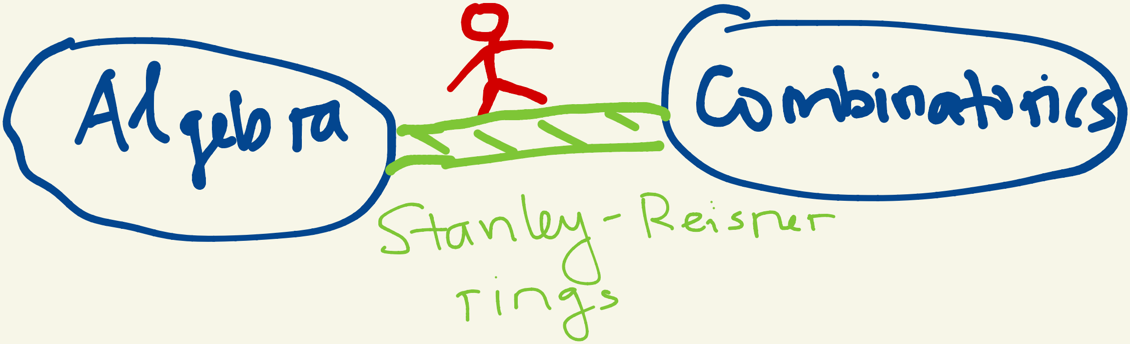


Part 1 Counting polynomials

$K = \overline{k}$, $S = k[x_1, \ldots, x_n]$, $I \subseteq (\underline{x})^2$ homogeneous

$c = ht(I)$, $d = \dim S/I$

$\boxed{\text{Theme: If } R = S/I \text{ is "nice", how many independent quadrics, cubics, etc can } I \text{ contain?}}$

ie: what can be $\dim_k I_e$, $e \geq 2$

Fundamental problem in Comm. alg, alg. geo. & combinatorics

Two special cases:

Q1 If Δ is a triangulation of $(d-1)$ sphere, w/ n vertices, how many i-dim faces can Δ contain?

Q2 $I \subseteq k[x_1, \ldots x_n]$ is prime how many ind. quadrics can I contain?

A1 $|f_i(\Delta)| \leq |f(C(n,d))|$

cyclic polytope

A2 $\dim_k I_2 \leq \binom{c+1}{2}$

$= \mathrm{conv}(r(t_1), \ldots r(t_n))$

$r(t) = (t, t^2 \ldots t^d)$

$\boxed{\text{h-vector}}$ $R = S/I$

$$H_R(t) = \sum \dim_k R_i \, t^i$$

$$= \frac{h_0 + h_1 t + \dots + h_s t^s}{(1-t)^d} \leftarrow \dim R$$

$h_i(R) = h_i$, $\vec{h} = (h_0, h_1, \dots, h_s)$

<u>Examples</u>

① $I = (x,y) \cap (y,z) \cap (z,x)$
$= (xy, yz, zx)$

$R = k[x,y,z]/I$ $R_i = \langle x^i, y^i, z^i \rangle$

$$H_R = 1 + 3t + 3t^2 + \dots = \frac{1+2t}{(1-t)}$$

$\vec{h} = (1,2)$

Ex2 $I = (x,y) \cap (z,w)$

$$= (xz, xw, yz, yw)$$

$R = S/I \quad 0 \to S/J\cap K \to S/J \oplus S/K \oplus S/J+K \to 0$

$$0 \to S/I \to S/(x,y) \oplus S/(z,w) \to k \to 0$$

$$H_R = H_{S/(x,y)} + H_{S/(z,w)} - H_k$$

$$= \frac{2}{(1-t)^2} - 1 = \frac{1+2t-t^2}{(1-t)^2}$$

$$\vec{h} = (1, 2, -1)$$

Ex3 $R = k[x,y,z,w] / (xw - yz,\ y^3 - x^2 z,\ z^3 - y w^2,\ x z^2 - y^2 w)$

$\cong k[a^4, a^3 b, a b^3, b^4]$ (missing $a^2 b^2$)

(non CM domain)

