REFLECTION ARRANGEMENTS, SYZYGIES AND THE CONTAINMENT PROBLEM

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1. Reflection groups, invariant theory, and subspace arrangements

1.1. Reflection groups.

- A **reflection** is a linear transformation that fixes a hyperplane = **reflecting hyperplane**.
- A **reflection group** is a finite group generated by reflections.
- A **reflection** (hyperplane) **arrangement** $\mathcal{A}(G)$ is the set of reflecting hyperplanes of all reflections in a reflection group G.

Examples: The symmetric $S_n = \langle (i,j) \mid 1 \le i < j \le n \rangle$

$S_n = \{ n \times n \text{ permutation matrix} \} \le GL_n(k)$

$H_{ij} = V(x_i - x_j) \rightsquigarrow$ fixed by (i,j)

$$\mathcal{A}(S_n) = \{ H_{ij} \} = V\Big(\prod_{1 \le i < j \le n} x_i - x_j \Big)$$

Example: The generalized symm. gp (monomial) gp

$G(m,m,n) = C_m \wr S_n$ wreath prod

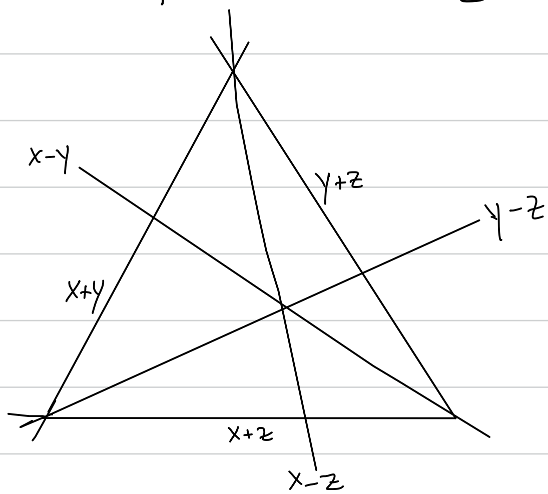
$G(m,m,n) = \{ n \times n$ permutation-like matrices w/ nonzero entries $= m^{th}$ roots of 1 & $\det = 1 \}$

$G(1,1,n) = S_n$

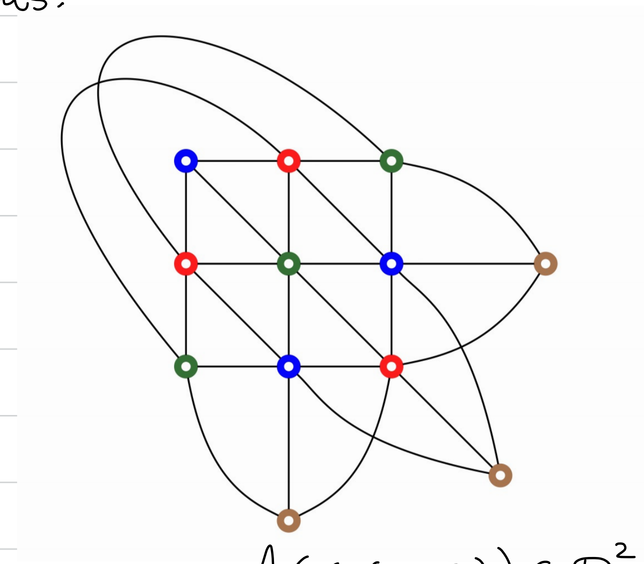
$H_{ij} = V(x_i - \zeta x_j)$ where $\zeta^m = 1$

$$\mathcal{A}(G(m,m,n)) = V\Big(\prod_{\substack{1 \le i < j \le n \\ \zeta^m = 1}} (x_i - \zeta x_j) \Big) = V\Big(\prod_{1 \le i < j \le n} (x_i^m - x_j^m) \Big)$$

Some reflection arrangements:



$\mathcal{A}(G(2,2,3)) \subseteq \mathbb{P}^2$



$\mathcal{A}(G(3,3,3)) \subseteq \mathbb{P}^2$

<u>Thm</u> [Shephard – Todd]
The irreducible complex reflection groups belong to
- 1 infinite family $G(m,p,n)$ monomial
- 34 sporadic groups $G_3 - G_{37}$ of rank 2 to 8

<u>Examples</u>: monomial gps include S_n, C_n, D_n
sporadic gps include Weyl gp E_6, E_7, E_8

1.2. **Invariant theory.**

- $G \leq GL_n(k)$ acts on $R = k[x_1, \ldots, x_n]$ as follows: G acts on Span$\{x_1, \ldots, x_n\}$

$$g = [g_{ij}] \text{ acts by } g \circ x_i = \sum_{j=1}^{n} g_{ij} x_j \text{ and } g \circ f(x_1, \ldots, x_n) = f(g \circ x_1, \ldots, g \circ x_n).$$

- The **ring of invariants** of G is $R^G = \{f \in R \mid g \circ f = f, \ \forall g \in G\}$.

