Rees algebras of ideals generated by 2x2 minors

(joint with Celikbas, Dufresne, Fouli, Gorla, Lin, and Swanson)

Fix V k-vector space of dim n

Let m fixed $1 \leq m \leq n$ $\quad \{W \leq V : \dim W = m\}$

$$W \rightsquigarrow M \quad m \times n \text{ matrix}$$

X $m \times n$ matrix of variables x_{ij}

$I_m = (\Delta_1, \ldots, \Delta_{\binom{n}{m}})$ the ideal generated by the maximal minors of X

$$\begin{tikzcd}
F: \mathbb{P}^{mn-1} \arrow[rr, dashed, "{[\Delta_1 : \ldots : \Delta_{\binom{n}{m}}]}"] \arrow[dr, dashed] & & \mathbb{P}^{\binom{n}{m}-1} \\
& \operatorname{Im} F := G_m(V) \arrow[ur] &
\end{tikzcd}$$

Plücker embedding

Homogeneous coordinate ring of $G_m(V)$ is

$$\underbrace{\mathcal{F}(I_m) = K[\Delta_1, \ldots, \Delta_{\binom{n}{m}}]}_{\text{Plücker algebra}} \subset R := K[x_{ij}]$$

- $G_m(V)$ is smooth of dim $m(n-m)$
- $\mathcal{F}(I_m)$ CM normal

- $\mathcal{F}(I_m)$ UFD Gorenstein
- $\mathcal{F}(I_m)$ has rational singularities in $ch(K)=0$

$$T = K[T_1, \dots, T_{\binom{n}{m}}] \twoheadrightarrow \mathcal{F}(I_m)$$

$$T_i \mapsto \Delta_i$$

what is Ker ?
What are the algebraic relations among the minors ?

- $\mathcal{F}(I_m)$ is defined by quadratic relations

<u>Plücker relations</u>

Ex: X $2\times n$ matrix $1 \le i < j < k < l \le n$

$$[ij][kl] - [ik][jl] + [il][jk] = 0$$

$$\Updownarrow$$

$$P_{ijkl} = T_{ij}T_{kl} - T_{ik}T_{jl} + T_{il}T_{jk}$$

$\mathcal{P} = \langle P_{ijkl} \; : \; 1 \le i < j < k < l \le n \rangle$ is the defining ideal of the Plücker algebra.

Homogeneous coordinate ring of the graph of F

$$\mathcal{R}(I_m) = R[\Delta_1 t, \dots, \Delta_{\binom{n}{m}} t] \subset R[t]$$

Rees algebra of I_m

$$\begin{array}{ccccc} \mathrm{Graph}(F) & \subset & \mathbb{P}^{mn-1} & \times & \mathbb{P}^{\binom{n}{m}-1} \\ \downarrow & & & \downarrow & \\ G_m(V) & \subset & & \mathbb{P}^{\binom{n}{m}-1} & \end{array}$$

corresponds to the diagram

$$
\begin{array}{ccccccc}
\mathcal{R}(I_m) & \twoheadleftarrow & R[T_1, \dots, T_{\binom{n}{m}}] = K[\overset{(1,0)}{\underline{x}}, \overset{(0,1)}{\underline{T}}] =: S & \leftarrow & \mathcal{J} & \leftarrow & 0 \\
\cup & & \cup & & \cup & & \\
\mathcal{F}(I_m) & \twoheadleftarrow & K[\underline{T}] = T & \leftarrow & \mathcal{P} & \leftarrow & 0 \\
 & & & & & & \\
\Delta_i t & \longmapsfrom & T_i & & & &
\end{array}
$$

In general: $I = (f_1, \dots, f_s) \subset R = K[\underline{x}]$

f_i forms of the same degree

$$\mathcal{R}(I) = R[It] = R \oplus It \oplus I^2t^2 \oplus \dots \subset R[t]$$

$$\cup$$

$$K[f_1 t, \dots, f_s t] \cong \mathcal{F}(I) = K[f_1, \dots, f_s]$$ special fiber ring

$$\Rightarrow \mathcal{F}(I) = [\mathcal{R}(I)]_{(0,*)}$$

Back to $\mathcal{R}(I_m)$

- [EH, '83] $\mathcal{R}(I_m)$ ASL on a wonderful poset $\Rightarrow$ CM
- $\mathcal{R}(I_m)$ normal $\Leftrightarrow$ $\overline{I_m^t} = I_m^t \quad \forall t$
- [Trung, '79] $I_m^{(t)} = I_m^t \quad \forall t$
- $\mathcal{R}(I_m)$ has rational singularities in $ch(K) = 0$

What is $\mathcal{J}$?

