GRASSMANNIAN CLUSTER (?) CATEGORIES OF INFINITE RANK & COUNTABLE COHEN-MACAULAY TYPE

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WINART 2

<u>Goal</u>: Grassmannian cluster combinatorics

$\updownarrow$

cat. of max. CM-modules $\mathbb{C}[x,y]/(x^k)$, $k \geq 2$

<u>$k=2$</u>: cluster structure

I Cluster algebras/categories, Grassmannian cluster cats

II Infinite constructions

1. Cluster algebras

$\to$ introduced by [Fomin-Zelevinsky ~2000]

Take a quiver = directed graph $Q = (Q_0, Q_1)$, $Q_0 = \{1, \dots, n\}$, Q_1 "arrows"

$\tilde{n}<\infty$

s.t. Q does not contain loops

not contain 2-cycles

To each vertex $i \to x_i$ cluster variables

$\underline{x}=(x_1,\dots,x_n)$ a cluster

$(\underline{x},Q)$ (initial) <u>seed</u>. get $\underbrace{\mathcal{A}_Q}_{\text{cluster alg. ass. to } \underline{x}} \subset \overset{k}{\mathbb{C}}(x_1,\dots,x_n)$

by <u>mutation</u>:

- initial cluster $\underline{x}$
- mutate in direction i: $\mu_i(\underline{x})=(x_1,\dots,x_i^*,\dots,x_n)$

$$x_i^* = \frac{\prod_{j\to i} x_j + \prod_{i\to j} x_j}{x_i}$$

exchange relation: $x_i \cdot x_i^* = \prod_{j\to i} x_j + \prod_{i\to j} x_j$

- $\mu_i(Q)$: - reverse all arrows at i
  - $j\to i\to k$: add $k\to j$
  - delete all 2-cycles

$\mathcal{A}_Q$: mutating in <u>all</u> possible directions

<u>ex</u>: $Q: 1\to 2$ "A_2-quiver"

$\underline{x}=(x_1,x_2)$

$\mu_1 Q: 1 \leftarrow 2$ $\quad (x_1^* = \frac{x_2+1}{x_1}, x_2)$

$\mu_2 Q: 1 \leftarrow 2$ $\quad (x_1, x_2^* = \frac{x_1+1}{x_2})$

$\mu_1\mu_2 Q: 1 \rightarrow 2$ $\quad (\frac{1+x_1+x_2}{x_1x_2}, \frac{1+x_1}{x_2})$

$\mu_2\mu_1 Q: \ldots$ $\quad \vdots$

$\Rightarrow \mathcal{A}_Q = \mathbb{C}[x_1, x_2, x_1^*, x_2^*, \frac{1+x_1+x_2}{x_1x_2}] \subset \mathbb{C}[x_1^{\pm}, x_2^{\pm}]$

- $Q = 1 \to 2 \to 3$: 14 cluster variables
- $Q = 1 \to 2 \to 3 \to 4$: 42 —"—

Rmk: In general: ∞-many cluster variables

But: $\mathcal{A}_Q$ is finitely gen. if Q is acyclic [FZ, III]

- $\exists$ fin. cluster vars $\Leftrightarrow$ Q is of type ADE, B_n, C_n, F_4, G_2 [FZ, II]

- If Q acyclic: $\mathcal{A}_Q$ is complete intersection, normal, has at most canonical singularities [Benito-Muller-Rajchgot-Smith]

Rmk: sometimes don't mutate all vertices "frozen vertices" variables

- ex [Scott '06] $\mathcal{A}_{k,n} = \mathbb{C}[Gr(k,n)]$ = homog. coord ring of $Gr(k,n)$

has a cluster structure

$\mathcal{A}_{k,n} = \mathbb{C}[\underset{\text{Plücker}}{p_I}, \; I \subset \{1,\dots,n\}, |I| = k] / \underset{\text{Plücker relat.}}{\mathcal{I}_p}$

- max. sets of compatible Plückers p_I are clusters
- exchange rel $\Leftrightarrow$ Plücker rel.

$\underline{k=2}$: $\mathcal{A}_{2,n}$ $\quad p_I \overset{1-1}{\longleftrightarrow}$ cluster vars

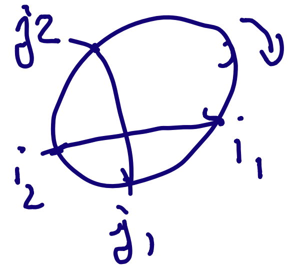
$\hookrightarrow Q = A_{n-3}$ + frozen vars.

Def: $I, J \subset \mathbb{Z}$, $|I| = |J| = k$ are crossing if $\exists\, i_1, i_2 \in I \setminus J$ and $\exists\, j_1, j_2 \in J \setminus I$ s.t.

