Steven Dale Cutkosky
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Multiplicities and Mixed Multiplicities of Filtrations

R, m_R a (Noetherian) local ring.

An m_R-filtration is a family of ideals $\mathcal{I} = \{I_n\}_{n \in \mathbb{N}}$

$R = I_0 \supset I_1 \supset I_2 \supset \cdots$

I_n m_R-primary for $n > 0$

$I_i I_j \subset I_{i+j}$ $\forall i, j$.

$\mathcal{I}$ is Noetherian if

$\bigoplus_{n \geq 0} I_n$ is a f.g. R-algebra

Ex1 I m_R-primary

$\mathcal{I} = \{I^n\}$.

Ex2 $R \subset S$ $m_S \cap R = m_R$

$\mathcal{I} = \{m_S^n \cap R\}$

Ex3 R local domain, μ = valuation with value group $\mathbb{Z}$. $R \subset \mathcal{O}_\mu$ $m_\mu \cap R = m_R$

$I(\mu)_n = \{f \in R \mid \mu(f) \geq n\}$

$\mathcal{I} = \{I(\mu)_n\}$

Ex4 R local excellent domain $\varphi : X \to \operatorname{spec} R$

the normalization of the blow up of an m_a - primary ideal with prime exceptional divisors $E_1, \ldots, E_r$

v_{E_i} = valuation with valuation ring $\mathcal{O}_{X,E_i}$. For $a_1, \ldots, a_r \in \mathbb{Z}_{\geq 0}$

$D = a_1 E_1 + \cdots + a_r E_r$

$I(mD) = \Gamma(X, \mathcal{O}_X(-ma_1E_1 - \cdots - ma_rE_r)) \cap R$

$= I(v_{E_1})_{ma_1} \cap \cdots \cap I(v_{E_r})_{ma_r}$

$\mathcal{I}(D) = \{I(mD)\}$

a divisorial m_a - filtration

The pair $\varphi: X \to \operatorname{Spec}(R)$ and Expression $D = \sum a_i E_i$ is called a representation of $\mathfrak{a}(D)$

Ex 1 is always Noetherian

Examples 2, 3, 4 are often not Noetherian, even in regular local rings.

R local ring of dim d

$I = m_R$-primary ideal

$\ell(R/I^n) =$ polynomial of degree d for $n \gg 0$

$$= \frac{e(\mathcal{I})}{d!} n^d + \cdots$$

$e(\mathcal{I}) \in \mathbb{Z}_{>0}$ is the multiplicity

$$e(\mathcal{I}) = \lim_{n \to \infty} \frac{e(R/I^n)}{n^d/d!}$$

Theorem A (—) Suppose R is a local ring of dim d and $N(\hat{R})$ = nilradical of $\hat{R}$. Then the limit

$$\lim_{n \to \infty} \frac{e(R/I_n)}{n^d}$$

exists for every m_R-filtration $\mathcal{I} = \{I_n\} \Leftrightarrow \dim N(\hat{R}) < d$

so limits always exist if R is analytically unramified ($\hat{R}$ is reduced) or if R is a reduced excellent local ring. (This limit was shown to exist in some cases by Ein, Lazarsfeld and Smith and by Mustata, and shown to exist for local rings of closed points on varieties over an alg. closed field by Lazarsfeld and Mustata, using methods of algebraic geometry. They use the method of Okounkov bodies, developed by Okounkov, Kaveh-Khovanskii and Lazarsfeld-Mustata. We also use this method. The fact that $\dim N(R) = d$ $\Rightarrow$ $\exists$ a filtration without a limit was observed by Dao and Smirnov)

Define the multiplicity

$$e(\mathcal{I}) = \lim_{n\to\infty} \frac{\ell(R/I_n)}{n^d/d!}$$

(when it exists)

Theorem (Bhattacharya, Rees, Rubin and Teissier)

