Syzygies for Products of Projective Spaces ~ Juliette Bruce (Berkeley/MSRI)

§1 - curves

$X \hookrightarrow \mathbb{P}^r$ $\rightsquigarrow$ $S = \mathbb{C}[x_0, x_1, \dots, x_r]$

smooth projective curve
genus $= g$

I_X = homg. defining ideal $X \subseteq \mathbb{P}^r$

$S(X)$ = homogeneous coord. ring

$$0 \leftarrow S(X) \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_r \leftarrow 0$$

$$\begin{tikzcd} & F_0 = S & F_1 \arrow[dl] \\ 0 & I_X \arrow[l] \arrow[u] & \\ & 0 \arrow[u] & \end{tikzcd}$$

graded minimal free res. of $S(X)$

$$\beta_{p,q}(X \subseteq \mathbb{P}^r) = \#\left\{\begin{matrix}\text{minimal generators}\\ \text{of } F_p\\ \text{of degree } q\end{matrix}\right\} = \#\left\{\begin{matrix}\text{syzygies of}\\ \text{degree } q\\ \text{homological deg. } p\end{matrix}\right\}$$

Betti table: $\beta_{p,p+q}(X \subseteq \mathbb{P}^r) \rightsquigarrow (p,q)$-spot

Ex: $\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$

$[s:t] \longmapsto [s^d : s^{d-1}t : \cdots : t^d]$

$S = \mathbb{C}[x_0, \dots, x_d]$

$$I_X = \left\langle 2\times 2 \text{ minors of } \begin{pmatrix} x_0 & \cdots & x_{d-1} \\ x_1 & \cdots & x_d \end{pmatrix} \right\rangle$$

$$S(\mathbb{P}^1, d) = S/I_X \cong \mathbb{C}[s^d, s^{d-1}t, \dots, t^d]$$

d = 3:

$$0 \leftarrow S(\mathbb{P}^1, 3) \leftarrow \overset{0}{S} \leftarrow \overset{1}{S(-2)^{\oplus 3}} \leftarrow \overset{2}{S(-3)^{\oplus 2}} \leftarrow 0$$

$\beta_{0,0} = 1$

$\beta_{1,2} = 3$

$\beta_{2,3} = 2$

$q \backslash p$	0	1	2
0	1	-	-
1	-	3	2
2	-	-	-

$d=4$:

	0	1	2	3
0	1	-	-	-
1	-	6	8	3
2	-	-	-	-

Facts:

① $\beta_{p,p+q}(X \subseteq \mathbb{P}^r) = 0 \qquad \forall p > r$

② $\beta_{p,p+q}(X \subseteq \mathbb{P}^r) = 0 \qquad \forall q > \underset{\dim X + 1}{2} \quad (*)$

	0	1	2	...	r
0	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	...	
1	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	...	
2	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	...	

special cases due to Castelnuovo

• Thm: (Green '84): Let $X \subseteq \mathbb{P}^{r_d}$ be a curve of degree d then:

$$\beta_{p,p+2}(X,d) = 0 \quad \text{for all } p \in [0, d-(2g+1)]$$

• Def:

$$\rho_q(X,d) = \frac{\#\{p \in \mathbb{N} \mid \beta_{p,p+q}(X,d) \neq 0\}}{r_d}$$

= percentage of entries in the qth row that are non-zero.

• Cor: (Green '84): With X, and L_d as above

$$\lim_{d \to \infty} \rho_2(X,d) = 0.$$

Pf: Note that since X is a curve $r_d = O(d)$. $\blacksquare$

$X \hookrightarrow \mathbb{P} H^0(L_d) \cong \mathbb{P}^{r_d}$

X: sm. projective variety of dim $= n$

$\{L_d\}_{d \in \mathbb{N}}$ v. ample l.b.

$S(X, L_d) = \bigoplus_k H^0(X, k \cdot L_d)$

$S = \mathrm{Sym}\, H^0(L_d) \cong \mathbb{C}[x_0, \ldots, x_{r_d}]$

$0 \leftarrow S(X, L_d) \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_{r_d} \leftarrow 0$

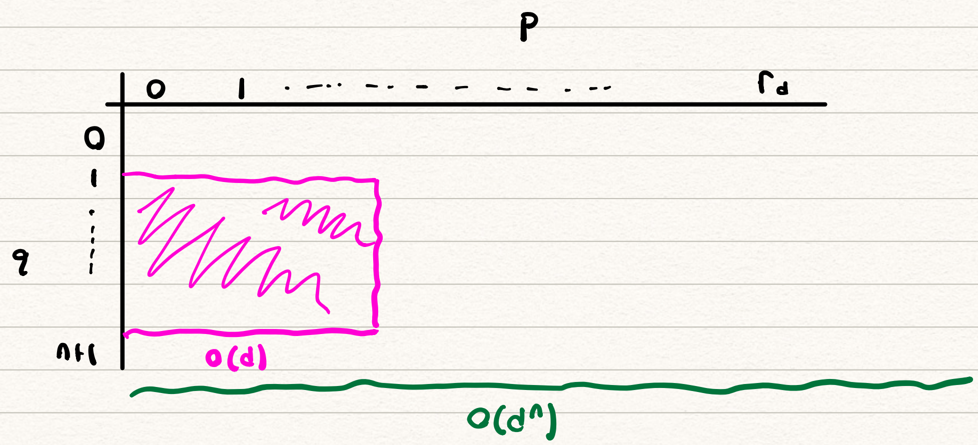
min. graded free resolution of $S(X, L_d)/S$