$i_1 < j_1 < i_2 < j_2$ or

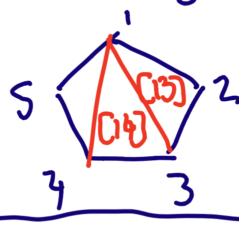
$j_1 < i_1 < j_2 < i_2$

$k=2$



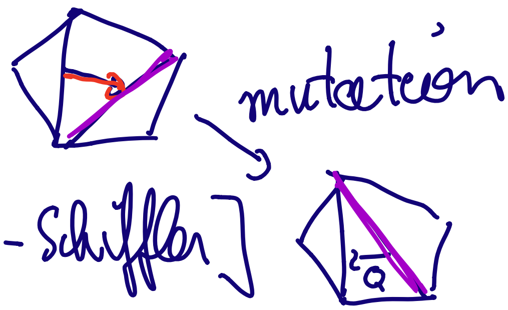
p_I, p_J are compatible if $I + J$ are non-crossing

2. Cluster cats of type A + triangulations

ALGEBRA	CATEGORY (cc-map)	COMBINATORICS
$\mathcal{A}_{A_{n-3}}$	$\mathcal{C}(A_{n-3}) := D^b(\mathrm{rep}A_{n-3}) / \tau^{-1} \circ [1]$ cluster cat. [Buan-Marsh-Reineke-Reiten-Todorov]	regular n-gon $n=5$ 
· cluster vars	· indec obj. in $\mathcal{C}(A_{n-3})$	· diagonals

• clusters	• cluster tilting obj $T=\bigoplus_{i=1}^{n-3} T_i$: max rigid $(\mathrm{Ext}^1(T_i,T_j)=0)$	• triangulation (= max. set of noncross. diagonals)
• mutation	• T-approximation	• flip of a diagonal

ex: $Q: 1 \to 2$



[Caldero-Chapoton-Schiffler]

Rmk: $\mathcal{A}_{2,n} \to$ all arcs in n-gon

Q: What happens for $n \to \infty$?

$\mathcal{A}_{2,\infty} \rightsquigarrow$ "∞-gon"

cluster structure on ∞-gon?

3. Grassmannian cluster cats $\mathcal{C}(k,n)$

∃ CC-map
• rigid indec. → cluster vars
• cluster tilting obs ⟷ clusters

[Jensen-King-Su '16]: additive categorification $\mathcal{C}(k,n)$ of $\mathcal{A}_{k,n}$

$2 \le k \le \frac{n}{2}$ and $S := \mathbb{C}[x,y]$ and $\mu_n = \{\xi \in \mathbb{C} : \xi^n = 1\}$

$\mu_n \circlearrowright S$ via $x \mapsto \xi x$
$y \mapsto \xi^{-1} y$

Consider $R_{k,n} = S/(x^k - y^{n-k})$

$k=2 \quad (x^2 - y^{n-2})$ "A_{n-3} curve sing."

$\mathcal{C}(k,n) := MCM_{\mu_n}(R_{k,n}) = MCM(\underbrace{R_{k,n} * \mu_n}_{=:A})$

skew-group

Iwanaga-Gorenstein

$\Rightarrow \mathcal{C}(k,n)$ Frobenius cat

$\underline{\mathcal{C}(k,n)}$ is Δed cat [Buchweitz]

- $\{$ rk 1 mod in $MCM(A)\} \xleftrightarrow{1-1} \{$Plücker coords$\}$

$M_I \longrightarrow p_I$

- $Ext^1(M_I, M_J) = 0 \iff p_I + p_J$ are compatible
- cluster tilting obj. $\longleftarrow$ max set of compatible Plückers

$[k=2 \qquad \xleftrightarrow{1-1} \qquad]$

$\underline{\mathrm{II}}$

4. Infinite $Gr(k,\infty)$

[Grabowski-Gratz '14] : $\mathcal{A}_k = \mathbb{C}[p_I, I \subset \mathbb{Z}, |I| = k] / I_p$

[Gratz '15]

can be endowed with

a cluster structure

$\mathcal{A}_k$ = colimit of cluster algebras of finite rank (in cat. of rooted cluster algebras)

[Groechenig '14] $\mathcal{A}_k$ can be seen as the coord ring of inf. rank Grassmannian

$\to$ Start [JKS] $\mathcal{C}(k,n)$

$\mu_n \rightsquigarrow \mathbb{G}_m = \mathbb{C}^*$ acts on $S = \mathbb{C}[x,y]$

$x \mapsto \xi x$

$y \mapsto \xi^{-1} y$, $\xi \in \mathbb{C}^*$

$\cancel{-y^{n-k}} \to 0$

$R_{k,\infty} = \mathbb{C}[x,y]/(x^k)$

We define $MCM_{\mathbb{G}_m}(R_{k,\infty}) =:$ Grassmannian cat of ∞-rank

$\mathbb{Z}$: $\mathrm{mod}_{\mathbb{G}_m} R_{k,\infty} \simeq \mathrm{gr}(\mathrm{mod}\, R_{k,\infty})$

