Lagrangian Geometry of Matroids

Federico Ardila Mantilla (SFSU/Los Andes)
Graham Denham (U. Western Ontario)
June Huh (Institute of Advanced Studies)

Fellowship of the Ring

July 30, 2020

1 Goals

1A Geometric

Initiate the study of the Lagrangian geometry of matroids

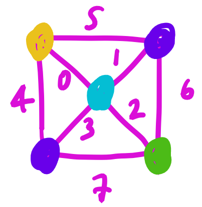
1B Combinatorial

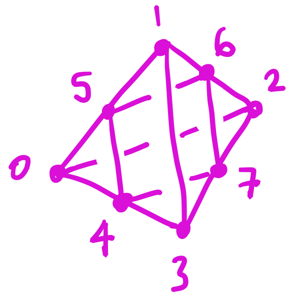
Prove some conjectural inequalities about matroids.

---

2 Matroids: abstraction of "independence" (lin., alg., graph)

graph $\xhookrightarrow{0\%}$ point config $\xhookrightarrow{0\%}$ matroid (flats)





$\emptyset$

0, 1, 2, ..., 7

015, 02, 034, 06, 07, ..., 67

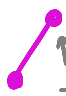
01256, 02347, 067, ..., 4567

01234567

$\downarrow$

Invariants:

$\chi(q) = \#$ proper q-colorings of vertices



$\uparrow \neq$ colors $\uparrow$

$q^5 - 8q^4 + 24q^3 - 31q^2 + 14q$

$h(q) = \chi(q+1)/(q+1)$

$q^4 - 4q^3 + 6q^2 - 3q$

hyperplane config

$\mathbb{F}_q$: $\chi(q) = \#$ pts in $\mathbb{F}_q^n \setminus A$

$\mathbb{R}$: $|\chi(-1)| = \#$ regions in $\mathbb{R}^n \setminus A$

$\mathbb{C}$: coeffs of $\chi(q)$ = Betti #s of $\mathbb{C}^n \setminus A$

coeffs of $\chi(q)$ = intersection #s in trop $\cap$ th of M

[AHK 20]

coeffs of $h(q)$ = intersection #s of lagrangian trop $\cap$ th of M

[ADH 20]

hyp arr

Combin. Hodge theory

$\Downarrow$

Goal 1

Theorems:

coeffs of $\chi(q)$ are unimodal | $a_1 \leq \dots \leq a_k \geq \dots \geq a_n$ | [AHK 20] Rota Mason Welsh 70s

log-concave | $a_{i-1}\, a_{i+1} \leq a_i^2$

flawless | $a_i \leq a_{n-i}$ | [ADH 20] Brylawski 80s Dawson

$\Uparrow$

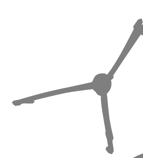
— — $h(q)$ ===

Goal 2

---

③ (Lagrangian) geometry of matroids

G $\hookrightarrow$ L.A. $\hookrightarrow$ M



Bergman Fan Σ_M $\overset{\circ\circ\circ}{=}$ tropical linear space

(A.-Klivans 06)

M matroid

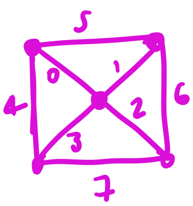
[AHK] $A^\bullet(\Sigma_M)$

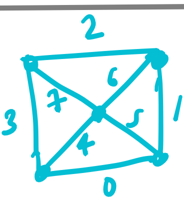
Thm (Fink)
The tropical vars of deg 1 are precisely the Bergman fans

Conormal Fan $\Sigma_{M,M^\perp}$ ∘∘∘ (conormal bundle)

[ADH 20]
$A^\bullet(\Sigma_{M,M^\perp})$

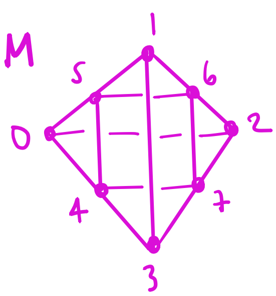
dim $r-1$	dim $n-1$		dim $n-2$	dim $2n-2$	
$\Sigma_M \subseteq$	Σ_E	braid fan (normal fan of permutahedron)	$\Sigma_{M,M^\perp} \subseteq$	$\Sigma_{E,E}$	bibraid fan (normal fan of bipermutahedron)