Let $I_1, \ldots, I_r$ be m_R-primary ideals. Then for $n_1, \ldots, n_r \in \mathbb{N}$

with $n_1 + \dots + n_r \gg 0$

$$e(R/I_1^{n_1} \cdots I_r^{n_r}) =$$

$$= \text{polynomial in } n_1, \dots, n_r \text{ of degree } d$$

$$= \underbrace{\sum_{d_1 + \dots + d_r = d} \frac{1}{d_1! \cdots d_r!} e(I_1^{[d_1]}, \dots, I_r^{[d_r]}) n_1^{d_1} \cdots n_r^{d_r}}_{\substack{\| \\ H(n_1, \dots, n_r)}} + \text{LOT}$$

$$\lim_{m \to \infty} \frac{e(R/I_1^{mn_1} \cdots I_r^{mn_r})}{m^d} = H(n_1, \dots, n_r) \quad \forall\, n_1, \dots, n_r \in \mathbb{N}$$

Theorem B (–, Sarkar, Srinivasan)

Suppose R is a local ring of dim d such that $\dim N(\hat{R}) < d$ and

$\mathcal{I}(1) = \{I(1)_n\}, \ldots, \mathcal{I}(r) = \{I(r)_n\}$

m_R-filtrations. Then

$$P(n_1, \ldots, n_r) = \lim_{m \to \infty} \frac{e(R/I(1)_{mn_1} \cdots I(r)_{mn_r})}{m^d}$$

is a homogeneous polynomial of degree d for $n_1, \ldots, n_r \in \mathbb{N}$.

Write

$$P(n_1, \ldots, n_r) = \sum_{d_1 + \cdots + d_r = d} \frac{1}{d_1! \cdots d_r!} e(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}) n_1^{d_1} \cdots n_r^{d_r}$$

mixed multiplicities

$$e(\mathcal{I}(i)) = e(\mathcal{I}(i)^{[i]}) \in \mathbb{R}_{\geq 0}$$

$\mathcal{I} = \{I_n\}$ an m_R-filtration

The integral closure

$$\overline{\sum_{n\geq 0} I_n t^n} \text{ of } \sum_{n\geq 0} I_n t^n \text{ in } R[t]$$

is $\sum_{n\geq 0} J_n t^n$

where

$J_n = \{ f \in R \mid f^r \in \overline{I_{rn}}$ for some $r > 0 \}$.

Theorem (Rees) Suppose $I' \subset I$ are $\mathfrak{m}_R$-primary ideals and R is formally equidimensional. Then the following are equivalent

1) $e(I') = e(I)$

2) $\overline{\sum_{n \geq 0} (I')^n t^n} = \overline{\sum_{n \geq 0} I^n t^n}$

3) $\overline{I'} = \overline{I}$

<u>Question:</u> Suppose $\mathcal{I}' = \{I_n'\} \subset \mathcal{I} = \{I_n\}$ are $\mathfrak{m}_R$-filtrations

Are

1) $e(\mathcal{I}') = e(\mathcal{I})$

2) $\overline{\sum_{n\geq 0} I_n' t^n} = \overline{\sum_{n\geq 0} I_n t^n}$

equivalent?

2) $\Rightarrow$ 1) is true for m_R-filtrations
(if $\dim N(\hat{R}) < d$)

1) $\Rightarrow$ 2) is false for arbitrary filtrations.

If $\mathcal{I}(D)$ is a divisorial m_R-filtration

then $\sum_{n\geq 0} I(nD)t^n$ is

integrally closed in $R[t]$?

Theorem C (—) (Rees theorem for filtrations) 1) $\Leftrightarrow$ 2)

Suppose R is an excellent local domain, let $\mathcal{I}$ be an m_R-filtration and $\mathcal{I}(D)$ a divisorial m_R-filtration such that $\mathcal{I}(D) \subset \mathcal{I}$. Then

$$e(\mathcal{I}(D)) = e(\mathcal{I}) \Leftrightarrow \mathcal{I} = \mathcal{I}(D).$$

Minkowski inequalities for mixed multiplicities of m_R-primary ideals in local rings were proven by Teissier and Rees and Sharp, Katz

Theorem D (—, Sarkar, Srinivasan)

Minkowski inequalities for filtrations.