Thm [Chevalley, Shephard–Todd & Serre]
G is a refl. gp, char $k \nmid |G|$

$\Longleftrightarrow$ R^G is a polynomial ring i.e.
$R^G = k[f_1, .., f_n]$ $f_1, ..., f_n$ alg. indep.

Examples: • $k[x_1, .., x_n]^{S_n} = k[e_1, .., e_n] = k[p_1, .., p_n]$
e_i = elem. sym poly
$p_i = \sum_j x_j^i$

• $k[x,y,z]^{G(3,3,3)} = k[xyz, x^3+y^3+z^3, x^6+y^6+z^6]$

Def: $\mathrm{Der}_k(R) = R\frac{\partial}{\partial x_1} \oplus \dots \oplus R\frac{\partial}{\partial x_n}$

$\mathrm{Der}_k^G(R) = \{ \varphi \in \mathrm{Der}_k(R) \mid g \circ \varphi = \varphi \ \forall g \in G \}$
$\parallel$
the R^G - mod of G-invariant k-deriv.

Thm If G is a refl. gp then $\mathrm{Der}_k^G(R)$ is free of rank = rank G.

Problems in invariant theory:

- find algebra generators (primary invariants) and relations (secondary invariants) for R^G
- find bounds on the degrees of the primary/secondary invariants
- find generators / relations / bounds on their degrees for $\mathrm{Der}_k^G(R)$.

1.3. Subspace arrangements and the containment problem.

- A **subspace arrangement** is a union of linear spaces $X_i \not\subseteq X_j$ for $i \neq j$

$$X = X_1 \cup \cdots \cup X_d, \qquad \text{so} \qquad I(X) = I(X_1) \cap \cdots \cap I(X_d).$$

Problems on subspace arrangements:

- find the generators for $I(X)$
- find the Hilbert function of $I(X)$
- find the Betti numbers/ projective dimension/ regularity of $I(X)$

Thm [Derksen - Sidman]
If X is an arrangement of d subspaces then $\operatorname{reg} I(X) \leq d$.

Def. The codim c truncation of a hyperplane arr. is the set of intersections of c lin. indep hyperplanes in the arrangement.

- The m^{th} fattening of a subspace arrangement X is the scheme defined by
$I(X)^{(m)} = I(X_1)^m \cap I(X_2)^m \cap \cdots \cap I(X_d)^m$

Results:

- X = union of codim c coord subspaces
Betti #'s of $I(X)^{(m)}$ ⤳ Biermann, De Alba, Galetto, Murai, Nagel, Römer, O'Keefe, S.

- X = star configuration
Betti #'s of $I(X)^{(m)}$ reduce to by Geramita, Harbourne, Migliore, Nagel
- X = star config of hypersurfaces P. Mantero

Containment Problem: For which r, m is $I(X)^{(m)} \subseteq I(X)^r$?

2. Reflection arrangements and the containment problem

joint with Ben Drabkin ArXiv: 2002.05353

Syzygies $\longleftrightarrow$ Reflection groups

Containment problem

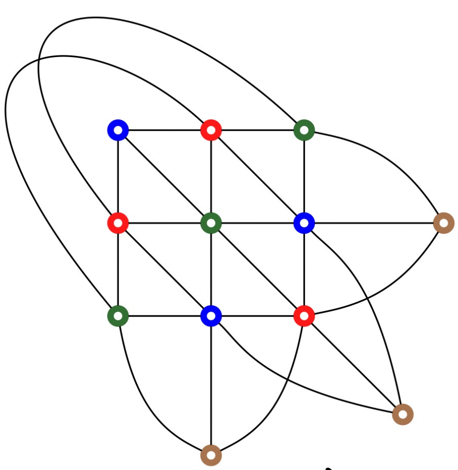
Containment problem: For which r, m is $I(X)^{(m)} \subseteq I(X)^r$?

Thm [Ein – Lazarsfeld – Smith, Hochster – Huneke, Ma – Schwede]
If I is an equidim ideal in a regular ring with $ht(I) = c$ then $I^{(cr)} \subseteq I^r$, $\forall r \geq 1$.

Say I is containment-tight if $I^{(cr-1)} \not\subseteq I^r$ for some $r \geq 1$.

