DDC Online Seminar, MSRI

October 9, 2020

Effective Ultrapowers

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Models and theories

- A non-standard model is a model of a theory, which is not isomorphic to the intended, standard model.

- Every model consists of a nonempty set of elements, called the *domain*, with certain *functions* (operations), *relations* and *constants* on the domain.

- For example, the *standard model* of Peano arithmetic, $\mathcal{N} = (\omega, 0, S, +, \cdot)$, consists of the set $\omega = \{0, 1, 2, 3, \ldots\}$ with the constant 0, successor function S, and operations of addition $+$ and multiplication $\cdot$

- The standard order of natural numbers is $\mathbb{N} = (\omega, <)$.

- The *first-order formulas* are built using function, relation and constant symbols, variables (denoting the elements of the domain), Boolean propositional symbols (for negation, conjunction, disjunction, implication and equivalence), and universal and existential quantifiers for variables.

- A *sentence* is a formula with no free variables (i.e., all variables are quantified).

- A *theory* is a set of sentences. Every model has its (complete) theory.

- For example, *complete number theory* is the set of all first-order sentences true in the standard model of arithmetic.

Ultraproduct

- Let $(\mathcal{A}_i)_{i\in\omega}$ be a sequence of structures for the same language L.

  Note: ω can be replaced by another nonempty index set I.

- An ultrafilter U over ω is a certain set of subsets of ω.

- An *ultraproduct* is a direct product of structures $(\mathcal{A}_i)_{i\in\omega}$ modulo U, in symbols $\mathcal{B} = \prod_U \mathcal{A}_i$.

- For $f, g \in \mathcal{B}$, we have $f =_U g$ iff $\{i : f(i) = g(i)\} \in U$

  The equivalence class of f is denoted by $[f]$.

- An *ultrafilter* U over ω satisfies the following properties for all X, Y:

  1. $\omega \in U$
  2. $(X \in U \ \& \ X \subseteq Y \subseteq \omega) \Rightarrow Y \in U$
  3. $(X \in U \ \& \ Y \in U) \Rightarrow X \cap Y \in U$
  4. $X \in U \Leftrightarrow \overline{X} = \omega - X \notin U$

- U is a *principal* ultrafilter if there is $i \in \omega$ such that:

  $U = \{X : X \subseteq \omega \ \& \ i \in X\}$

- An ultrafilter U is *nonprincipal* iff for every $i \in \omega$, $\{i\} \notin U$.

- A nonprincipal ultrafilter U over ω contains all co-finite subsets of ω.

- **Theorem** (Zorn's Lemma)

  There is a nonprincipal ultrafilter over any infinite set I.

- In $\mathcal{B} = \prod_U \mathcal{A}_i$, we define its functions, relations and constants as follows.

- For an n-ary function symbol F :

  $F^{\mathcal{B}}([f_1], \ldots, [f_n]) = [g]$, where for every i,

  $g(i) = F^{\mathcal{A}_i}(f_1(i), \ldots, f_n(i))$

- For an m-ary relation symbol R :_

  $R^{\mathcal{B}}([f_1], \ldots, [f_m])$ iff $\{i \in \omega : R^{\mathcal{A}_i}(f_1(i), \ldots, f_m(i))\} \in U$

- For a constant symbol c :

  $c^{\mathcal{B}} = [f]$ where for every $i, f(i) = c^{\mathcal{A}_i}$

- **Fundamental Theorem** (Łoś)

  If $\alpha(x_1, \ldots, x_n)$ is a formula in L, then:

  $\mathcal{B} \models \alpha([f_1], \ldots, [f_n])$ iff $\{i : \mathcal{A}_i \models \alpha(f_1(i), \ldots, f_n(i))\} \in U$

- Hence, if σ is a sentence then: $\mathcal{B} \models \sigma$ iff $\{i : \mathcal{A}_i \models \sigma\} \in U$

- If $\mathcal{A}_i = \mathcal{A}$, then the ultraproduct $\prod_U \mathcal{A}$ is called *ultrapower*.

- *Corollary*

  $\prod_U \mathcal{A}$ and $\mathcal{A}$ are *elementarily equivalent.*
  (Have the same first-order theory.)

Effective ultrapower

- A structure is *computable*, and ultrafilters are replaced by infinite sets from computability theory, which are *indecomposable* with respect to *computably enumerable* sets.

  The elements of the product are equivalence classes of *partial computable* functions.

Computable structures

- A set is *computable* if there is a decision algorithm that recognizes its elements and non-elements.

