Genuine Equivariant Factorization Homology

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Before we start

Factorization homology	(Ayala-Francis)
Parametrized ∞-categories	(Barwick-Dotto-Glasman-Nardin-Shah)

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Appropriate context for equivariant factorization homology (in stable equivariant homotopy)

Factorization homology [Ayala-Francis]

Have: $\mathbb{E}_n$-algebra A, framed n-manifold M ($TM \cong M \times \mathbb{R}^n$)
Want: $\int_M A$

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- D_n little disk operad

$\mathbb{E}_n = N(D_n)$

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$$\mathbb{E}_n = N(D_n) \xrightarrow{A} \mathbf{Sp}$$

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Theorem [Ayala-Francis]

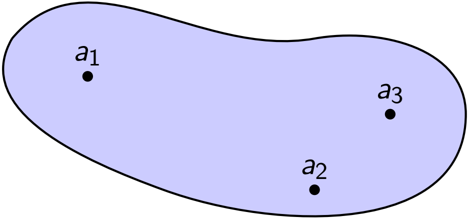
∃! SM extension of A satisfying axioms

- Construct $\int_{-} A$ as LKE,

$$\int_M A \simeq \underrightarrow{colim}\Big((\mathbf{Disk}^{fr}_n)_{/M} \to \mathbf{Disk}^{fr}_n \xrightarrow{A} \mathbf{Sp}\Big)$$

Visualizing factorization homology

Think: $\int_M A \simeq$ labeled configurations,



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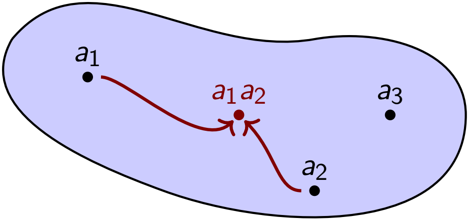

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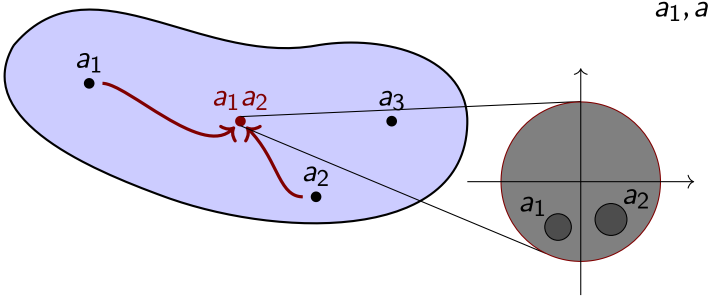

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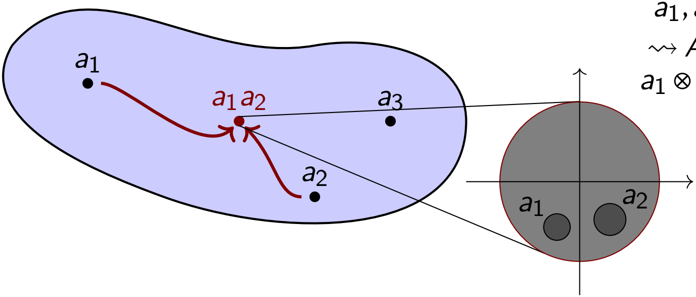
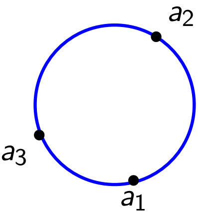


$$a_1, a_2, a_3 \in A$$
$$\rightsquigarrow A \otimes A \to A$$
$$a_1 \otimes a_2 \mapsto a_1 a_2$$

A guiding example: THH

- $A \in Alg(\mathbf{Sp})$, $M = S^1$:
  $\int_{S^1} A \simeq THH(A)$



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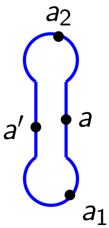

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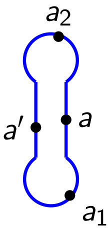

A
↑
A-A bimodule

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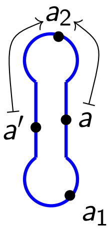

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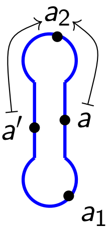
- A ring w/ anti-involution $a \mapsto \overline{a}$:
  $\rightsquigarrow THR(A) \in \mathbf{Sp}_{C_2}$
  Real THH [Hesselholt-Madsen]
  (e.g. $A = M_n(\mathbb{C})$, $M \mapsto \overline{M}^T$)



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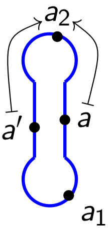

