The homotopy Lie algebra and the conormal module

Part 1

The Conormal module / Kähler differentials / etc

Conjectures of Vasconcelos.

Part 2

Idea of proof:

The homotopy Lie algebra

Part 1

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr] & S \arrow[d] \\ & k \end{tikzcd}$$

$\varphi: R \to S$ is a surjective local map of local rings

k residue field. $I = \ker(\varphi)$

$\to I/I^2 =$ conormal module of φ.

(an S module)

Name is from geometry:

$\operatorname{Spec} S = V(I) \hookrightarrow \operatorname{Spec} R$ closed embedding

$I/I^2 \rightsquigarrow$ conormal sheaf.

Ring theoretic properties of $\varphi: R \to S$ $\overset{?}{\leftarrow}$ $\rightsquigarrow$ module theoretic properties of I/I^2

Example:

$\varphi: R \to S$ is complete intersection (c.i.) $\Longrightarrow$ $\Longleftarrow$ I/I^2 is free S-module $\cong \bigoplus f_i S$

IE $I = (f_1 \dots f_n)$ is generated by a regular sequence

IE f_{i+1} is nonzero divisor on $R/(f_1 \dots f_i)$ $\forall i$

Theorem (Ferrand 67, Vasconcelos 67)

Assume I has finite projective dim. "φ has finite proj dim"

If I/I^2 is free $/S$ then φ is complete int.

Remark $\operatorname{projdim}_R I < \infty$ is needed:

$R \longrightarrow k$ residue field

always has conormal module m/m^2 free over k

but this is c.i. $\iff$ R regular.

Conjecture (Vasconcelos 78) ①

If $R \longrightarrow S$ has finite proj dim and

$\operatorname{projdim}_S I/I^2 < \infty$ then $R \longrightarrow S$ is comp. int.

$\operatorname{projdim}_S I/I^2 < \infty \Rightarrow ci \Rightarrow \operatorname{projdim}_S I/I^2 = 0.$

IE only $\operatorname{projdim} I/I^2 = 0, \infty$ is possible

True in the following cases

- if $\mathrm{pd}_R I \leq 1$
- I Gorenstein of height ≤ 3 and $\frac{1}{2} \in R$
- I is almost complete int.

Vasconcelos 78

(ie $\mu(I) \leq \mathrm{grade}(I)+1$)

Aoyama 77 / Masuoka 77

$\Leftarrow$ projdim = 1 is impossible

- Gulliksen 69 / Vasconcelos 85 : $\mathrm{projdim}_S I/I^2 \leq 1$
- Herzog 81 : I in linkage class of c.i. (licci)
- other cases of low ht/pd, Vasconcelos 85
- Avramov Herzog '94 : graded, characteristic zero and R smooth.

"absolute"

Methods from rational homotopy theory ...

Theorem (-20)

Vasconcelos' conjecture is true: if $R \xrightarrow{\iota} S$ has finite projdim, $\operatorname{projdim}_S I/I^2$ can only be 0 or ∞

complete int

not comp. int.

Note: this is a relative result: don't assume R regular

Question: Must the resolution of I/I^2 have maximal growth?
eg radius of convergence of Poincaré series of I/I^2
equal to that of residue field?

(true in graded char 0 setting by AH)

Module of Kähler differentials

Assume $S = \dfrac{k[x_1 \dots x_n]_m}{I}$, $R = k[x_1 - \cdot x_n]_m$, $I = \sqrt{I}$.

$\Omega_{S/k}$ usually defined by

$$\mathrm{Hom}_S(\Omega_{S/k}, M) \cong \mathrm{Der}_k(S, M).$$

"Jack Jeffries talk for eg."