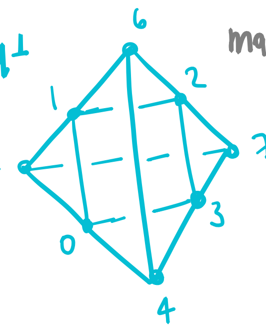
$M =$ 

$M^\perp =$ 

$E = \{0, 1, \dots, 7\}$

sets | matroids

M 

$M^\perp$ 

S subset: 014

$S|T$ bisubset: 014 | 023567

$S \cup T = E \quad S \cap T \neq \emptyset$

$\mathcal{S}$ flag of subsets $1 \subset 014 \subset 01347$

$S_1 \subset \dots \subset S_k$

$\mathcal{S}|\mathcal{T}$ biflag of bisubsets

$1 \subseteq 014 \subseteq 014 \subseteq 01347$

$E \supseteq 235(67 \supseteq 023567 \supseteq 2567$

$S_1 \subseteq \dots \subseteq S_k$

$T_1 \supseteq \dots \supseteq T_k$

$S_i \cup T_{i+1} \neq E$ for some i

F flat: 01345

$F|G$ biflat: 01345 | 23467

bisubset: F flat G coflat

$\mathcal{F}$ flag of flats $\emptyset \subset 1 \subset 015 \subset 01345 \subset E$

$\mathcal{F}|\mathcal{G}$ biflag of biflats

$\emptyset \subseteq 1 \subseteq 015 \subseteq 01345 \subseteq 01345 \subseteq E$

$E \supseteq E \supseteq E \supseteq 23467 \supseteq 267 \supseteq \emptyset$

M matroid

- Σ_M [AK 06]
  - ambient space: $\mathbb{R}^E/\mathbb{R}$
  - rays: e_F, F flat, $e_F = \underbrace{110010}_{F}$
  - cones: $e_{F_1}, \dots, e_{F_k}$, $F_1 \dots F_k$ flags

- $\Sigma_{M,M^\perp}$ [ADH 20]
  - ambient space: $\mathbb{R}^E/\mathbb{R} \times \mathbb{R}^E/\mathbb{R}$
  - rays: $e_F + f_G$, $F|G$ biflat, $\underbrace{110010}_{F} \,|\, \underbrace{011101}_{G}$
  - cones: $e_{F_1|G_1}, \dots, e_{F_k|G_k}$ biflags

Def The conormal Chow ring of a matroid M is $A^\bullet(\Sigma_{M,M^\perp})$

$$A^\bullet(\Sigma_{M,M^\perp}) = \mathbb{Z}[x_{F|G} : F|G \text{ biflat}] / (I+J)$$

$$I = \langle x_{F_1|G_1} \cdots x_{F_k|G_k} : F_1|G_1, \dots, F_k|G_k \text{ not a biflag} \rangle$$

$$J = \langle \alpha_i - \alpha_j,\ \alpha_i^\perp - \alpha_j^\perp : i \neq j \rangle$$

$$\alpha_i = \sum_{F \ni i} x_{F|G}$$

$$\alpha_i^\perp = \sum_{G \ni i} x_{F|G}$$

Def.

$\alpha = \alpha_i$ for any i

$\delta = \delta_i$ for any i

$$\delta_i = \sum_{F,G \ni i} x_{F|G}$$

(Ex: $\delta_i = \delta_j$)

Fact: $A^\bullet = A^0 \oplus A^1 \oplus \cdots \oplus A^{n-2}$, with $A^0 \cong \mathbb{Z}$, $A^{n-2} \cong \mathbb{Z}$

$$\boxed{\textbf{Thm [ADH 2020]} \quad \alpha^k \delta^{n-2-k} = \text{coeff of } q^{r-k} \text{ in } h_M(q)}$$

$\Downarrow$ old+new combin Hodge th.

$$\boxed{\textbf{Thm} \quad \text{log-concavity of } h}$$

Pfs of Thm:

① Geometric [ADH: "Lagrangian geometry of matroids", 2020]

CSM classes of matroids / trop. man. fold ... Conormal fan

CSM classes of matroids / [illegible]

[LRS]

2 Combinatorial [ADH: "Lagrangian combin. of matroids", 2020]

"Gröbner" combinatorics + basis activities

[LV]