- A countable structure $\mathcal{A}$ for a finite (more generally, computable) language L is *computable* if its domain is computable and its relations and functions are computable (uniformly computable).

- *Examples of computable structures*

  The ordered set of natural numbers, $\mathbb{N}$ (of order type ω)
  The ordered set integers, $\mathbb{Z}$
  The ordered set of rational numbers, $\mathbb{Q}$
  The additive group of integers, $(Z, +, 0)$
  The field of rational numbers, $(Q, +, \cdot, 0, 1)$

- Example of a non-computable structure

  Let X be a non-computable set.

  Define a linear order $(\{0, 1, 2, \ldots\}, \prec)$ isomorphic to $\mathbb{N}$ (of order type ω):

  $2n \prec 2n+1$ if $n \in X$
  $2n+1 \prec 2n$ if $n \notin X$

  $2n,\ 2n+1 \prec 2n+2, 2n+3$

- If this order were computable, X would be computable, which is a contradiction.

- (Tennenbaum)
  There is no computable non-standard model of Peano arithmetic.

Computably enumerable sets

- A set $X \neq \varnothing$ of natural numbers is *computably enumerable* (abbreviated by c.e.) if there is an algorithm that generates it by enumerating (listing) its elements.

- If X is finite or its elements can be algorithmically enumerated in strictly increasing order, then X is *computable*.

- C.e. sets coincide with Diophantine sets.

- A set X is computable iff X and its complement $\overline{X}$ are both c.e.

  There are many non-computable c.e. sets.

Partial computable functions

- Let $P_0, P_1, ..., P_e, ...$ be an algorithmic enumeration (given by systematic listing) of all Turing machine programs.

- Turing machine program P_e computes a *partial computable* (possibly total, thus computable) function φ_e:

  on input x, it halts and outputs its value, in symbols $\varphi_e(x)\downarrow$, when $x \in dom(\varphi_e)$, or it computes forever, in symbols $\varphi_e(x)\uparrow$, when $x \notin dom(\varphi_e)$.

- It can be shown that c.e. sets are exactly the domains of these partial functions.

- Hence we have algorithmic enumeration of all c.e. sets as domains of partial functions computed by Turing machine programs:

  $W_0, W_1, ..., W_e, ...$

- Turing (diagonal) *halting set*, $H = \{e : e \in W_e\}$ is a non-computable c.e. set.

- **Proof that H is non-computable**. Assume otherwise.

  Then $\overline{H}$ is c.e., so $\overline{H} = W_j$ for some j.

  $j \in \overline{H} \Leftrightarrow j \in W_j \Leftrightarrow j \in H$

C.e.-indecomposable sets

- A set $C \subseteq \omega$ is *c.e.-indecomposable* if C is infinite and for every c.e. set W, either $W \cap C$ or $\overline{W} \cap C$ is finite.

  Hence

  $W \cap C$ is infinite $\Rightarrow C \subseteq^* W$
  $\overline{W} \cap C$ is infinite $\Rightarrow C \subseteq^* \overline{W}$

  $\subseteq^*$ stands for inclusion of all but finitely many elements

- Every infinite set of natural numbers has a c.e.-indecomposable subset.

Effective ultrapowers

- Let $\mathcal{A}$ be a computable structure with domain A, and let $C \subseteq \omega$ be a c.e.-indecomposable set.

  The *effective ultrapower* of $\mathcal{A}$ over C, in symbols $\mathcal{B} = \Pi_C \mathcal{A}$, has the domain ($D$ mod $=_C$) where

  $D = \{\varphi \mid \varphi : \omega \to A$ is partial computable and $C \subseteq^* dom(\varphi)\}$.

  For $\varphi, \psi \in D$, define

  $$\varphi =_C \psi \quad \text{iff} \quad C \subseteq^* \{i : \varphi(i) \downarrow = \psi(i) \downarrow\}.$$

  The equivalence class of φ is denoted by $[\varphi]$.

- If F is an n-ary function symbol, then

$$F^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n]) = [\varphi],$$

where for every $i \in \omega$,

$$\varphi(i) = F^{\mathcal{A}}(\varphi_1(i), \ldots, \varphi_n(i)),$$

equal as partial functions.

- If R is an m-ary relation symbol, then

$$R^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_m]) \quad \text{iff} \quad C \subseteq^* \{i \in \omega : R^{\mathcal{A}}(\varphi_1(i), \ldots, \varphi_m(i))\}$$

- If c is a constant symbol, then $c^{\mathcal{B}}$ is the equivalence class of the computable function with constant value $c^{\mathcal{A}}$.