<u>Alternative</u> $\Omega_{S/k} \stackrel{?}{=}$ conormal module of mult. map

$= J/J^2 \qquad J \subseteq S \otimes_k S \longrightarrow S.$

$dx = x\otimes 1 - 1\otimes x.$

Geometrically: $\operatorname{Spec} S \xrightarrow{\Delta} \operatorname{Spec} S \times_k \operatorname{Spec} S$

$\Omega_{S/k} \rightsquigarrow$ cotangent sheaf

Jacobian Criterion

$\Omega_{S/k}$ is free of rank $= \dim S$ $\iff$ S is <u>smooth</u> $\iff$ $S \otimes_k S \longrightarrow S$ is complete intersection

<u>Theorem</u> (Ferrand 67)

Assume S is generically separable over k & $k \to k(p) = S_p / pS_p$ separable $\forall$ min p. Eg char 0

then S is a reduced complete int. $\iff$ $\mathrm{projdim}_k \Omega_{S/k} \leq 1$

(2) <u>Conjecture</u> (Vasconcelos 78) $\operatorname{char} k = 0$

If $\operatorname{projdim}_S \Omega_{S/k} < \infty$ then S is reduced complete int.

<u>Note:</u> Conormal theorem doesn't apply since $S \otimes_k S \to S$ almost never has finite projective dim.

But Ferrand / Vasconcelos kind of works because of Jacobian Crit.

<u>Evidence</u>: • Known in cases mentioned above

• Also Platte '80: if $\operatorname{projdim} \Omega_{S/k}$ is finite then S is quasi-Gorenstein (i.e. canonical module $\cong S$)

Need extra ingredient:

$$R = k[x_1 \ldots x_n]_m$$

<u>Conjecture</u> (Eisenbud Mazur 97)

Assume $\operatorname{char} k = 0$, then $I^{(2)} \subseteq m I$

where $I^{(2)} := \bigcap_{p \in \operatorname{Ass} R/I} I^2 R_p \cap R$

"Symbolic square"
eg Eloísa Grifo's talk

"a function vanishing to order ≥ 2 on $V(I)$ cannot be a min gen of I."

Theorem (-20) $I \subseteq R$ radical, $S = R/I$,

and $I^{(2)} \subseteq \mathfrak{m} I$, then $\operatorname{projdim}_S \Omega_{S/k} < \infty \Rightarrow S$ complete intersection

$\Leftarrow$ EM conj $\Rightarrow$ Conj 2 (both still open)

$\longrightarrow$ and new proof of graded char 0

Idea of proof: use conormal sequence

$$0 \to \frac{I^{(2)}}{I^2} \to I/I^2 \to \Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to 0$$

$$\Omega_{R/k} \otimes_R S \cong S^n$$

If $I^{(2)} = I^2$ then $I/I^2 = \operatorname{Syz} \Omega_{S/k}$.

$$\operatorname{projdim}_S \Omega_{S/k} = \operatorname{projdim}_S I/I^2 + 1$$

so Conormal theorem $\Rightarrow$ done

$\longrightarrow$ $I^{(2)} \subseteq \mathfrak{m} I$ is enough to get this to work

$\square$

The first Koszul homology

$I = (f_1 \dots f_n) \subseteq R$ ideal of finite proj dim.

$H = H_1(\mathrm{Kos}^R(f_1 \dots f_n))$ the first Koszul homology.

Classical: $H = 0 \iff f_1 \dots f_n$ is a regular seq.

Question: if $\mathrm{projdim}_S H < \infty$, $\Rightarrow$?

(Vasconcelos 85)

Evidence: • Gulliksen 69 projdim $= 0$ (free) $\Rightarrow$ yes.

• Vasconcelos 85 $\Rightarrow$ other cases

• Avramov Herzog 94 $\Rightarrow$ dim 0, graded

Theorem (-20) $I \subseteq R$ ideal of finite proj dim,

and $H = H_1(\mathrm{Kos}^R(I))$ finite projdim over $S = R/I$

$\longrightarrow$ I is complete int.

ie $H = 0$ or $\mathrm{projdim}_S H = \infty$

André-Quillen cohomology

(Assume knowledge of this)

$D^i(S/R; M) =$: ith AQ cohomology group with coeffs in M.