- $\mathcal{P} \subset \mathcal{J}$

- $\mathcal{J}_{(0,*)} \subset \mathcal{J}$ is the defining ideal of the special fiber
- $\mathcal{J}_{(*,1)} \subset \mathcal{J}$ is obtained by the syzygies of I

$$\begin{array}{ccc} \mathrm{Sym}(I) & \longrightarrow\!\!\!\!\rightarrow & \mathcal{R}(I) \\ \| & & \\ S/\mathcal{L} & & \end{array}$$

where $\mathcal{L}$ is obtained from a presentation matrix φ of I

$$\mathcal{L} = \langle [\underline{I}] \cdot \varphi \rangle$$

$\mathrm{Sym}(I_m) = S/\mathcal{L}$ Eagon-Northcott relations

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} \\ X \\ \\ \end{bmatrix} \\ & \begin{bmatrix} \quad \end{bmatrix} \end{array}$$

Take $m+1$ minors

$\leftarrow$ any row of X

Ex: X $2 \times n$ matrix $u = 1, 2$ $1 \le i < j < k \le n$

$$x_{ui}[jk] - x_{uj}[ik] + x_{uk}[ij] = 0$$

$$\mathcal{L} = \langle x_{ui}T_{jk} - x_{uj}T_{ik} + x_{uk}T_{ij} := \ell_{uijk} \rangle$$

- $J = \mathcal{L} \Rightarrow \mathcal{R}(I) \cong \operatorname{Sym}(I)$ and $\mathcal{F}(I)$ is a polynomial ring

  I is of linear type

- $J = (\mathcal{L}, J_{(0,*)})$ then I is of fiber type

- relation type of I, $\operatorname{rtype}(I)$, is the highest T-degree of a minimal homogeneous generator of J

Ex I of linear type $\Rightarrow \operatorname{rtype}(I) = 1$

- $\mathcal{R}(I_m)$ is of fiber type $\Rightarrow \mathcal{R}(I_m) = S/(\mathcal{L}, P)$ [EH]

  $\Rightarrow \operatorname{rtype}(I_m) \leq 2$

For non maximal minors $\quad 1 < t < m \leq n$

$I_t = I_t(X)$

Bruns ['91], B-Conca ['98, '01], B-C-Varbaro ['13] [BCV]

OPEN: What are the defining ideals of $\mathcal{F}(I_t)$ and $\mathcal{R}(I_t)$?

Conjectures

(1) [BCV] $\mathcal{F}(I_t)$ is defined by quadrics and cubic relations

(2) I_t is of fiber type

(3) $\operatorname{rtype}(I_t) \leq 3$

[Huang, Perlman, P, Raicu, Sammartano] $ch(K)=0$

- The defining ideal of $\mathcal{F}(I_2)$ is generated in degree 2 and 3 by the relation described in [BCV]
- I_2 is of fiber type for any generic $m \times n$ matrix $\Leftrightarrow$ it is of fiber type for a generic $m \times (m+2)$ matrix

§2 SAGBI BASES [Robbiano - Sweedler, Kapur - Madlener]

$A \subset K[x_1, \dots, x_n] := R$ f.g. K-subalgebra

τ term order on R

$$A \rightsquigarrow \mathrm{in}_\tau(A) = K[\{\mathrm{in}_\tau(f) : f \in A\}]$$

A set of elements $a_i \in A$ $i \in \mathcal{I}$ is a SAGBI basis of A if

$$\mathrm{in}_\tau(A) = K[\mathrm{in}_\tau(a_i) : i \in \mathcal{I}]$$

Assume A has a finite SAGBI basis

$$\Rightarrow \mathrm{in}_\tau(A) = \mathrm{gr}_{\mathcal{F}_\alpha}(A)$$

for a suitable degree filtration $\mathcal{F}_\alpha$

Conca-Herzog-Valla ['96] [CHV]

