Differential Powers of Ideals

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Joint work with:
DDSGHNB: Dao, De Stefani, Grifo, Huneke, NB
BJNB: Brenner, Jeffries, NB
DNB: Duarte, NB

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Setting: $K = \overline{K}$ alg. closed field

R = f.g. K-alg. & domain

$d = \dim(R)$

$\mathfrak{m} \subseteq R$ maximal.

1.- Differential Operators

Def: We define the diff. operators of order $\leq n$ inductively as follows:

i) $D^0_{R|K} = \underset{\substack{\cong \\ R}}{\mathrm{Hom}_R(R,R)} \subseteq \mathrm{Hom}_K(R,R)$

ii) $D^n_{R|K} = \{ \delta \in \mathrm{Hom}_K(R,R) \mid \delta \cdot r - r\delta \in D^{n-1}_{R|K} \quad \forall r \in R \}$

Prop:
- $D^n_{R|K} \subseteq D^{n+1}_{R|K}$
- $D^a_{R|K} \, D^b_{R|K} \subseteq D^{a+b}_{R|K}$

Conj (Nakai)

Suppose $char(K)=0$. Then,

$D^a_{R|K} D^b_{R|K} = D^{a+b}_{R|K} \iff R$ is a reg. ring.

$\forall a,b \in \mathbb{N}$

Prop: $D^1_{R|K} \cong R \oplus Der_{R|K}$

$$Der_{R|K} = \{ \delta : R \to R \text{ } K\text{-linear} \mid \delta(fg) = f\delta(g) + g\delta(f) \}$$

Def: The ring of differential op. of R is defined by

$$D_{R|K} = \bigcup_{n \in \mathbb{N}} D^n_{R|K}$$ ← filtered ring

$\subseteq Hom_K(R,R)$

Obs: R is a left $D_{R|K}$-mod.

Ex: $S = K[x_1, \dots, x_t]$

$\mathrm{char}(K) = 0 \implies D_{S|K} = R\langle \partial_1, \dots, \partial_t \rangle$

left & right Noeth

$$gr(D_{S|K}) = \bigoplus_{n \in \mathbb{N}} D^n_{R|K} / D^{n-1}_{R|K}$$

$$\cong K[x_1, \dots, x_t, y_1, \dots, y_t]$$

$$\left(\begin{array}{l} \partial_1 x_1 (1) = \partial_1 (x_1) = 1 \\ x_1 \partial_1 (1) = x_1 \cdot 0 = 0 \end{array} \implies \begin{array}{c} D_{S|K} \\ \text{is} \\ \text{Not Comm} \end{array} \right)$$

$\mathrm{char}(K) = p.$

prob: $\partial^p x^p = p! = 0$

Sol: $\frac{1}{p!} \partial^p (x^p) = 1$

$$\frac{1}{\alpha!}\partial_x^{\alpha}(x^{\beta}) = \begin{cases} \binom{\beta}{\alpha} x^{\beta-\alpha} & \alpha \le \beta \\ 0 & \alpha > \beta \end{cases}$$

$$D_{S|k} = S\left\langle \frac{1}{\alpha_1!}\partial_1^{\alpha_1}, \dots, \frac{1}{\alpha_t!}\partial_t^{\alpha_t} \right\rangle_{\alpha_i \in \mathbb{N}}.$$

Not Noeth

2) $R = S/I$ polynomial ring.

$$D_{R|k} = \frac{\{\delta \in D_{S|k} \mid \delta(I) \subseteq I\}}{I\, D_{S|k}}.$$

3) $R = \dfrac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$ Bernstein

Gel'fand

Gel'fand

Not Noeth

4) $R = \dfrac{\mathbb{C}[x,y,z,w]}{(x^3+y^3+z^3+w^3)}$ KLT

Mallory: • R is not $\mathcal{D}_{R|K}$-simple.
• There are not dif. op. that lower degree

<u>Recall</u>: $\mathcal{D}'_{R|K} = R \oplus \mathrm{Der}_{R|K}$

$\mathrm{Der}_{R|K} \cong \mathrm{Hom}_R(\Omega_{R|K}, R)$

$T = R \otimes_K R \qquad \mu: T \longrightarrow R$

$f \otimes g \longrightarrow f \cdot g$

$\Delta = \mathrm{Ker}(\mu) \qquad T/\Delta \cong R$

$\Omega_{R|K} = \Delta/\Delta^2$

Def: The module of principal parts of R is defined by

$$P^n_{R|K} = T/\Delta^{n+1}$$

Obs:

$$0 \to \Omega_{R|K} \cong \Delta/\Delta^2 \to P^1_{R|K} = T/\Delta^2 \to T/\Delta \cong R \to 0$$

$$\Rightarrow \quad P^1_{R|K} \cong R \oplus \Omega_{R|K}$$

Prop: $D^n_{R|K} \cong \operatorname{Hom}_R(P^n_{R|K}, R)$

$$\left(\Rightarrow \quad D^1_{R|K} \cong R \oplus \operatorname{Der}_{R|K} \right)$$

2.- Diff. powers

Def (DDSGHNB)

Let $I \subseteq R$ be an ideal, and $n \in \mathbb{N}$.

