The dual graph of a ring

Fellowship of the ring seminar, MSRI

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The dual graph of a ring

Fix a Noetherian ring R of dimension $d < \infty$. Its **dual graph** (a.k.a. **Hochster-Huneke graph**) $G(R)$ is the simple graph with:

- The minimal prime ideals of R as vertices.
- As edges, $\{\mathfrak{p}, \mathfrak{q}\}$ where $R/(\mathfrak{p} + \mathfrak{q})$ has Krull dimension $d - 1$.

Example A

If $R = \mathbb{C}[X, Y, Z]/(XYZ)$, then $\mathrm{Min}(R) = \{(\overline{X}), (\overline{Y}), (\overline{Z})\}$ and $G(R)$ is a triangle, indeed R has dimension 2 and:

- $R/(\overline{X}, \overline{Y}) \cong \mathbb{C}[Z]$ has dimension 1.
- $R/(\overline{X}, \overline{Z}) \cong \mathbb{C}[Y]$ has dimension 1.
- $R/(\overline{Y}, \overline{Z}) \cong \mathbb{C}[X]$ has dimension 1.

The dual graph of a ring: basic properties

Exercise

The following properties come directly from the definition:

- $G(R) = G(R/\sqrt{\{0\}})$.
- If R is a domain, $G(R)$ consists of a single point.
- If $\mathfrak{p} \in \mathrm{Min}(R)$ is such that $\dim R/\mathfrak{p} < d$, then $\mathfrak{p}$ is an isolated vertex in $G(R)$ (i.e. it does not belong to any edge). In particular, if R is not equidimensional, $G(R)$ is not connected.

The dual graph of a ring: deep properties

Theorems

Let $(R, \mathfrak{m})$ be a Noetherian local ring.

- (Hartshorne, 1962) If R is Cohen-Macaulay, then $G(R)$ is connected.
- (Grothendieck, 1968) If R is complete and $G(R)$ is connected, then $G(R/xR)$ is connected for any nonzero-divisor $x \in R$.
- (Hochster and Huneke, 2002) If R is complete, then $G(R)$ is connected if and only if R is equidimensional and $H_{\mathfrak{m}}^{\dim R}(R)$ is an indecomposable R-module.

The dual graph of a ring: which graphs?

Our first aim is to understand which finite simple graphs can be realized as the dual graph of a ring. Several examples come from **Stanley-Reisner rings**, so let us quickly introduce them:

Let n be a positive integer and $[n] := \{0, \ldots, n\}$. A **simplicial complex** Δ on $[n]$ is a subset of $2^{[n]}$ such that:

$$\sigma \in \Delta,\ \tau \subset \sigma \ \Rightarrow\ \tau \in \Delta.$$

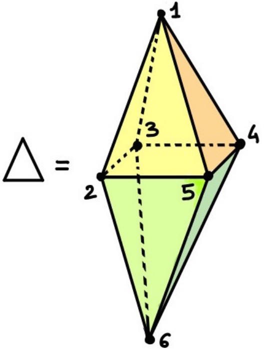
Any element of Δ is called *face*, and a face maximal by inclusion is called *facet*. The set of facets is denoted by $\mathcal{F}(\Delta)$. The dimension of a face σ is $\dim \sigma := |\sigma| - 1$, and the dimension of a simplicial complex Δ is

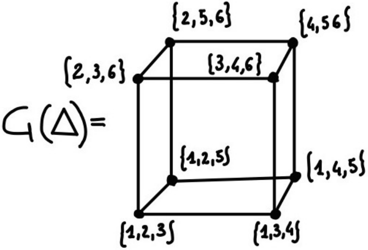
$$\dim \Delta := \sup\{\dim \sigma : \sigma \in \Delta\} = \sup\{\dim \sigma : \sigma \in \mathcal{F}(\Delta)\}.$$

The *dual graph* of a d-dimensional simplicial complex Δ is the simple graph $G(\Delta)$ with:

- The facets of Δ as vertices.
- As edges, $\{\sigma, \tau\}$ where $\dim \sigma \cap \tau = d - 1$.

Example B





Stanley-Reisner correspondence

Let K be a field and $S = K[X_0, \dots, X_n]$ be the polynomial ring.