"the nonabelian derived functor of $\mathrm{Der}_R(S, M)$"

Theorem (Avramov 99) **Conjecture** (Quillen 68)

$\varphi: R \to S$ finite projdim, and $D^i(S/R; k) = 0$ for $i \gg 0$ then $R \to S$ is complete intersection.

Remarks
- Says cotangent complex has finite proj dim $\Rightarrow$ c.i.; similar to other conjectures.
- if char $k = 0$, proven by Avramov–Halperin 87
- uses homotopy Lie algebra.

Theorem (–, Iyengar '20)

If $D^i(S/R; k) = 0$ for ∞ many odd i and ∞ many even i, then $R \to S$ is complete i.t. (so $D^{\geq 2}(S/R, k) = 0$)

Main point: new proof (simpler) using same method as Vasconcelos' conj.

Also: rigidity of cotangent complex?

Part 2 Idea of how to prove this:

> **Theorem** (~20)
>
> Vasconcelos' conjecture is true; if $R \to S$ has finite projdim, $\operatorname{projdim}_S I/I^2$ can only be 0 or ∞

Methods from rational homotopy theory

Rapid history: late 70s Avramov & Roos started exploiting methods from rational homotopy theory ... gradually realized that the connection ran deep.... made contacts with RHT in 80s...

Roos: interested in question of Serre/Kaplansky.

- must $P_R(t) = \sum t^i \dim_k \operatorname{Tor}_i^R(k,k)$ be a rational function for all local $R \to k$?
- And $P_X(t) = \sum t^i \dim H_i(\Omega X; \mathbb{Q})$ rational? X = finite CW complex
- Anick 80: no for spaces $\xrightarrow{\text{Roos}}$ no for rings (bad rings)

not superficial: structural similarities accomodated same framework.

Avramov: The homotopy Lie algebra.

local ring R $\rightsquigarrow$ graded Lie algebra $\pi^*(R)$

Name from topology: X space (simply connected.) $\rightsquigarrow \pi_*(\Omega X) \otimes \mathbb{Q}$

This means: each π^i is f.d. vect space over k

with bracket $\pi^i \times \pi^j \longrightarrow \pi^{i+j}$

$x \;\; y \longmapsto [x,y]$

bilinear and anti symmetric $[x,y] = -(-1)^{ij}[y\,x]$

+ Jacobi identity $[x[y\,z]] = [[x\,y]\,z] + (-1)^{ij}[y[x\,z]]$

+ in char 2 and 3 need a bit more! (not needed here)

Quick!

Construction of $\pi^*(R)$ due to Assmus, Levin, Schoeller, Milnor-Moore, André, Sjödin: $\mathcal{U}\pi^*(R) \underset{\text{def.}}{\cong} \operatorname{Ext}_R(k,k)$.

$\Rightarrow$ Koszul dual!!

championed by Avramov, Halperin, ...

$k \longrightarrow \ell$

Also relative version for local $\varphi : R \longrightarrow S$

$\pi^*(\varphi)$ graded Lie alg over ℓ

if char $\ell = 0 \Rightarrow \pi^*(\varphi) \cong D^{*+1}(S/R ; \ell)$ AQ coh

$\longleftarrow$ comp. otherwise quite different.

Theorem of Avramov: If $\varphi : R \to S$ has finite flat dimension

$k \to \ell$ residue fields

then there is an exact sequence

$$\cdots \to \pi^i(\varphi) \to \pi^i(S) \longrightarrow \pi^i(R) \otimes_k \ell \xrightarrow{\delta^i} \pi^{i+1}(\varphi) \to \cdots$$

$\Rightarrow$ Avramov resolved Grothendieck's localization problem

(R complete int $\Rightarrow R_p$ complete int.)