Example [Dumnicki – Szemberg – Tutaj – Gasińska]

X = sing locus of $\mathcal{A}(G(3,3,3))$ so $c = 2$



$\mathcal{A}(G(3,3,3))$

then $I(X)^{(4)} \subseteq I(X)^2$
but $I(X)^{(3)} \not\subseteq I(X)^2$

so $I(X)$ is containment tight

Question: which reflection arrangements have cont-tight singular loci?

Notation: $J(G)$ = ideal dof. the sing-locus of $\mathcal{A}(G)$

2.1. Generators & syzygies for $J(G)$

Thm Let G be a refl. gp w/ $R^G = k[f_1, \dots, f_n]$

Then
$$\det \left| \frac{\partial f_j}{\partial x_i} \right| = \prod_{H \in \mathcal{A}(G)} \ell_H^{|g_H|}$$

Example $p_i = \sum x_j^i$
$$\left| \frac{\partial p_j}{\partial x_i} \right| = \begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & & \vdots \\ \vdots & \vdots & & \\ 1 & x_n & & x_n^{n-1} \end{vmatrix}$$
$$= \prod (x_i - x_j)$$

Theorem A. *Let G be an irred. complex reflection group. Then $J(G)$ is minimally generated by the maximal minors of either* perfect, ht $J(G) = 2$

cont. tight = *(1) the jacobian matrix for a set of* $\operatorname{rank}(G) - 1$ *basic invariants of lowest degrees or*

(2) the coefficient matrix for a set of $\operatorname{rank}(G) - 1$ *basic derivations of lowest degrees expressed in terms of* $\frac{\partial}{\partial x_i}$.

Example of case (1) for $J(G(3,3,3))$ ← containment-tight

$$\left| \frac{\partial f_j}{\partial x_i} \right| = \left| \nabla xyz \quad \nabla x^3+y^3+z^3 \right| = \begin{vmatrix} yz & x^2 \\ xz & y^2 \\ xy & z^2 \end{vmatrix}$$

$$J(G(3,3,3)) = I_2(\ \ \ )$$

Remark on case (2)

The Euler derivation $\partial^E = x_1 \frac{\partial}{\partial x_1} + \dots + x_n \frac{\partial}{\partial x_n}$ is always a basic derivation

so if $J(G)$ is in case (2), it has a linear syzygy $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

2.2 (Non)containments

case $r=2$, $c=2$

Theorem B (Drabkin-S.). *Let $J(G)$ be the ideal defining the singular locus of $\mathcal{A}(G)$ where G is a reflection group. Then $J(G)^{(3)} \subseteq J(G)^2$ if and only if no irreducible factor of G is isomorphic to one of the following groups*

$$G_{24}, G_{27}, G_{29}, G_{33}, G_{34}, \text{ or } G(m,m,n) \text{ with } m,n \geq 3.$$

Stable cont holds

$m=3$ previously known

$m>3$

new examples of $J(G)^{(3)} \nsubseteq J(G)^2$ ideals in $\mathbb{P}^4, \mathbb{P}^5, \mathbb{P}^6, \mathbb{P}^{n-1}$

cont-tight

- Reduction to low emb. dim

Lemma (Drabkin)

$$J(G)^{(m)} \subseteq J(G)^r \iff J(H)^{(m)} \subseteq J(H)^r$$

for $H \leq G$ with $3 \leq \operatorname{rank} H \leq \operatorname{bight}(J(G)^r) \leq r+1$

- Syzygy criteria in emb. dim 3

Thm [Grifo-Huneke-Mukundan]

If $J \subseteq k[x,y,z]$, char $k \neq 3$, J perfect, $\operatorname{ht}(J)=2$

then

μ(ideal entries of Hilb-Burch mat of J) ≤ 5

$\Rightarrow J^{(3)} \subseteq J^2$.

case (2)

$= m$

2.3 Stable containments

Conjecture 2.9 (Stable Harbourne conjecture). *For a radical equidimensional ideal J of $\mathrm{ht}(J) = c$ the containments $J^{(cr-c+1)} \subseteq J^r$ hold for $r \gg 0$.*

For us $c=2$, so stable conj is $J^{(2r-1)} \subseteq J^r$, $\forall r \gg 0$

Theorem C (Drabkin-S.). *Let G be a finite complex reflection group with irreducible factors of rank three and let $J(G)$ be the ideal defining the singular locus of $\mathcal{A}(G)$. Then the containment*

$$J(G)^{(2r-1)} \subseteq J(G)^r \text{ holds for } r \geq 3.$$

Question: For rank $G > 3$, does the stable containment conj hold for $J(G)$?

In particular is $J(G)^{(5)} \subseteq J(G)^3$?