-module), then G is quadratic.

(3) R not Koszul $\Rightarrow$ G not Koszul (Gulliksen)

So we need a quadratic, superlevel Artinian ideal with arbitrarily large degree 1st syzygy.

Take $I = (\underbrace{x_1^2, \dots, x_{2s}^2}_{\text{C.I.}}, (\underbrace{x_1 + \dots + x_{2s}}_{\text{Lefschetz element}})^2)$

Aside: (1) Bonus: For $s \geq 7$ get quadratic Gorenstein ideals with non-unimodal HFs. (See also Gondim-Zappala)

(2) We also construct quadratic Gorenstein ideals that are not Koszul

but have linear resolutions of K over R for arbitrarily many steps.

<u>Open Q</u>: What does the full resolution of (defining ideal of) G look like?

What about more general result?

<u>Note</u>: If $\ell \in S_1$ is regular on S/I,

Betti table of S/I

$=$ Betti table of $\bar{S}/\bar{I}$ for $\bar{S} = S/(\ell)$

May assume $\operatorname{depth}(S/I) = 0$ ie.

$\operatorname{pd}(S/I) = n$.

Will do this from now on.

Thm (Eisenbud - Huneke - Ulrich '06)

① If $\dim(S/I) \le 1$, then

$$\overline{t}_a(S/I) + \overline{t}_b(S/I) \ge \overline{t}_n(S/I)$$

$\forall a,b$ with $a+b=n$.

"weak convexity"

② If $\dim(M) \le 1$, $\mathrm{Ann}(M)$ contains a reg. seq of degree $d_1, \dots, d_c$ then

$$\overline{t}_n(M) \le \overline{t}_{n-c}(M) + \sum_{i=1}^{c} d_i.$$

Open Q: Is "$\dim(S/I) \le 1$" necessary?

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① above $\Longrightarrow$

$$\star\quad \bar{t}_n(S/I) \le \min_{a+b=n} \{\bar{t}_a(S/I) + \bar{t}_b(S/I)\}$$

Thm (-) '12

$$\bar{t}_n(S/I) \le \max_{a+b=n} \{\bar{t}_a(S/I) + \bar{t}_b(S/I)\}$$

Thm (Herzog - Srinivasan '13)

$$\bar{t}_n(S/I) \le \bar{t}_1(S/I) + \bar{t}_{n-1}(S/I)$$

They also showed if I is monomial then $\bar{t}_a(S/I) + \bar{t}_1(S/I) \ge \bar{t}_{a+1}(S/I)$

Thm (-) '18 : $I \subseteq S$ any graded ideal
$c = \operatorname{codim}(I)$

Then $\operatorname{reg}(S/I) \le \max_{0 \le i \le n-c} \{\bar{t}_i(S/I) + (n-i)\bar{t}_1(S/I)\}$

$-n$

in particular $\bar{t}_n(S/I) \le$

<u>Recall</u>: Ullery's designer ideals / idealizations gave arbitrary $\bar{t}_1, \ldots, \bar{t}_{n-c}$ with linear tail c steps long.

<u>Idea</u>: I contains a complete intersection of forms $f_1, \ldots, f_c$ of degree $\le \bar{t}_1(S/I)$. Write $R = \dfrac{S}{(f_1, \ldots, f_c)}$

Form a SES:

$$\begin{array}{lccccccccc} & 0 \to & K & \to & R & \to & S/I & \to 0 \\ & & \uparrow & & \uparrow & & \uparrow & \\ pd_S: & & n-1 & & c & & n & \end{array}$$

Do reverse induction on pd_S then apply EHU ②

Need an inductive statement for modules, --- gets messy.

Open Questions:

① Subadditivity fails for CM ideals when $a+b=2$.

Taking sums can make it fail for even $a+b \leq \frac{n}{2}$

By EHU ①, it holds when $a+b=n$.

In between?

(1b) Is $t_i(S/I) \leq \max\{i \cdot t_1(S/I), \frac{i}{2} \cdot t_2(S/I)\}$

for I CM?

② Question (Constantinescu-Kahle-Varbaro)

Is there a family of quadratic ($t_1(S/I)=2$) linearly presented

$(T_2(S/I) = 3)$ ideals with

$$\lim_{n\to\infty} \frac{\operatorname{reg}(S/I)}{n} > 0?$$