Facts [CHV]

(1) $\mathrm{in}_\tau(A)$ CM of dim d and type t $\Rightarrow$ A CM of dim d and type $\leq t$

(2) $\mathrm{in}_\tau(A)$ normal $\Rightarrow$ A has rational singularities in $\mathrm{ch}(K)=0$

(3) $A = \bigoplus_{i \geq 0} A_i \Rightarrow \mathrm{in}_\tau(A) = \bigoplus_{i \geq 0} \mathrm{in}_\tau(A_i)$

NB for (3) one does not assume that A has a finite SAGBI basis

(3) $\Rightarrow$ HF of A and $\mathrm{in}_\tau(A)$ are the same

(4) SAGBI basis $= \{f_1, \dots, f_s\}$ f_i forms of the same degree

$\mathrm{in}_\tau(A)$ Koszul $\Rightarrow$ A Koszul

Recall A standard graded K-algebra is Koszul if the residue field $K = A/A_+$ has a linear resolution over A

Back to $\mathcal{R}(I)$:

Assume $I \subset K[\underline{x}] := R$ generated by forms of the same degree

τ term order on R $\rightsquigarrow$ τ' term order on $R[t]$

u, v monomials in R

$$ut^i \underset{\tau'}{<} vt^j \iff i<j \text{ or } i=j \quad u \underset{\tau}{<} v$$

Thm [CHV]

$$\mathrm{in}_\tau(\mathcal{R}(I)) = \bigoplus_{i \geq 0} \mathrm{in}_\tau(I^i)\, t^i$$

$$\begin{array}{c}\mathrm{in}_\tau(I^i) = \mathrm{in}_\tau(I)^i \\ \forall i\end{array} \iff =$$

$$\cup\!\!\!\mid$$

$$\mathcal{R}(\mathrm{in}_\tau(I)) = \bigoplus_{i \geq 0} \mathrm{in}_\tau(I)^i\, t^i$$

Criterion: $\mathrm{in}_\tau(I^i) = \mathrm{in}_\tau(I)^i \quad 1 \leq i \leq \mathrm{rtype}(\mathrm{in}_\tau(I))$

$$\Rightarrow \mathrm{in}_\tau(I^i) = \mathrm{in}_\tau(I)^i \quad \forall i$$

Ex: Take τ diagonal term order on X $m \times n$

(1) [Sturmfels] $\Delta_1, \ldots, \Delta_{\binom{n}{m}}$ is a SAGBI basis for $\mathcal{Y}(I_m)$

(2) [Conca] $\mathrm{in}_{\tau'}(\mathcal{R}(I_m)) = \mathcal{R}(\mathrm{in}_\tau(I_m))$

Criterion for a set to be a SAGBI basis [Sturmfels]

How are the defining ideals of A and $\mathrm{in}_\tau(A)$ related?

$A = K[f_1, \ldots, f_s] \subset K[x_1, \ldots, x_n] = R$

τ term order on R

$\mathrm{in}_\tau(f_i) = \underline{x}^{b_i} \rightsquigarrow B = [b_1 \ \ldots \ b_s]$ $n \times s$ matrix with entries in $\mathbb{N}$

$A = K[\underline{T}]/\mathcal{J}$ $\qquad A' = K[\underline{x}^{b_1}, \ldots, \underline{x}^{b_s}] = K[\underline{T}]/\mathcal{J}'$

Let ω be any weight vector representing τ for f_i's

$\Rightarrow$ the weight vector $B^T \cdot \omega$ gives a weight on $K[\underline{T}]$

Thm: (1) $f_1, \ldots, f_s$ is a SAGBI basis for A

$$\Longleftrightarrow$$

$$\mathrm{in}_{B^T\omega}(\mathcal{J}) = \mathcal{J}'$$

(2) Every reduced GB of $\mathcal{J}'$ lifts to a reduced GB of $\mathcal{J}$

§3 SPARSE MATRICES

X $2\times n$ sparse matrix : the entries of X are either zero or variables

Ex :

$$X = \begin{bmatrix} x_{11} & x_{12} & 0 & 0 & x_{15} & x_{16} \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} & 0 \end{bmatrix}$$