The n-th diff. of I is defined by

$$I^{\{n\}} = \{ f \in R \mid D^{n-1}_{R|K} f \in I \}$$

Prop: i) $I^{\{n\}}$ is an ideal

ii) $I^{\{1\}} = I$

iii) $I^{\{n\}} \supseteq I^{\{n+1\}}$

iv) $I^{n} \subseteq I^{\{n\}}$

v) P is prime $\Rightarrow$ $P^{\{n\}}$ is P-primary

vi) $I = \sqrt{I} \Rightarrow I^{(n)} \subseteq I^{\{n\}}$

$$\bigcap_{I \subseteq P_{min}} (I^n R_P) \cap R$$

Thm (DDS GHNB)

Let $R = \dfrac{K[X]}{I_{t+1}(X)}$ $\quad X = (x_{i,j})$

$m = (\underline{x})$.

If char$(K)=0$, then $J^{(2n)} \subseteq m^n$

$\forall\, J = \sqrt{J} \subseteq m$

Proof: $J^{(2n)} \subseteq J^{\{2n\}} \subseteq m^{\{2n\}} \subseteq m^n$ //

Prop (BJNB)

$m^{\{n\}} = m^n \iff R_m$ is regular

Obs: $I = \sqrt{I} \subseteq R = K[\underline{x}]$

$$I^{\{n\}} = \Big(\bigcap_{I \subseteq m} m\Big)^{\{n\}} = \bigcap_{I \subseteq m} m^{\{n\}}$$

$$= \bigcap_{I \subseteq m} m^n = I^{(n)}$$

Eisenbud-Hochster

Thm (Zariski-Nagata)

Let $R = k[\underline{x}]$, and, $I = \sqrt{I}$

Then, $I^{\{n\}} = I^{(n)}$

3.- Differential Signature

Thm (BJNB)

$$\lambda(R/m^{\{n\}}) = \text{f.r.}(P^{n-1}_{R_m|k}) = a$$

$$(\Leftarrow P^{n-1}_{R_m|k} = R^a \oplus N)$$

no free summand

$$\left[\Rightarrow P^{n-1}_{R_m|k} \text{ is free} \Leftrightarrow R_m \text{ is regular}\right]$$

<u>Def</u> (BJNB): The differential signature of R at m is defined by

$$0 \leq s^{\text{diff}}(R,m) = \limsup_{n\to\infty} \frac{d!\,\lambda(R/m^{\langle n\rangle})}{n^d}$$

$$= \limsup \frac{d!\ \text{f.r.}(P^{n-1})}{\text{rk}(P^{n-1})}$$

$$\leq 1$$

Why not a limit?

$\{m^{\langle n\rangle}\}$ are not always a graded system

<u>Ex</u>: $R = K[t^2, t^3]$

$$m^{\langle 3\rangle} m^{\langle 3\rangle} \not\subseteq m^{\langle 6\rangle}$$

Q: Is $\{ \mathfrak{m}^{\{n\}} \}$ a graded system if R is normal?

Thm (BJNB)

If $gr(\mathcal{D}_{R_m|K}) = \bigoplus_{n \in \mathbb{N}} \mathcal{D}^n / \mathcal{D}^{n-1}_{R_m|K}$,

is a f.g. R_m-algebra, then

$$s^{diff}(R_m) = \lim_{n \to \infty} \frac{d! \, \lambda(R_m / m^{\{n\}})}{n^d} \in \mathbb{Q}$$

Ex: $R = S^G \longrightarrow S = K[\underline{x}]$

$\circlearrowleft G \quad |G| \in K^\times$

$m = (\underline{x}) \cap R$

$$s^{diff}(R_m) = 1/|G|$$

Ex: $R = \dfrac{K[X]}{I_{t+1}(X)}$ $X = (x_{i,j})$ $a \times b$

$m = (x_{i,j})$ $\operatorname{char}(K) = 0$

$$s^{diff}(R_m) = \frac{e(R_m)}{2^{t(a+b-t)}}$$

Thm (BJNB)

If $\operatorname{char}(K) = p$ & R is F-pure, then

$s^{diff}(R_m) > 0 \iff R$ is strongly F-reg.