Suppose R is a d-dim local ring with $\dim N(\hat{R}) < d$. Let $\mathcal{I}(1)$ and $\mathcal{I}(2)$ be m_R-filtrations. Set.

$$e_i = e(\mathcal{I}(1)^{[d-i]}, \mathcal{I}(2)^{[i]}). \text{ Then}$$

$$e_i^2 \leq e_{i-1} e_{i+1} \quad \text{for } 1 \leq i \leq d-1$$

$\Rightarrow$ a series of other inequalities and "The Minkowski inequality"

$$(*) \quad e(\underbrace{\mathcal{I}(1)\mathcal{I}(2)}_{\{I(1)_n I(2)_n\}})^{\frac{1}{d}} \leq e(\mathcal{I}(1))^{\frac{1}{d}} + e(\mathcal{I}(2))^{\frac{1}{d}}$$

((*) was proven by Mustata for $\overline{\text{RLRs}}$ with alg. closed residue fields)

Theorem (Teissier, Rees and Sharp, Katz) Suppose R is a d-dim formally equidim. local ring and $I(1)$, $I(2)$ are m_R-primary ideals. Then TFAE

1) The Minkowski equality
$$e(I(1)I(2))^{\frac{1}{d}} = e(I(1))^{\frac{1}{d}} + e(I(2))^{\frac{1}{d}}$$
holds

2) $\exists$ positive integers a, b such that
$$\overline{\sum I(1)^{an} t^n} = \overline{\sum I(2)^{bn} t^n}$$

3) $\exists$ positive integers a, b such that
$$\overline{I(1)^a} = \overline{I(2)^b}$$

<u>Question</u>

Suppose $\mathcal{I}(1)$, $\mathcal{I}(2)$ are m_R-filtrations ~~are~~

1) The Minkowski equality

$$e(\mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d}} = e(\mathcal{I}(1))^{\frac{1}{d}} + e(\mathcal{I}(2))^{\frac{1}{d}}$$

2) $\exists$ positive integers a, b such that

$$\overline{\sum_{n\geq 0} I(1)_{an}t^n} = \overline{\sum_{n\geq 0} I(2)_{bn}t^n}$$

equivalent?

$2) \Rightarrow 1)$ is true for arbitrary m_R-filtrations (if $\dim N(\hat{R}) < d$)

But $1) \Rightarrow 2)$ is false for arbitrary m_R-filtrations

Theorem E (—) (The Teissier Rees Sharp Katz theorem is true for divisorial filtrations)

Suppose R is an excellent local domain. Let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be divisorial m_R-filtrations. Then the Minkowski equality holds between $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ $\iff$ $\exists\ a, b \in \mathbb{Z}_{>0}$ such that

$$I(amD_1) = I(bmD_2) \text{ for } m \in \mathbb{N}$$

Let R be a 2-dim Normal excellent local domain. $\mathcal{I} = \{I_n\}$ an m_R-filtration,

μ an m_R-valuation

Let $\tau_{\mu,m}(\mathcal{I}) = \mu(I_m)$

$= \min\{\mu(f) \mid f \in I_m\}$

Define $\gamma_\mu(\mathcal{I}) = \inf_m \dfrac{\tau_{\mu,m}(\mathcal{I})}{m}$.

Let $\mathcal{I}(D)$ be a divisorial

m_R-filtration,

$\varphi: X \to \mathrm{Spec}(R)$ be the blow up of an m_R-primary ideal with prime exceptional divisors

$E_1, \ldots, E_r$ such that X is normal

$D = a_1 E_1 + \cdots + a_r E_r$ is a representation of $\mathcal{O}(D)$. Let v_{E_i} be the

μ_i - Valuation with valuation ring $\mathcal{O}_{X, E_i}$

Let $\gamma_{E_i}(D) = \gamma_{v_{E_i}}(\mathcal{O}(D))$

$\gamma_{E_i}(D) \geq a_i \quad \forall i$

$\lceil x \rceil$ = roundup of a real number x,

$$\Gamma(X, \mathcal{O}_X(-\lceil m \gamma_{E_1}(D) E_1 + \cdots + m \gamma_{E_r}(D) E_r \rceil))$$

$$= \Gamma(X, \mathcal{O}_X(-mD)) = I(mD)$$

$\forall m \in \mathbb{N}$

$m a_i$ is the prescribed order of vanishing of elements of $I(mD)$ along E_i

$m \gamma_{E_i}(D)$ is asymptotically the actual order of vanishing.