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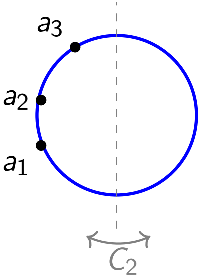

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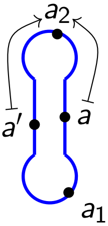

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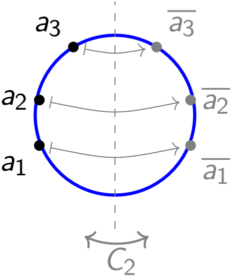

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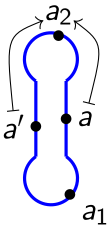

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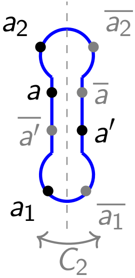

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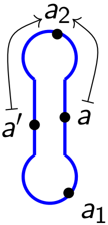

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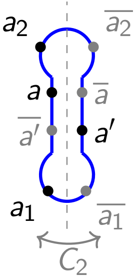

$$\begin{tikzcd} A \arrow[loop above, "C_2"] \\ A\otimes A = NA \arrow[u] \arrow[d] \arrow[loop left, "C_2"] \\ A \arrow[loop below, "C_2"] \end{tikzcd}$$

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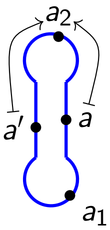
$THR(A) \simeq A \otimes_{NA} A$



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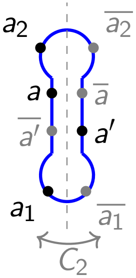

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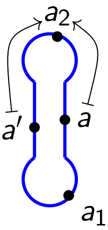
<u>Want</u>: $\int_{S^1} A \simeq A \otimes_{NA} A$ as C_2-spectra.



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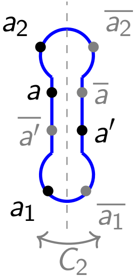

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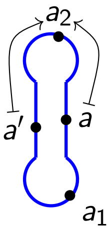

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Want: $\int_{S^1} A \simeq A \otimes_{NA} A$ as C_2-spectra.

Fixed points:

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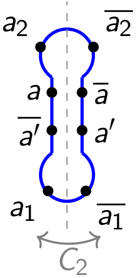

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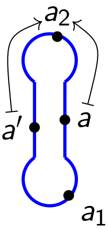
Takeaway: Think of $(\int_M A)^{\Phi G}$ as labeled configurations of orbits!



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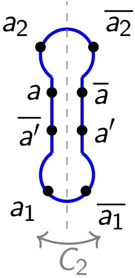

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\mathbb{E}_n = N(D_n) \arrow[r, "A"] \arrow[d, hook] & \mathbf{Sp} \\
Env^{\otimes}(\mathbb{E}_n) \arrow[ur, dashed, "\exists ! SM"'] \arrow[d, dash, "\simeq"] & \\
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- D_n little disk operad
- $Env^{\otimes}(\mathbb{E}_n)$ symmetric monoidal envelope.
- $\mathbf{Mfld}_n^{fr}$ = framed n-manifolds and open embeddings
- $\{\sqcup_I \mathbb{R}^n | I \text{ finite}\} = \mathbf{Disk}_n^{fr} \subset \mathbf{Mfld}_n^{fr}$, full
  Fact: $Env^{\otimes}(\mathbb{E}_n) \simeq \mathbf{Disk}_n^{fr}$.

Theorem [Ayala-Francis]

∃! SM extension of A satisfying axioms

- Construct $\int_- A$ as LKE,

$$\int_M A \simeq \underrightarrow{colim}\Big((\mathbf{Disk}_n^{fr})_{/M} \to \mathbf{Disk}_n^{fr} \xrightarrow{A} \mathbf{Sp}\Big)$$

Definition: Equivariant framing [H,Weelinck]

Definition: V-framed G-manifold

$G \curvearrowright M$ smooth $\boxed{TM \cong M \times V}$ $V = (G \curvearrowright \mathbb{R}^n)$ linear

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- Note that $S^1 \times \mathbb{R}^1 \cong \mathbb{R}^2 \setminus \{0\}$ can be framed differently:
  $\lambda = (C_n \curvearrowright \mathbb{R}^2)$ the standard rep $\Rightarrow \mathbb{R}^2 \setminus \{0\}$ is λ-framed.

Definition: G-disks [H]

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- Capture this structure using *parametrized ∞-categories* [Barwich-Dotto-Glasman-Nardin-Shah]

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Theorem [H]

$\exists !$ G-SM extension of A, A.-F.-axioms

What is $\int_M A$ again?