To a simplicial complex Δ on $[n]$ we associate the ideal of S:

$$I_\Delta = (X_{i_1} \cdots X_{i_k} : \{i_1, \dots, i_k\} \notin \Delta) \subset S.$$

I_Δ is a square-free monomial ideal, and conversely to any such ideal $I \subset S$ we associate the simplicial complex on $[n]$:

$$\Delta(I) = \{\{i_1, \dots, i_k\} \subset [n] : X_{i_1} \cdots X_{i_k} \notin I\} \subset 2^{[n]}.$$

It is straightforward to check that the operations above yield a 1-1 correspondence:

$\{$simplicial complexes on $[n]\} \leftrightarrow \{$square-free monomial ideals of $S\}$

For a simplicial complex Δ on $[n]$:

(i) $I_\Delta \subset S$ is called the **Stanley-Reisner ideal** of Δ;

(ii) $K[\Delta] = S/I_\Delta$ is called the **Stanley-Reisner ring** of Δ.

Lemma

$I_\Delta = \bigcap_{\sigma \in \mathcal{F}(\Delta)} (X_i : i \in [n] \setminus \sigma)$. Hence $\dim K[\Delta] = \dim \Delta + 1$.

Proof: For any $\sigma \subset [n]$, the ideal $(X_i : i \in \sigma)$ contains I_Δ if and only if $[n] \setminus \sigma \in \Delta$. Being I_Δ a monomial ideal, its minimal primes are monomial prime ideals, i.e. ideals generated by variables. So, since I_Δ is radical, $I_\Delta = \bigcap_{\sigma \in \Delta} (X_i : i \in [n] \setminus \sigma)$. In the above intersection only the facets matter, so we conclude. $\square$

Exercise

The previous proof shows that there is a 1-1 correspondence between the facets of Δ and the minimal prime ideals of $K[\Delta]$. As it turns out, this correspondence gives an isomorphism of graphs $G(\Delta) \cong G(K[\Delta])$.

To the simplicial complex Δ of Example B corresponds the ideal

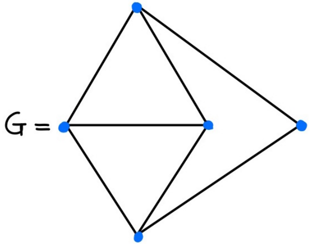
$$\begin{aligned} I_\Delta &= (X_1X_6,\ X_2X_4,\ X_3X_5) \subset K[X_1,\dots,X_6] \\ &= (X_1,X_2,X_3)\cap(X_1,X_2,X_5)\cap(X_1,X_4,X_3)\cap(X_1,X_4,X_5) \\ &\cap(X_6,X_2,X_3)\cap(X_6,X_2,X_5)\cap(X_6,X_4,X_3)\cap(X_6,X_4,X_5) \end{aligned}$$

and one can directly check that the dual graph of $K[\Delta]$ is the same described in Example B.

Dual graph of a simplicial complex

The previous discussion shows that any finite simple graph which is dual to some simplicial complex is the dual graph of a ring. However, not all finite simple graphs are dual to some simplicial complex. Some discussions on this issue can be found in a paper by Sather-Watsgaff and Spiroff and in [BBV].

Example C - Exercise



Not dual to any simplicial complex.

Dual graph of a projective line arrangement

Let $\mathbb{P}^n$ denote the n-dimensional projective space over the field K, and $X \subset \mathbb{P}^n$ a union of lines. Precisely,

$$X = \bigcup_{i=1}^{s} L_i \subset \mathbb{P}^n,$$

where the L_i are projective lines, i.e. projective varieties defined by ideals generated by $n-1$ linear forms of $S = K[X_0, \ldots, X_n]$.

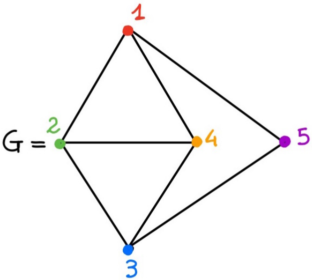
The *dual graph* of X is the simple graph $G(X)$ with:

- The lines L_i as vertices.
- As edges, $\{L_i, L_j\}$ if L_i and L_j meet in a point.

Dual graph of a projective line arrangement

Example D

The simple graph of Example C, which was not dual to any simplicial complex, is dual to a line arrangement:



Choose:

- $L_1, L_2, L_4 \in \mathbb{P}^3$ coplanar (H) meeting in 3 different points (P_{12}, P_{24}, P_{14}).
- L_3 not in H but passing through P_{24};
- L_5 any line meeting L_3 in a point different from P_{24} and L_1 in a point different from P_{12}, P_{14}.