Theorem F (—) Suppose R is a d-dim normal excellent local domain. Let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be divisorial m_R-filtrations.

Let $X \to \operatorname{spec}(R)$ be a representation of $D_1 = \sum a_i E_i$ and $D_2 = \sum \beta_i E_i$.

Then $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ satisfy the Minkowski equality

$\iff$

(1) $$\frac{\partial_{E_i}(D_2)}{\partial_{E_i}(D_1)} = \frac{\partial_{E_j}(D_2)}{\partial_{E_j}(D_1)}$$

for all $1 \le i, j \le r$

when this happens

(2) $$\frac{\gamma_{E_i}(D_2)}{\gamma_{E_i}(D_1)} = \frac{e(\mathcal{L}(D_2))^{1/d}}{e(\mathcal{L}(D_1))^{1/d}}$$

$$= \frac{a}{b} \in \mathbb{Q} \quad (a, b \in \mathbb{Z}_{>0})$$

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and so

$$I(amD_1) = \Gamma\left(X, \mathcal{O}_X\left(-\left\lceil \sum_{i=1}^{r} ma\gamma_{E_i}(D_1)E_i \right\rceil\right)\right)$$

$$= \Gamma\left(X, \mathcal{O}_X\left(-\left\lceil \sum_{i=1}^{r} mb\gamma_{E_i}(D_2)E_i \right\rceil\right)\right)$$

$$= I(bmD_2)$$

Outline of proof of Statement (1) of Theorem F

Minkowski equality $\Rightarrow$

$$f(n_1,n_2) = \frac{1}{d!}\left(e_0^{1/d} n_1 + e_d^{1/d} n_2\right)^d$$

$$\shortparallel$$

$$\lim_{m\to\infty} \frac{e(R/I(mn_1D_1)\,I(mn_2D_2))}{m^d}$$

$e_0 = e(\mathcal{L}(D_1))$, $e_d = e(\mathcal{L}(D_2))$

μ an m_R - valuation

Let $\nu : R \to \mathbb{Z}^d$ be a valuation such that $\nu = (\mu, \ldots)$

$\Gamma(n_1,n_2)$ = semigroup

$$\{(\nu(f), m) \mid f \in I(mn_1D_1)\,I(mn_2D_2)\} \subset \mathbb{N}^{d+1}$$

$\Delta(n_1,n_2)$ = intersection in $\mathbb{R}^{d+1}$

of the closure of the real cone generated by $C(n_1, n_2)$ with $\mathbb{R}^d \times \{1\}$

$$C(R) = C(0,0), \quad \Delta(R) = \Delta(0,0)$$

Theorem $\exists\ \varrho \in \mathbb{R}_{>0}$ such that letting

$$\overline{H}_{\phi, n_1, n_2} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 + \ldots + x_d \leq \varrho \varrho_0^{g} n_1 + \varrho \varrho_0^{g} n_2\}$$

$$\Delta_\phi(n_1, n_2) = \Delta(n_1, n_2) \cap \overline{H}_{\phi, n_1, n_2}$$

$$\widetilde{\Delta}_\phi(n_1, n_2) = \Delta(R) \cap \overline{H}_{\phi, n_1, n_2}$$

we have

$$f(n_1, n_2) = \delta\left[\mathrm{Vol}(\widetilde{\Delta}_\phi(n_1, n_2)) - \mathrm{Vol}(\Delta_\phi(n_1, n_2))\right]$$

$\forall\ n_1, n_2 \in \mathbb{N}$ $\quad \delta = [\mathcal{O}_{\theta}/m_2 : \mathcal{O}/m_0]$

$\Delta(\mathbb{R})$ a closed convex cone with vertex at the origin $(v(1)=0)$

$$\mathrm{Vol}(\tilde{\Delta}_\phi(n_1,n_2)) = (n_1 e_0^{1/d} + n_2 e_d^{1/d})^d \tau,$$