$\mathbb{E}_V$-algebra is a G-SM functor

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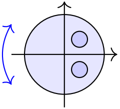

$\longmapsto (NA \to A)$

norm map

What is $\int_M A$ again?

$\mathbb{E}_V$-algebra is a G-SM functor

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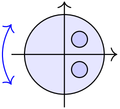

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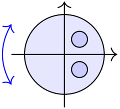

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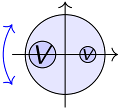

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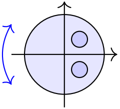

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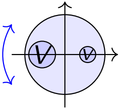

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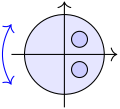

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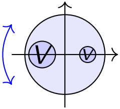

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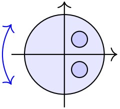

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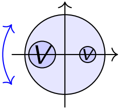

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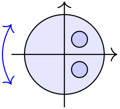

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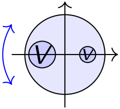

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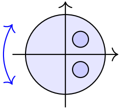

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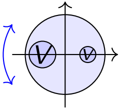

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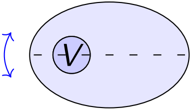

$\longmapsto A$

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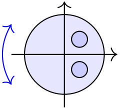

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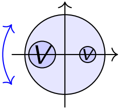

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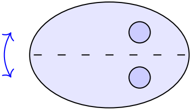

$\longmapsto NA$

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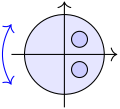

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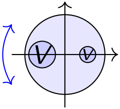

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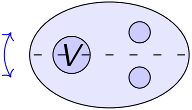

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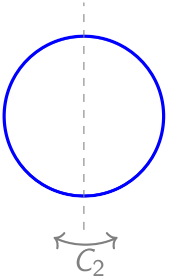
A factorization homology theory (Ayala-Francis)

Example 1: Real THH [Hesselholt-Madsen]

- Let $G = C_2$
- $V = \sigma$ the sign representation
- $M = S^1$ with reflection action is σ-framed



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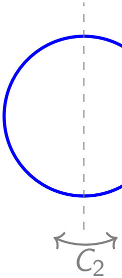

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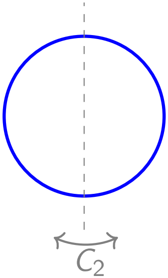

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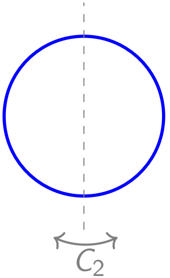

Theorem [H]

$\int_{S^1} A \simeq THR(A)$ as genuine C_2-spectra

Example 2: “Twisted” THH [Angeltveit-Blumberg-Gerhardt-Hill-Lawson-Mandell]

A ring C_n-spectrum, $THH_{C_n}(A) := |B_\bullet^{cyc, C_n}|$

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$$B_q^{cy,C_n} = A^{\otimes q+1},$$

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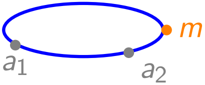

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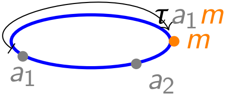

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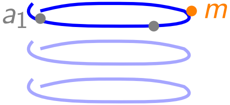

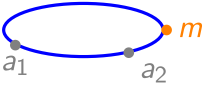

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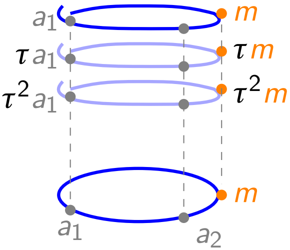

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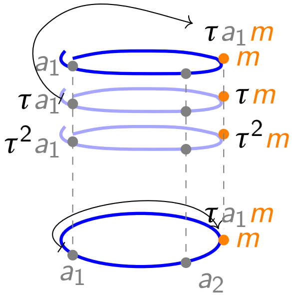

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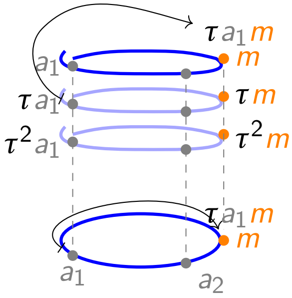

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$THH_{C_n}(A) \simeq (\int_{S^1} A)^{\Phi C_n}$
Circle action: rotate S^1.

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where B is a Mackey functor.

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- [Hahn-Wilson] calculate $THR(H\underline{\mathbb{Z}})$ as $H\underline{\mathbb{Z}}$-module.

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