$\Rightarrow$ Normal domain

C-M $\simeq$ KLT

($D_{R|K}$-simple)

Comment: Jeffries and Smirnov used diff. signature to bound the local étale fundamental group at a singular point.

4.- Nash Blowups

Let X be an irreducible alg. variety over k of dim d

$(X \cong \mathrm{Spec}(R))$

Suppose that $x_0 \in X$ is not ~~A~~ singular

$$\lambda\left(\mathcal{O}_{x_0} / m_{x_0}^{n+1}\right) = \binom{d+n}{n}$$

$$\Rightarrow \quad \mathcal{O}_{X,0}/\mathfrak{m}^{n+1} \in \mathrm{Hilb}_{\binom{d+n}{n}}(X)$$

<u>Def</u> (Semple, Nash, Yasuda)
The n-th Nash Blowup of X is defined by

$$\mathrm{Nash}_n(X) = \overline{\left\{ (x, \mathcal{O}_x/\mathfrak{m}_x^{n+1}) \in X \setminus \mathrm{Sing}(X) \times \mathrm{Hil}_{\binom{d+n}{n}}(X) \right\}} \subseteq X \times \mathrm{Hil}_{\binom{d+n}{n}}(X)$$

$$\mathrm{Nash}_n(X) \xrightarrow{\pi_n} X$$

Q : Does the sequence,

$$\cdots \xrightarrow{\pi} X_2 \xrightarrow{\pi} X_1 \xrightarrow{\pi} X_0 ,$$

where $X_0 = X$
$X_{n+1} = Nash_1(X_n)$

give eventually a res. of singu?

True for curves if $char(k) = 0$
(Nobile)

Q: $\exists n \in \mathbb{N}$ s.t. $Nash_n(X)$ is nonsigular?

True for curves if $char(k) = 0$
Yasuda

False in general (Toh-Yama)

Thm (Spivakovsky)

Suppose that $\mathrm{char}(k)=0$ & $d=2$,

Then, the seq.

$$\cdots \to X_2 \to X_1 \to X_0,$$

where $X_0 = \overline{X}$

$X_{n+1} = \mathrm{Nash}_1(X_n)$,

gives a reso. of sing.

Thm (Nobile) If $\mathrm{char}(k)=0$,

then

$\mathrm{Nash}_1(X) \cong X \iff X$ non sigular

<u>Ex</u>: This fails for

$$\mathbb{X} = \operatorname{Spec}\left(\frac{\overline{\mathbb{F}_2}[x,y]}{(x^2 - y^3)} \right)$$

$$\operatorname{Nash}_n(\mathbb{X}) \cong \mathbb{X}$$

<u>Thm</u> (DNB): If $\mathbb{X}$ is normal, then

$$\operatorname{Nash}_1(\mathbb{X}) \cong \mathbb{X} \iff \mathbb{X} \text{ is nonsingular}$$

char $p>$

<u>Sketch</u>: Fix $x_0 \in \mathbb{X}$

$$\operatorname{Nash}_1(\mathbb{X}) \cong \mathbb{X} \underset{\text{Teissier}}{\Longrightarrow} \lambda\left(\mathcal{O}_{x_0} / m_{x_0}^{\{2\}} \right) = d+1$$

$$\Longrightarrow \varphi : \mathcal{O}_{x_0}^{1/p} \longrightarrow \mathcal{O}_{x_0}^{pd} \text{ is an isomorphism}$$

$$\underset{\text{Kunz}}{\Longrightarrow} \mathcal{O}_{x_0} \text{ is regular.} //$$

Thm (DNB): If X is F-pure, $\mathrm{char}(k) = p$, and $\mathrm{Nash}_n(X) \cong X$ for some $n \geq 1$, then X is strongly F-regular.

Sketch:

$$\mathrm{Nash}_n(X) \cong X \underset{\text{Teissier}}{\Longrightarrow} \lambda\left(\mathcal{O}_{x_0} / \mathfrak{m}_{x_0}^{n+1}\right) = \binom{n+d}{n}$$

$$\Longrightarrow s^{\mathrm{diff}}(\mathcal{O}_{x_0}) > 0$$

$$\Longrightarrow \mathcal{O}_{x_0} \text{ is strongly F-regular.} \quad //$$

Ideas from
Aberbach
Enescu
Aberbach
Leuschke
Smith