$\beta_{p,q}(X, L_d) = \#\{\text{min. gens of } F_p \text{ of deg } q\}$

$= \#\{\text{syzygies of deg } q \text{ \& homological deg } p\}$

• <u>Thm</u>: (Ein-Lazarsfeld '93): With X as above fix an index $1 < q \leq n$. If A is very ample and $L_d = K_X + (n+1+d)A$ then

$$\beta_{p,p+q}(X, L_d) = 0 \quad \text{for all } p \in [0, d].$$



• <u>Thm</u>: (Ein-Lazarsfeld '12): Let $n \geq 2$ and fix an index $1 \leq q \leq n$. If $L_{d+1} - L_d$ is a constant ample l.b. then

$$\lim_{d \to \infty} \underbrace{p_q(X, L_d)}_{} = 1$$

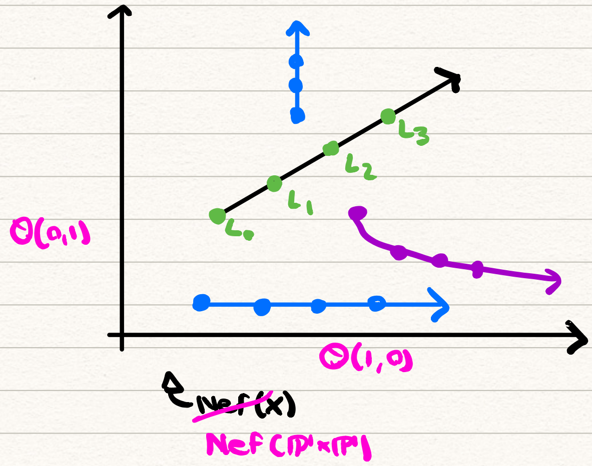
percentage of entries in the qth row that are non-zero

$L_{d+1} - L_d = A$ for all d

$\Leftrightarrow L_d = dA + B$

"$\Leftrightarrow$" Let $X \subseteq \mathbb{P}^r$ have homg. coord. ring $S_X = S/I_X$

$S(X, L_d) = S_X^{(d)}$



$L_{d+1} - L_d$ is semi-ample

- **Def:** A l.b L is semi ample $\Leftrightarrow$ $|kL|$ b.p.f for some $k \gg 0$.

- **Ex:** $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ $\mathcal{O}(1,0)$ or $\mathcal{O}(0,1)$ are semi-ample

- **Thm:** (Juliette Bruce): Consider $\mathbb{P}^n \times \mathbb{P}^m$ and fix $1 \leq q \leq n+m$ then

$$P_q(\mathbb{P}^n \times \mathbb{P}^m, \mathcal{O}(d_1,d_2)) \geq 1 - \sum_{\substack{s+t=q \\ 0 \leq s \leq n \\ 0 \leq t \leq m}} \left(\frac{C_{s,t}}{d_1^s d_2^t} + \frac{D_{s,t}}{d_1^{n-s} d_2^{n-t}} \right) + O\left(\text{lower order terms}\right).$$

the $C_{s,t}$ and $D_{s,t}$ are explict constants.

- **Ex:**

$$P_2(\mathbb{P}^1 \times \mathbb{P}^5, \mathcal{O}(d_1,d_2)) \geq 1 - \frac{20}{d_2^2} - \cancel{\frac{60}{d_1 d_2^3}} - \cancel{\frac{5}{d_1 d_2}} - \frac{120}{d_2^4} - \text{lower order terms}$$

If d_2 is fixed then as $d_1 \longrightarrow \infty$ $\quad \geq 1 - \frac{20}{d_2^2} - \frac{120}{d_2^4}$

If $d_2 = 5$ $\quad \geq \frac{1}{125}$

Proof sketch:

$$\Lambda^{p+1} \bar{S}_d \otimes \bar{S}_{(q-1)d} \longrightarrow \Lambda^{p} \bar{S}_d \otimes \bar{S}_{qd} \longrightarrow \Lambda^{p-1} \bar{S}_d \otimes \bar{S}_{(q+1)d}$$

$$\rotatebox{90}{$\in$}$$

$$m_1 \wedge m_2 \wedge \cdots \wedge m_p \otimes f$$

① quotient by a monomial reg seq.

② pick a monomial $f \in S_{qd}$ that has exactly p monomial divisors of deg d: $m_1, m_2, \ldots, m_p$

③ consider the element above.

$$
\left\langle
\begin{array}{c}
x_0^{d_1} y_0^{d_2} \\[1em]
x_0^{d_1} y_1^{d_2} + x_1^{d_1} y_0^{d_2} \\[1em]
x_0^{d_1} y_2^{d_2} + x_1^{d_1} y_1^{d_2} + x_2^{d_1} y_2^{d_2} \\
\vdots \\
x_2^{d_1} y_4^{d_2}
\end{array}
\right\rangle
$$