Then $X = \bigcup_{i=1}^{5} L_i$ is such that $G(X) = G$

Dual graph of a projective line arrangement

If $X = \cup_{i=1}^{s} L_i \subset \mathbb{P}^n$ and L_i is defined by the ideal

$$I_i = (\ell_{1,i}, \ldots, \ell_{n-1,i}) \subset S = K[X_0, \ldots, X_n],$$

then $G(X)$ is isomorphic to the dual graph of the 2-dimensional ring S/I where $I = \cap_{i=1}^{s} I_i$ under the correspondence between the L_i's and the minimal prime ideals $\overline{I_i}$ of S/I. Indeed T.F.A.E.:

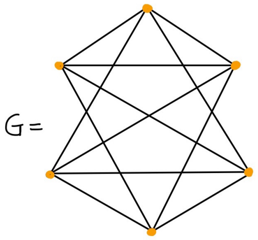
- L_i and L_j meet in a point.
- $S/(I_i + I_j)$ has dimension 1.

Dual graph of a projective line arrangement

If a simple graph is dual to a d-dimensional simplicial complex Δ, then it is also dual to a projective line arrangement:

Indeed, if $d \geq 1$ and all the facets of Δ has the same dimension, just take the Stanley-Reisner ring $K[\Delta]$ and go modulo $d - 1$ general linear forms. The resulting ring R will be the coordinate ring of a line arrangement and have the same dual graph as $K[\Delta]$. However...

Example E - Exercise



Not dual to any line arrangement.

Summing up, so far we proved the following inclusions:

$$\left\{\begin{array}{c}\text{dual graphs}\\ \text{of simplicial}\\ \text{complexes}\end{array}\right\} \subsetneq \left\{\begin{array}{c}\text{dual graphs}\\ \text{of projective}\\ \text{line arr'ts}\end{array}\right\} \subsetneq \left\{\begin{array}{c}\text{dual graphs}\\ \text{of rings}\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{all finite}\\ \text{simple graphs}\end{array}\right\}.$$

It turns out that the last inclusion is an equality. We show how to get a ring R such that $G(R)$ is the graph of Example E, and this will give the flavour for the proof of the general case. The details can be found in [BBV].

Assuming that K is infinite, we can pick 6 linear forms $\ell_1, \dots, \ell_6$ of $S = K[X, Y, Z]$ such that ℓ_i, ℓ_j, ℓ_k are linearly independent for all $1 \leq i < j < k \leq 6$. With this choice the corresponding 6 lines of $\mathbb{P}^2$ will meet in 15 distinct points.

Consider the ideal $J = (\ell_2, \ell_3) \cap (\ell_5, \ell_6) \subset S$ and the ring $A = K[J_3] \subset S$. Now let I be the ideal $(\ell_1 \cdots \ell_6) \cap A \subset A$. Then the dual graph of $R = A/I$ is isomorphic to the one of Example E (check it as exercise!).

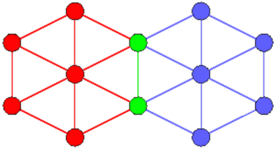
Remark

Geometrically, A is the coordinate ring of the blow-up of $\mathbb{P}^2$ along the intersection points P_{23} and P_{56}, and R is the coordinate ring of the strict transform of the line arrangement given by the original 6 lines of $\mathbb{P}^2$.

Notions from graph theory

Given a simple graph G on s vertices and an integer r less than s, we say that G is r-**connected** if the removal of less than r vertices of G does not disconnect it. The **valency** of a vertex v of G is:

$$\delta(v) = |\{w : \{v, w\} \text{ is an edge of } G\}|.$$



- 2-connected, not 3-connected.
- $\delta(\bullet) = 5$.
- $\delta(\text{inner}) = \delta(\text{inner}) = 6$.
- $\delta(\text{boundary}) = \delta(\text{boundary}) = 3$.

Notions from graph theory

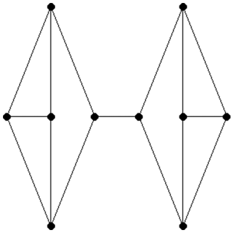
Remark

(i) G is 1-connected $\Leftrightarrow$ G is connected and has at least 2 vertices.

(ii) G is r-connected $\Rightarrow$ G is r'-connected for all $r' \leq r$.

(iii) G is r-connected $\Rightarrow$ $\delta(v) \geq r$ for all vertices v of G.

G is said to be r-**regular** if each vertex has valency r.



3-regular, 1-connected, not 2-connected.