$\overline{R} = k[a^4, a^3 b, a^2 b^2, a b^3, b^4]$

$$0 \to R \longrightarrow \overline{R} \longrightarrow k(-1) \longrightarrow 0$$

$$H_{\overline{R}}(t) = 1 + 5t + 9t^2 + \cdots$$

$\boxed{M(-i)_j = M_{j-i}}$

$$H_R(t) = H_{\overline{R}}(t) - t$$

$$= \frac{1+3t}{(1-t)^2} - t = \frac{1+2t+2t^3-t^3}{(1-t)^2}$$

$$\vec{h} = (1,2,2,-1)$$

h-vectors give a good way to phrase the results A1, A2

$\boxed{\text{Th1}}$ R is CM then

$$0 \leq h_i \leq \binom{c+i-1}{i}$$

(implies UBC/T)

$\boxed{\text{Th2}}$ R is domain $\Rightarrow$ $h_2 \geq 0$

Th1 (proof)

<u>Lemma</u> l is a linear NZD on M

$$\Rightarrow \vec{h}(M) = \vec{h}(M/lM)$$

$$0 \to M(-1) \xrightarrow{\times l} M \to M/lM \to 0$$

$$H_M(t)(1-t) = H_{M/lM}$$

$\boxtimes$

$R = S/I$ CM dim d, $|k| = \infty$

$\Rightarrow l_1, l_2, \ldots, l_d$, $R' = R/(l_1, \ldots, l_d)$

$\vec{h}(R) = \vec{h}(R')$

$\qquad \qquad \qquad \qquad \qquad \| $

$\qquad \qquad \qquad \qquad \qquad k[y_1, \ldots, y_c]/J$

$h_i(R') = \dim R'_i \qquad (n = c + d)$

$0 \leq h_i \leq \dim k[y_1, \ldots, y_c]_i = \binom{c+i-1}{c}$

How does 1 $\Rightarrow$ UBC ?

Δ simp clx, $I_\Delta = \langle$ min non-face of $\Delta \rangle$

$R = S/I_\Delta = k[\Delta]$ Stanley-Reisner

There is Hochster's formula for $H^i_m(R)$

$R = k[\Delta]$ is CM $\Leftrightarrow$

$\widetilde{H}_i(\mathrm{lk}_\Delta F, k) = 0$, $i < \dim \mathrm{lk}\, F$

$\Rightarrow \Delta$ tri. of sphere $\Rightarrow k[\Delta]$ is CM

② UBC $\Leftrightarrow h_i(k[\Delta]) \leq \binom{c+i-1}{i}$

<u>Theorem 2</u> R domain

$\Rightarrow h_2 \geq 0$ (Castelnuovo 1897)

<u>Proof</u> (geometric, hand-wavy

$X = \mathrm{Proj}\, R$

$X \cap \mathbb{P}^c$ $\quad (c = \operatorname{codim} X)$

$\uparrow$ general

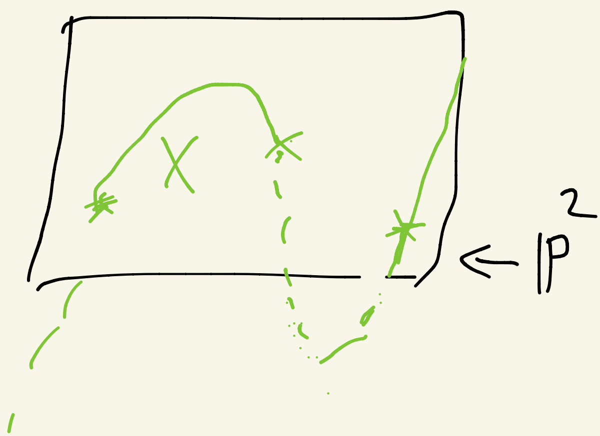
$\|$

$\{\text{pts}\} \supseteq \left\{ \begin{array}{l} c+1 \text{ pts} \\ \text{in gen position} \end{array} \right\}$