In general for a $m\times n$ sparse matrix X, let $I = I_m(X)$

Giusti - Merle ['81] • CMness, primeness, codim

Boocher ['13] • minimal free resolution

• maximal minors form a UNIVERSAL GB for I

WLOG

$$X = \left[\begin{array}{ccc:ccc:ccc} x_{11} & \cdots & x_{1r} & x_{1\,r+1} & \cdots & x_{1\,n-s} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & x_{2\,r+1} & \cdots & x_{2\,n-s} & x_{2\,n-s+1} & \cdots & x_{2n} \end{array}\right]$$

with $\overbrace{0 \cdots 0}^{s \text{ zeros}}$ in the first row, $\underbrace{0 \cdots 0}_{r \text{ zeros}}$ in the second row, and $\underbrace{x_{2\,r+1} \cdots x_{2\,n-s}}_{\text{generic block}}$

$r \geq s$ with $r > 0$ [$r=0$ X is generic 1^{st} part the talk]

$$I = I_2(X) = (f_{ij} = [ij] \;/\; 1 \leq i < j \leq n) \subset R = K[x_{ij}]$$

τ "diagonal term order" $\overset{\text{Boocher}}{\Longrightarrow}$ $\mathrm{in}_\tau(I) = \langle S_1 \cup S_2 \cup S_3 \rangle$

$$S_1 = \{ x_{1i} x_{2j} \;/\; 1 \leq i \leq r \quad r+1 \leq j \leq n \}$$

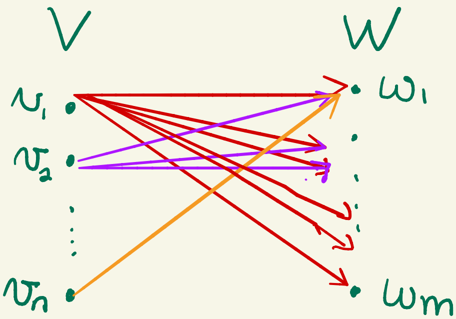
$$S_2 = \{ x_{1i} x_{2j} \;/\; r+1 \leq i \leq n-s \quad n-s+1 \leq j \leq n \}$$

$$S_3 = \{ x_{1i} x_{2j} \;/\; r+1 \leq i \leq n-s-1 \quad i+1 \leq j \leq n-s \}$$

KEY FACT: $\mathrm{In}_<(I)$ is a Ferrer ideal, that is the edge ideal of a Ferrer bipartite graph

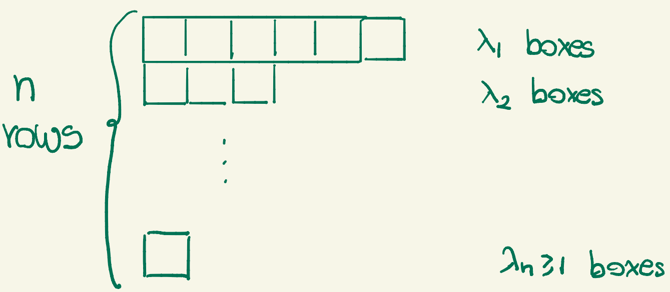
A Ferrer bipartite graph is a graph on two distinct set of vertices $V=\{v_1,\dots,v_n\}$ $W=\{w_1,\dots,w_m\}$ that can be described by a partition

$$\lambda=(m,\lambda_2,\dots,\lambda_n) \quad \text{with} \quad m=\lambda_1\geq\dots\geq\lambda_n\geq 1$$