Theorem (Félix-Halpern-Thomas 82, Halperin 87, Avramov 99)

If R is c.i. then $\pi(R) = \pi^1 \oplus \pi^2$

If R is not c.i. then $\pi^i(R)$ never zero $i \geq 1$
and $\dim \pi^i(R)$ grows exponentially

These numbers $\varepsilon_i(R) := \dim \pi^i(R)$ the deviations of R.
show up naturally by other methods.
(read from Poincaré series)

Example

$$\frac{k[[x\ y]]}{xy} \Rightarrow \pi^* = \begin{array}{c} 1 \\ ak \\ bk \end{array} \oplus \begin{array}{c} 2 \\ ck \end{array}$$

with $[a,b] = c$. (Sjödin)

Example

$$\frac{k[x,y]}{(x,y)^2} \Rightarrow \pi^* = \text{free Lie algebra on 2 gens degree 1.}$$

"Big gap between ci and non-ci"

In topology: Elliptic vs Hyperbolic spaces.

$z \in \pi^i(\varphi)$ is <u>central</u> if $[z, \pi^*(\varphi)] = 0$

is <u>radical</u> if there is some N such that $[z, \pi^{\geq N}(\varphi)] = 0$.

not usual definition

<u>Theorem</u> (Avramov Halperin 87)

$\varphi : R \to S$ finite proj dim. If every elt of $\pi^2(\varphi)$ is radical, then φ is ci $\longrightarrow \pi^{\geq 3} = 0$

<u>Corollary</u>: Quillen's conjecture in char 0.

<u>Remark</u>: Theorem can be proven using <u>Cohomological Support theory</u> ← Developed by Josh Pollitz.

Rough idea of proof of Vasconcelos' conjecture:

there is an isomorphism $\pi^2(\varphi) \cong \operatorname{Hom}_S(I/I^2, k)$

<u>Iyengar 01</u>: free summands of I/I^2 give rise to central elements of $\pi^2(\varphi)$.

<u>— '20</u>: finite proj dim summands of I/I^2 give rise to <u>radical</u> elements of $\pi^2(\varphi)$.

Key object "dg Kähler differentials"

$\tilde{\Omega}_{S/R}$ complex of free S modules, deg $i \geq 1$.

$$\left(\tilde{\Omega}_{S/R}\right)_i = S^{\varepsilon_{i-1}(\varphi)}$$

showed up in Avramov–Herzog '94.

if char $k = 0$ $\tilde{\Omega}_{S/R} \simeq$ cotangent complex of φ.

(simplicial methods)

otherwise different

Minimal free dg R-algebra resolution

$$A \xrightarrow{\simeq} S$$

"resolvent"
"minimal model"
"Sullivan model"

$\longrightarrow$ $\tilde{\Omega}_{S/R} = \Omega_{A/R} \otimes_A S$

Derived version of Kähler differentials

(but different to cotangent complex)

Proposition (AH 94)

$$\mathrm{Ext}^{*-1}_S(\Omega_{A/k}\otimes_A S,\ k) \cong \pi^*(\varphi)$$

Connection with Part 1:

Proposition (AH 94)

Syzygies of $\widetilde{\Omega}_{S/R}$:

$$\begin{tikzcd}
\vdots \arrow[d, "\partial_3"'] & \vdots \\
(\widetilde{\Omega}_{S/R})_3 \arrow[r, hook] \arrow[d, "\partial_2"'] & \operatorname{coker}\partial_3 \quad \text{“higher conormal modules”} \\
(\widetilde{\Omega}_{S/R})_2 \arrow[r] \arrow[d, "\partial_1"'] & \operatorname{coker}\partial_2 \cong H_1(\mathrm{kos}^R(I)) \\
(\widetilde{\Omega}_{S/R})_1 \arrow[r] \arrow[d, "\partial_0"'] & \operatorname{coker}\partial_1 \cong I/I^2 \\
(\widetilde{\Omega}_{S/R})_0 \arrow[r] & \operatorname{coker}\partial_0 \cong \Omega_{S/R}.
\end{tikzcd}$$

nothing here in syzygetic situation

All the modules which showed up so far are syzygies of $\widetilde{\Omega}_{S/R}$!

these two props structurally connect I/I^2 with $\pi^*(\varphi)$

Thanks !