Γ

$I_X \subseteq I_\Gamma, \quad \dim (I_\Gamma)_2 = \binom{c+1}{2}$

$X =$ curve in $\mathbb{P}^3$, $c = 2$



$\vec{h} = (h_0, h_1, \dots)$

$h_0 = 1,\ h_1 = c,\ e(R) = \sum_{i \geq 0} h_i$

Thm 2 (more algebraic)

assume $d = 2$, domain

ℓ : linear NZD on R

$R' = R/\ell R \leftarrow$ dim 1

$\vec{h}(R') = h(R)$, WTS: $h_2(R') \geq 0$

$$0 \to \underbrace{H^0_m(R')}_{L} \to R' \to \underset{\text{CM}}{T} \to 0$$

$$\underbrace{(1-t)H_L}_{\text{need } h_2 \geq 0} = h(R') - \underset{h_2 \geq 0}{h'(T)}$$

$\Rightarrow$ need $L_{\leq 1} = 0$

Recall, need $H^0_m(R/\ell R)_{\leq 1} = 0$

$$0 \to R(-1) \xrightarrow{\times \ell} R \to \underset{\text{“}R'\text{”}}{R/\ell R} \to 0$$

$\Gamma(\ ) \Rightarrow 0 \to H^0(R') \to H^1_m(R(-1)) \to H^1_m(R)$ ..

$\Rightarrow$ need $\boxed{H^1_m(R)_{\leq 0} = 0}$

($X = \operatorname{Proj} R$, X connected)

$$0 \to R \to \bar{R} \to \bar{R}/R \to 0$$

$\bar{R}$ ← normalization

CM

$H^1_m(R) = H^0_m(\bar{R}/R)$

$(\bar{R}/R)_{\leq 0} = 0$ (use $k = \bar{k}$)

Generalizations of Thm 1 & 2

(Alg proof of Thm 2)

* Need cohomological vanishing

* May need to compare with coh. of desingularization

* Same proof $\Rightarrow$ $e(R) \geq c+1 \Leftrightarrow h_0 + h_1$

$$\Updownarrow$$

$$h_2 + h_3 + \dots \geq 0$$

* Also $h_1 = 0 \Rightarrow$ R is CM.

Murai-Terai (2009) Δ simp clx.

$R = S/I_\Delta$ is (S_r) ← Serre's condition

then $h_r(R) \geq 0$.

$$\boxed{\operatorname{depth} R_p \geq \min\{r, \dim R_p\}}$$

- CM $\Rightarrow S_r$ (any r) $\Rightarrow S_{r-1} \Rightarrow \dots$

- R normal $\Leftrightarrow (R_1) + (S_2)$

- $S_2 \Rightarrow$ pure

<u>Thm</u> (D, Ma, Varbaro)

$R = S/I$, If R is (S_r) + reasonable sing

$\Rightarrow h_r \geq 0$

R is F-pure char > 0

cohomological vanishing $\Leftarrow$ or Proj R has at worst Du Bois sing

(Stanley-Reisner rings are F-pure / Du Bois)

Key $K_i := \mathrm{Ext}_S^{n-i}(R, \omega_S)$ $\quad \omega_S = S(-n)$

"deficiency modules" $\quad (= D(H^i_m(R))$ Matlis dual

. We say that R satisfies (MT_r) if $\mathrm{reg}\, K_i \leq i - r$, $0 \leq i \leq d-1$

(Note: R CM $\Rightarrow$ (MT_r) all r)

<u>Thm</u> (DMV, MT)

1). R is $(MT_r) \Rightarrow R/\ell R = R'$, $R'/H^0_m(R')$ also (MT_r)

2) R is $(MT_r) \Rightarrow h_r \geq 0$
If $h_r = 0 \Rightarrow R$ is CM

$\boxed{\text{Lemma}}$ If R is pure then:

R is $(S_r) \iff \dim K_i(R) \leq i - r$

<u>Facts</u> If R has "reasonable singularities" $\Longrightarrow$

$$H^j_m(K_i)_{>0} = 0 \quad \forall j$$

$$\operatorname{reg}(K_i) \leq \dim K_i$$

$$+(S_r) \qquad \leq i - r$$

$$\Rightarrow (MT_r)$$

Well-known : $\ell(H^i_m(R))$

is $< \infty$ & R is F-pure

action of Frobenius $\Rightarrow$

$$H^i_m(R)_{<0} = 0$$

$$\Rightarrow K_i(R)_{>0} = 0$$

$$\Rightarrow \operatorname{reg} K_i \leq 0$$

Open Question

Suppose $R = S/I$ is (R_1) + (S_3). Is $h_3(R) \geq 0$?