λ_i = # edges between v_i and the first λ_i vertices in W

$\lambda \rightsquigarrow$ Young diagram, called Ferrer diagram



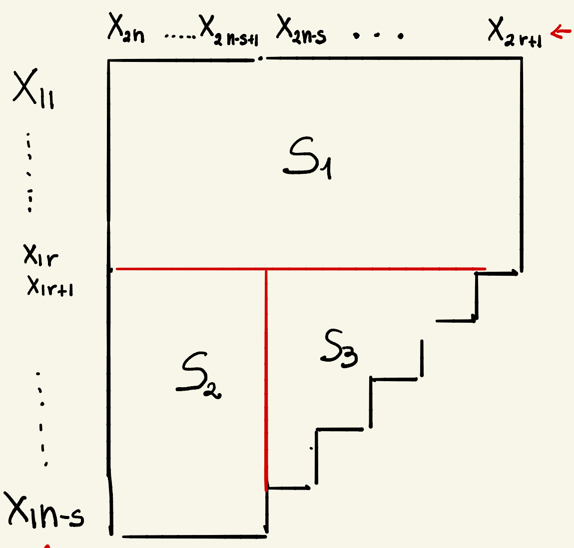
KEY FACT $in_\tau(I)$ is Ferrer ideal

$in_\tau(I)$ is the edge ideal of the Ferrer bipartite graph whose vertex sets (ordered sets)

$$V = \{x_{11}, \dots, x_{1\,n-s}\} \qquad W = \{\underbrace{x_{2n}, \dots, x_{2\,r+1}}_{\text{reverse order}}\}$$

and whose partition is

$$\lambda = (\underbrace{n-r, \dots, n-r}_{r \text{ times}}, n-r-1, \dots, s) \text{ if } r>0$$



← Vertices in W (in order)

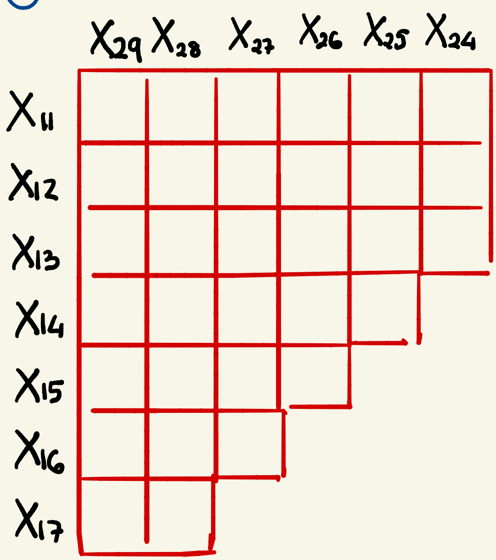
Ferrer diagram associated to $in_\tau(I)$

↑ Vertices in V

Ex: $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & \overbrace{0 \quad 0}^{s=2} \\ \underbrace{0 \quad 0 \quad 0}_{r=3} & & & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} & x_{29} \end{bmatrix}$

$I = I_2(X)$ $\quad$ $in_\tau(I)$ is the Ferrer ideal associated to $\lambda = (6,6,6,5,4,3,2)$

whose Ferrer diagram is:



There are two LADDERS associated to $in_\tau(I)$

a <u>ladder</u> is a subset of a matrix that has a shape similar to the Ferrer diagram the staircase can be in both "directions"

We put a NEW variable T_{ij} in the box corresponding to the generator $x_{1i}x_{2j}$

one-sided ladder L_λ

	X_{29}	X_{28}	X_{27}	X_{26}	X_{25}	X_{24}
X_{11}	T_{19}	T_{18}	T_{17}	T_{16}	T_{15}	T_{14}
X_{12}	T_{29}	T_{28}	T_{27}	T_{26}	T_{25}	T_{24}
X_{13}	T_{39}	T_{38}	T_{37}	T_{36}	T_{35}	T_{34}
X_{14}	T_{49}	T_{48}	T_{47}	T_{46}	T_{45}	
X_{15}	T_{59}	T_{58}	T_{57}	T_{56}		
X_{16}	T_{69}	T_{68}	T_{67}			
X_{17}	T_{79}	T_{78}				