$|x| = 1$

$|y| = -1$

$\rightsquigarrow$ "$\mathcal{C}(k,\infty)$" $= MCM_{\mathbb{Z}}(R_{k,\infty})$

$\to$ [Eisenbud] $\cong \underline{MF}(x^k)$ [matrix fact. (A,B) s.t. $S^m \xrightarrow{B} S^m \xrightarrow{A} S^m$, $AB = BA = x^k \cdot \mathbb{1}_m$

$k=2$: $\mathbb{C}[x,y]/(x^2)$ is countable CM-type

[Buchweitz - Greuel - Schreyer '1987] ☺

Bad news: $k \geq 3$: $R_{k,\infty}$ is of wild CM-type ☹

But: describe Plückers $(R = R_{k,\infty})$

Def: $\mathcal{F} = R_{(x)}$

$M \in MCM_{\mathbb{Z}}(R)$ is generically free of rk n :$\Leftrightarrow$

$M \otimes_R \mathcal{F}$ is a free $\mathcal{F}$-mod. of rk n.

Prop: (1) If $M \in MCM_{\mathbb{Z}}(R)$ is gen. free, then $M = \Omega(N)$

N is a R-mod of finite length.

(2) M is gen. free of rk 1 $\Leftrightarrow$ $M \cong$ an ideal of R on $y^n \in M$, $n > 0$

(3) Every hom. ideal in R can be generated by monomials

Thm [ACFGS]: $I \in MCM_{\mathbb{Z}}(R)$ gen. free of rk 1

$\Leftrightarrow$ $I = (x^{k-1}, x^{k-2}y^{i_1}, \ldots, xy^{i_{k-2}}, y^{i_{k-1}})(i_k)$

$0 \leq i_1 \leq \ldots \leq i_{k-1}$, $i_k \in \mathbb{Z}$

$I \longleftrightarrow$ Plücker $P_{\underline{\ell}(I)}$

$\underline{\ell}(I) = (-i_{k-1} - i_k, -i_{k-2} - i_k + 1, \ldots, -i_k + k - 1)$

$\Rightarrow$ $\{$gen. free rk 1 mod in $MCM_{\mathbb{Z}}(R)\} \longleftrightarrow \{P_{\underline{\ell}(I)}$ in $\mathcal{A}_k\}$

- $\operatorname{Ext}^1(I,J)=0 \iff p_{\underline{e}(I)}$ + $p_{\underline{e}(J)}$ are compatible

  $\overset{\|}{\operatorname{Ext}^1}(J,I)$ *

  $\operatorname{Ext}^1(\bar{I},I)=0$

Idea of pf: arXiv. $\square$

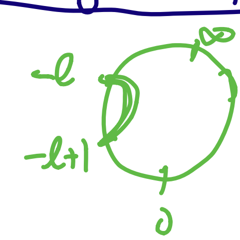
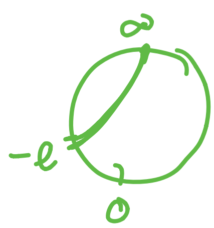
5. $k=2$: $\quad MCM_{\mathbb{Z}}(\mathbb{C}[x,y]/(x^2)) \qquad |x|=1, \; |y|=-1$

Rmk: [Holm-Jørgensen] $D^f_{dg}(\mathbb{C}[y])$: A_∞ behaviour, arcs in ∞-gon

[Paquette-Yildirim] discrete cluster cats

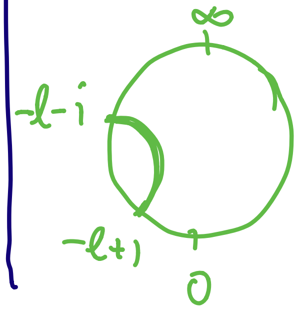
Thm [BGS]: Let $S=\mathbb{C}[x,y]$ and $R=\mathbb{C}[x,y]/(f)$
R is of countable CM-type (not finite) $\iff$ $f=x^2$ (A_∞) or $f=x^2y$ (D_∞)

A_∞ $\quad R=\mathbb{C}[x,y]/(x^2)$

MF	MCM-modules	∞-gon 
rk 1: $(x^2,1)(\ell)$ $S \xrightarrow{1} S \xrightarrow{x^2} S$	$\operatorname{coker}(x^2)=R(\ell)$	
$(x,x)(\ell)$ $S \xrightarrow{x} S \xrightarrow{x} S$	$\operatorname{coker}(x)=\mathbb{C}[y](\ell)$	

rk 2

$[\begin{pmatrix} x & y^i \\ 0 & -x\end{pmatrix}, \begin{pmatrix} x & y^i \\ 0 & -x \end{pmatrix}](\ell)$ | coker $\cong (x, y^i)(\ell)$ |



⇝ Calculate Exts: $\mathbb{C}[y](\ell)$ are not symmetric

⇝ triang ⟷ cluster tilting subcat of $MCM_{\mathbb{Z}}(R)$

max rigid + funct. finite

Thm [ACFGS]

∃ cluster tilting