Line arrangements algebraically

Given a line arrangement $X = \cup_{i=1}^{s} L_i \subset \mathbb{P}^n$ there is a unique radical ideal $I \subset S = K[X_0, \ldots, X_n]$ defining X. The ideal I has the form

$$I = I_1 \cap I_2 \cap \ldots \cap I_s$$

where $I_j \subset S$ is the ideal generated by the $n-1$ linear forms defining the line L_j. For simplicity we will call such ideals $I \subset S$ *line arrangement* ideals.

Of course there are many other homogeneous ideals $J \subset S$ defining $X \subset \mathbb{P}^n$ set-theoretically, namely those for which $\sqrt{J} = I$, but to our purposes the interesting one is I...

Gorenstein line arrangements

Theorem [BV]

Let $I \subset S$ be a line arrangement ideal such that S/I is Gorenstein. Then $G(S/I)$ is r-connected where $r = \operatorname{reg} S/I$.

The main ingredient of the proof is *liaison theory*.

Somewhat in contrast, we have the following:

Theorem (Mohan Kumar, 1990)

For any connected line arrangement ideal $I \subset K[X_0, X_1, X_2, X_3] = S$, there is a homogeneous complete intersection $J = (f, g) \subset S$ such that $\sqrt{J} = I$ (in particular $G(S/J) = G(S/I)$). Hence, any connected simple graph which is dual to a line arrangement is also dual to a complete intersection.

In view of the result of Mohan Kumar it is natural to ask the following:

Question

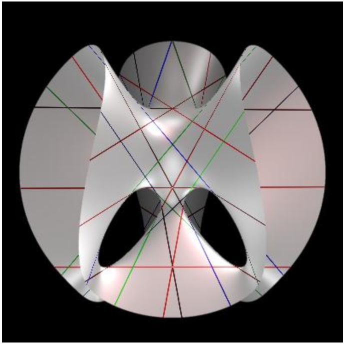
Is any connected simple graph dual to a complete intersection?

Coming back to our purposes, the previous result in [BV] is not optimal, in the sense that one can easily produce examples of line arrangement ideals $I \subset S$ such that S/I is Gorenstein of regularity r and $G(S/I)$ is k-connected for $k > r$.

However there are natural situations where the result is actually optimal ...

Let $Z \subset \mathbb{P}^3$ be a smooth cubic, and $X = \bigcup_{i=1}^{27} X_i$ be the union of all the lines on Z. Below is a representation of the Clebsch's cubic:

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = (x_0 + x_1 + x_2 + x_3)^3.$$



One can realize that the line arrangement ideal $I \subset S$ defining $X \subset \mathbb{P}^3$ is a complete intersection of the cubic defining Z and a product of 9 linear forms. So S/I is Gorenstein of regularity $3 + 9 - 2 = \mathbf{10}$. From the description of a smooth cubic as the blow up of $\mathbb{P}^2$ along 6 points one can check that:

- $G(S/I) = G(X)$ is **10**-connected (we already knew this from the theorem of [BV]).
- $G(X)$ is **10**-regular (in particular $G(X)$ is not 11-connected).

Line arrangements with planar singularities

A line arrangement $X \subset \mathbb{P}^n$ has planar singularities if all the lines of X meeting at a single point are co-planar. This is automatically satisfied if no more than two lines meet at the same point, or if X lies on a smooth surface.

Theorem [BDV]

Let $I \subset S$ be a line arrangement ideal such that S/I is Gorenstein. If the corresponding line arrangement has planar singularities, then $G(S/I)$ is r-regular where $r = \operatorname{reg} S/I$. In particular $G(S/I)$ is not $(r+1)$-connected (though it is r-connected).

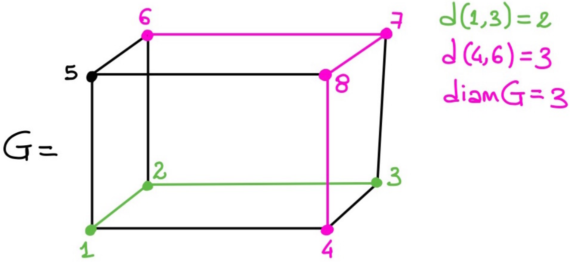
Once again, the main ingredient of the proof is *liaison theory*.

The diameter of a graph

Given two vertices v, w of a simple graph G, their *distance* $d(v, w)$ is the minimum length of a path connecting them; if such a path does not exists, $d(v, w) = +\infty$. The *diameter* of G is then

$$\operatorname{diam} G = \max\{d(v, w) : v \neq w \text{ are vertices of } G\}$$

($\operatorname{diam} G = -\infty$ if G consists of a single vertex).