(two-sided) ladder L'_λ

	X_{29}	X_{28}	X_{27}	X_{26}	X_{25}	X_{24}
X_{11}	T_{19}	T_{18}	T_{17}	T_{16}	T_{15}	T_{14}
X_{12}	T_{29}	T_{28}	T_{27}	T_{26}	T_{25}	T_{24}
X_{13}	T_{39}	T_{38}	T_{37}	T_{36}	T_{35}	T_{24}
X_{14}	T_{49}	T_{48}	T_{47}	T_{46}	T_{46}	
X_{15}	T_{59}	T_{58}	T_{57}	T_{56}		
X_{16}	T_{69}	T_{68}	T_{67}			
X_{17}	T_{79}	T_{78}				

$$\mathcal{R}(\mathrm{in}_\tau(I)) \qquad \mathcal{F}(\mathrm{in}_\tau(I))$$

$$\varphi' : S = R[T_{ij}] \twoheadrightarrow \mathcal{R}(\mathrm{in}_\tau(I))$$

$$T_{ij} \mapsto \mathrm{in}_\tau(f_{ij}t) = x_{1i}x_{2j}t$$

$$\varphi' : T = K[T_{ij}] \twoheadrightarrow \mathcal{F}(\mathrm{in}_\tau(I))$$

$$T_{ij} \longmapsto \mathrm{in}_\tau(f_{ij}) = x_{1i}x_{2j}$$

Narasimhan ['86] [N], Herzog-Trung ['92], Conca ['95] [C]

Villarreal ['95] [V], Simis-Vasconcelos-V ['94] [SVV],

Corso-Nagel ['09] [CN]

Thm: (1) $\operatorname{Ker}\varphi' = I_2(L_\lambda)$

(2) $\operatorname{Ker}\rho' = I_2(L'_\lambda)$

(3) $\operatorname{rtype}(\operatorname{in}_\tau(I)) = 2$

(4) $\mathcal{R}(\operatorname{in}_\tau(I))$ is normal CM

(5) $\mathcal{F}(\operatorname{in}_\tau(I))$ is normal CM of dim $\min\{2n-3, 2n-r-s-1\}$

(6) wrt a diagonal term order on L_λ and L'_λ the gens of $I_2(L_\lambda)$ and $I_2(L'_\lambda)$ form a GB for $\operatorname{Ker}\varphi'$ and $\operatorname{Ker}\rho'$

(7) $\mathcal{R}(\operatorname{in}_\tau(I))$ and $\mathcal{F}(\operatorname{in}_\tau(I))$ are Koszul

Pf: (1) [CN]

(2) [V] $\Rightarrow$ $\operatorname{in}_\tau(I)$ is of fiber type $\Rightarrow$ $\operatorname{Ker}\rho' = (\underbrace{\mathcal{L}, I_2(L_\lambda)}_{= I_2(L'_\lambda)})$

(4) and (5) [SVV], [CN], or [C]

[6] [N] $\blacksquare$

We want to study $\mathcal{R}(I)$ and $\mathcal{F}(I)$ using SAGBI basis theory. We extend $\tau \rightsquigarrow \tau'$ on $R[t]$

$$\mathcal{R}(\operatorname{in}_\tau(I)) \overset{?}{=} \operatorname{in}_{\tau'}(\mathcal{R}(I))$$

Thm : (1) $\mathrm{in}_\tau(I)^2 = \mathrm{in}_\tau(I^2)$

(2) $\mathrm{in}_{\tau'}(\mathcal{R}(I)) = \mathcal{R}(\mathrm{in}_\tau(I))$

Pf: $\mathrm{in}_\tau(I)^2 \subset \mathrm{in}_\tau(I^2)$ and $HF(\mathrm{in}_\tau(I^2)) = HF(I^2)$

ETS

$\mathrm{in}_\tau(I)^2$ and I^2 have the same HF

By induction on n. We can assume $r>0$ by Conca

$I_1 = \langle W \rangle \qquad I_2 = I_2$ (X' = matrix obtained by X deleting the first column)

Since $r>0$

$$I = x_{11}\underbrace{I_1}_{\text{monomial ideal}} + I_2 \quad \Rightarrow \quad I^2 = (x_{11}^2 I_1^2 + x_{11} I_1 I_2) + I_2^2$$

$$\mathrm{in}_\tau(I) = x_{11} I_1 + \mathrm{in}_\tau(I_2) \quad \mathrm{in}_\tau(I)^2 = (x_{11}^2 I_1^2 + x_{11} I_1 \mathrm{in}_\tau(I_2)) + \mathrm{in}_\tau(I_2)^2$$

$\blacksquare$

Using the results [CHV] reviewed in the 2nd part of the talk:

Thm : (1) $\{x_{ij}\} \cup \{f_{ij}t\}$ is a SAGBI basis for $\mathcal{R}(I)$

(2) $\mathcal{R}(I)$ has rational singularities in $ch(k)=0$

(3) $\mathcal{R}(I)$ is a Koszul algebra

(4) $\mathrm{reg}(I^\ell) = 2\ell \quad \forall \ell$

Defining Equations

$$0 \to J \to S = R[T_{ij}] \to Q(I) \to 0$$
$$T_{ij} \mapsto f_{ij}t$$

$$0 \to H \to T = K[T_{ij}] \to \mathcal{F}(I) \to 0$$
$$T_{ij} \mapsto f_{ij}$$

NB We refer to other X_{ui} or T_{ij} to make the notation easier with

IDENTIFICATION

$$X_{ui} = 0 \quad \text{if} \quad u=2 \quad 1 \le i \le r$$
$$\qquad \text{if} \quad u=1 \quad n-s+1 \le i \le n$$

$$T_{ij} = 0 \quad \text{if} \quad i,j \in \{1, \dots, r\}$$
$$\qquad \text{if} \quad i,j \in \{n-s+1, \dots, n\}$$

Recall from the 1st part of the talk:

Plücker ideal:

$$Q = \langle P_{ijkl} = T_{ij}T_{kl} - T_{ik}T_{jl} + T_{il}T_{jk} : 1 \le i < j < k < l \le n \rangle$$

Linear ideal:

$$\mathcal{L} = \langle l_{uijk} = X_{ui}T_{jk} - X_{uj}T_{ik} + X_{uk}T_{ij} : \begin{array}{c} u = 1,2 \\ 1 \le i < j < k \le n \end{array} \rangle$$

Thm: (1) $J = (\mathcal{L}, \mathcal{P})$

(2) $H = \mathcal{P}$

In particular I is of fiber type

(3) $\{l_{uijk}, P_{ijkl}\}$ and $\{P_{ijkl}\}$ are a GB for J and H, respectively

Pf: (1) [S] suitable weight ω

ω is defined by the degree matrix

$$\begin{bmatrix} 1 & \cdots & & 1 \\ 1 & 2 & \cdots & n \end{bmatrix}$$

that is

$$\deg_\omega(x_{1i}) = 1 \qquad \deg_\omega(x_{2j}) = j$$

we extend ω to T and S

$$\deg_\omega(T_{ij}) = \deg_\omega(\text{in}_\tau(f_{ij})) = \deg_\omega(x_{1i}x_{2j}) = j+1$$

This weight represents τ since

$$\text{in}_\omega(f_{ij}) = \text{in}_\tau(f_{ij}) \quad \forall i,j$$

$$[S] \quad \Longrightarrow \quad \text{in}_\omega(J) = \overbrace{I_2(L'_\lambda)}^{\text{defining ideal of } \text{in}_{\tau'}(\mathcal{R}(I))}$$

since $\text{in}_{\tau'}(\mathcal{R}(I)) = \mathcal{R}(\text{in}_\tau(I))$

Since $(\mathcal{L}, \mathcal{P}) \subset J$ ETS that

$$\mathrm{in}_\omega(\mathcal{L}, \mathcal{P}) = I_2(L'_\lambda)$$

that is each 2×2 minor of L'_λ is the leading form wrt ω of a P_{ijke} or ℓ_{uijk}.

(3) [S] $\blacksquare$

<u>COR</u> (1) $\{f_{ij}\}$ is a SAGBI basis for $\mathcal{J}(I)$

hence $\mathrm{in}_{\tau'}(\mathcal{J}(I)) = \mathcal{J}(\mathrm{in}_\tau(I))$

(2) $\mathcal{J}(I)$ Koszul

(3) $\mathcal{J}(I)$ has rational singularities in $ch(k)=0$

(4) $\mathcal{J}(I)$ is Gorenstein if $r \leq 2$

(5) $\mathrm{reg}\, \mathcal{J}(I) = \min\{n-3, n-r-1\}$

$\vdots$

<u>Conjecture</u> Same results are valid for any m

[Lin-Shen] $m=3$ 3-dim Ferrer diagrams with projection property