$\tau \in \mathbb{R}$

Define

$$h(n_1,n_2) = \mathrm{Vol}(\Delta_\phi(n_1,n_2)) =$$

$$\mathrm{Vol}(\tilde{\Delta}_\phi(n_1,n_2)) \sim \frac{f(n_1,n_2)}{\delta d!}$$

$$= \lambda (e_0^{1/d} n_1 + e_d^{1/d} n_2)^d \text{ for some } \lambda \in \mathbb{R}_{>0}$$

Let $g(n_1,n_2) = \mathrm{Vol}(n_1\Delta_\phi(1,0) + n_2\Delta_\phi(0,1))$

a homogeneous real polynomial of degree d

$n_1\Delta_\phi(1,0) + n_2\Delta_\phi(0,1) \subset \Delta_\phi(n_1,n_2)$

$\Rightarrow g(n_1,n_2) \leq h(n_1,n_2)$

$g(1,0) = h(1,0) > 0 \quad g(0,1) = h(0,1) > 0$

$$h(1-t,t)^{1/d} = (1-t)\,h(1,0)^{1/d} + t\,h(0,1)^{1/d}$$

$$= (1-t)\,g(1,0)^{1/d} + t\,g(0,1)^{1/d}$$

Brunn-Minkowski Inequality

$$\leq g(1-t,t)^{1/d}$$

$$\leq h(1-t,t)^{1/d}$$

$0 < t < 1$

$\Rightarrow$ equality in the Brunn-Minkowski inequality.

$\Rightarrow$ $\Delta_\varphi(1,0)$ and $\Delta_\varphi(0,1)$ are homothetic

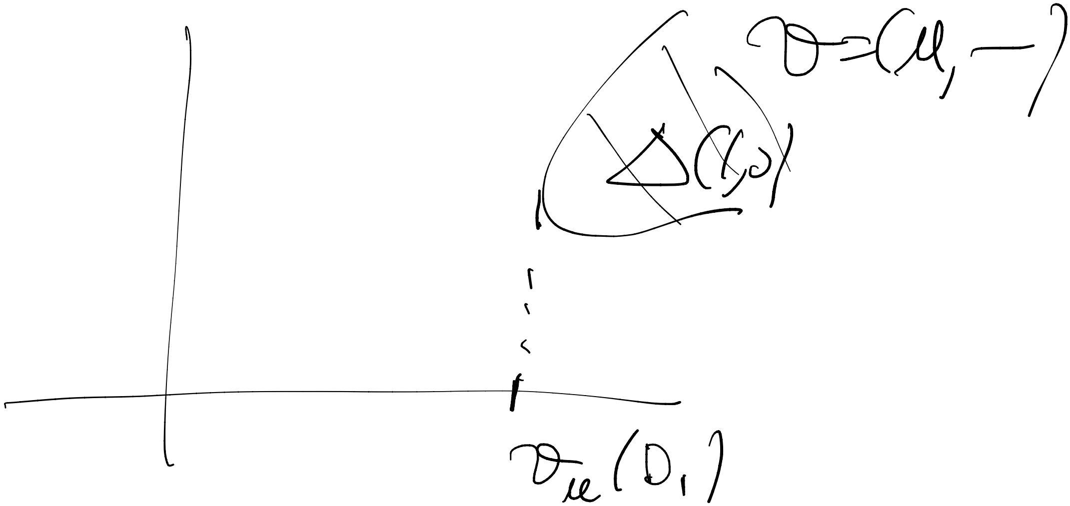
$\exists\ T : \mathbb{R}^d \to \mathbb{R}^d \quad T(\vec{x}) = c\vec{x} + \vec{\gamma}$

$c \in \mathbb{R}_{>0}$ such that

$T(\Delta_\phi(1,0)) = \Delta_\phi(0,1)$

$c = \dfrac{e_j^{1_j}}{e_0^{1_j}}$, $\vec{\gamma} = 0$

$\Rightarrow e_j^{1_j}\Delta_\phi(1,0) = e_0^{1_j}\Delta_\phi(0,1)$



Take $u = v_{E_j}$

$\dfrac{v_{E_j}(D_1)}{e_0^{1_j}} = \dfrac{v_{E_j}(D_2)}{e_j^{1_j}}$ $1 \le j \le r$

$\Rightarrow$ statement (1) of Theorem H.