Hirsch conjecture

The Hirsch conjecture is a conjecture from 1957 in discrete geometry. An equivalent formulation of it is that, if Δ is the boundary of a simplicial d-polytope with vertex set $[n]$, then

$$\operatorname{diam} G(\Delta) \leq n + 1 - d.$$

This conjecture has recently been disproved by Francisco Santos, however the statement is known to be true in some special cases:

Theorem (Adiprasito, Benedetti)

The conjecture of Hirsch is true if Δ is flag.

Algebraic Hirsch conjecture

If Δ is the boundary of a d-polytope with vertex set $[n]$, then $K[\Delta]$ is Gorenstein of dimension d, and $I_\Delta \subset S = K[X_0, \ldots, X_n]$ has height $n + 1 - d$. Furthermore, Δ being flag means that I_Δ is generated by quadrics.

In view of such considerations, it is natural to define a homogeneous ideal $I \subset S = K[X_0, \ldots, X_n]$ **Hirsch** if

$$\operatorname{diam} G(S/I) \leq \operatorname{ht} I.$$

We proposed the following, maybe too pretentious, conjecture:

Conjecture [BV]

Let $I \subset S$ be a radical homogeneous ideal generated by quadrics. If S/I is Cohen-Macaulay, then I is Hirsch.

Algebraic Hirsch conjecture

Theorem [DV]

The above conjecture is true if S/I is Gorenstein and $\mathrm{ht}\, I \leq 4$.

Once again, the main ingredient of the proof is *liaison theory*.

Actually we can also prove that the conjecture is true if S/I is Gorenstein, $\mathrm{ht}\, I = 5$, but I is not a complete intersection. If I is a radical complete intersection of 5 quadrics, we are only able to say that $\mathrm{diam}\, G(S/I) \leq 7$ ...

Hirsch ideals

If K has characteristic $p > 0$, let us recall that by the Fedder criterion the following are equivalent for a homogeneous ideal $I \subset S = K[X_0, \ldots, X_n]$:

- S/I is F-pure.
- There exists a polynomial $f \in I^{[p]} : I$ with $X_0^{p-1} X_1^{p-1} \cdots X_n^{p-1}$ in its support.

If, furthermore, $X_0^{p-1} X_1^{p-1} \cdots X_n^{p-1}$ is the initial monomial of f with respect to some monomial order, then one can show that the respective initial ideals of I and of all the intersections of the minimal prime ideals of I are square-free monomial ideals. So, as a consequence, by the results in [DV] one gets...

Hirsch ideals

Proposition

If S/I is F-pure and $\mathrm{in}(f) = X_0^{p-1}X_1^{p-1}\cdots X_n^{p-1}$, then

$$\mathrm{diam}\, G(S/I) \leq \mathrm{diam}(G(S/\mathrm{in}(I))).$$

Corollary

Let S/I be F-pure and $\mathrm{in}(f) = X_0^{p-1}X_1^{p-1}\cdots X_n^{p-1}$. If S/I is Cohen-Macaulay and $\mathrm{ht}\, I \leq 3$, then I is Hirsch.

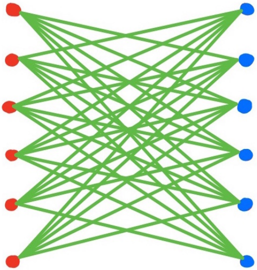
The proof uses the recent results of Conca and myself on square-free Gröbner degenerations and a study by Brent Holmes on the diameter of simplicial complexes of small codimension...

Conjecture

Let $I \subset S$ be a height 2 homogeneous ideal such that S/I is F-pure and Cohen-Macaulay. Then I is Hirsch.

If, among the 27 lines on a smooth cubic, we take only the 6 corresponding to the exceptional divisors and the 6 corresponding to the strict transforms of the conics, we get a line arrangement $X \subset \mathbb{P}^3$ known as **Schläfli double six**. One can check that the corresponding line arrangement ideal $I \subset S$ is a complete intersection of the cubic and of a quartic; we have the following:

$G(S/I) =$



As predicted, $G(S/I)$ is 5-regular and 5-connected ($\mathrm{reg}(S/I) = 5$).

Furthermore, I is a height 2 radical homogeneous ideal such that S/I is Gorenstein.

Though, I is not Hirsch ($\mathrm{diam}\, G(S/I) = 3$).