- Canonical embedding of $\mathcal{A}$ into $\Pi_C \mathcal{A}$: $a \to [\theta_a]$, where $\theta_a = (a, a, \ldots)$.

- For a finite structure $\mathcal{A}$, we have $\Pi_C \mathcal{A} \cong \mathcal{A}$.

  Proof. Let $[\varphi] \in \Pi_C \mathcal{A}$.
  For $a \in A$, let $X_a = \{i \in dom(\varphi) : \varphi(i) = a\}$.

  X_a is c.e. because it is enumerated by the procedure that simultaneously runs Turing machine program for φ on $0, 1, \ldots, k$ for bigger and bigger k's and for more and more computational steps, and whenever φ on i halts and outputs a, we enumerate i into X_a.

  Since A is finite, C is infinite, and $C \subseteq^* dom(\varphi)$, for some $b \in A$, $X_b \cap C$ is infinite.
  Hence $C \subseteq^* X_b$, so $[\varphi] = [\theta_b]$.

- For an infinite computable structure $\mathcal{M}$, the effective ultrapower $\Pi_C \mathcal{M}$ and $\mathcal{M}$ are not necessarily elementarily equivalent.

- If $\mathcal{A}$ and $\mathcal{B}$ are *computably isomorphic*, then $\Pi_C \mathcal{A} \cong \Pi_C \mathcal{B}$.

- **Proof**. Let $f : \mathcal{A} \mapsto \mathcal{B}$ be a computable isomorphism.

  Let $[\varphi] \in \Pi_C \mathcal{A}$.

  Define an isomorphism G of effective ultrapowers by $G([\varphi]) = [f \circ \varphi]$.

Preservation of satisfaction

- **Fundamental Theorem** (Dimitrov)

  (i) If $\alpha(x_1, \dots, x_n)$ is a formula that is a Boolean combination of $\forall$ (or $\exists$) formulas, then

  $$\Pi_C \mathcal{A} \models \alpha([\varphi_1], \dots, [\varphi_n]) \text{ iff } C \subseteq^* \{i : \mathcal{A} \models \alpha(\varphi_1(i), \dots, \varphi_n(i))\}$$

  (ii) If σ is a $\forall\exists$ (or $\exists\forall$) sentence, then

  $$\Pi_C \mathcal{A} \models \sigma \quad \text{iff} \quad \mathcal{A} \models \sigma$$

  (iii) If σ is a $\forall\exists\forall$ sentence, then

  $$\text{if } \Pi_C \mathcal{A} \models \sigma \quad \text{then} \quad \mathcal{A} \models \sigma$$

- If a computable structure $\mathcal{A}$ is from one of the following classes, then so is its effective ultrapower $\Pi_C\mathcal{A}$:
  - rings
  - (algebraically closed) fields
  - lattices
  - (atomless) Boolean algebras
  - (dense) linear orders (without endpoints)

- There are $\forall\exists\forall$ sentences true in some computable $\mathcal{A}$, but not in $\Pi_C\mathcal{A}$ for some C.

- (Feferman, Scott and Tennenbaum)

  There is a $\forall\exists\forall$ sentence (involving Kleene's T predicate), which is true in $\mathcal{N}$, the standard model of arithmetic, but not in $\Pi_C\mathcal{N}$ for some C.

- **Proof sketch.**

  Let P_e be the e-th Turing machine program.

  In Kleene's $T(e, x, z)$, x is the input, and z codes the output and the number s of computation steps.

  Consider the statement:

$$(\forall x)(\exists s)(\forall e \leq x)\ [P_e(x)\downarrow \Rightarrow\ P_{e,s}(x)\downarrow]$$

Effective ultrapowers of linear orders

- We use $+$ for the sum and $\times$ for the lexicographical product of two linear orders.

- We can show that for $\mathbb{N}$, we have $\Pi_C \mathbb{N} \cong \mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$.

- Assume that $\mathcal{L}$, $\mathcal{L}_1$, $\mathcal{L}_2$ are computable linear orders, and $\mathcal{L}^{rev}$ is the reverse of $\mathcal{L}$.

  $\Pi_C (\mathcal{L}_0 + \mathcal{L}_1) \cong \Pi_C \mathcal{L}_0 + \Pi_C \mathcal{L}_1$

  $\Pi_C (\mathcal{L}_0 \times \mathcal{L}_1) \cong \Pi_C \mathcal{L}_0 \times \Pi_C \mathcal{L}_1$

  $\Pi_C \mathcal{L}^{rev} \cong (\Pi_C \mathcal{L})^{rev}$

- For example,

  $$\Pi_C \mathbb{N}^{rev} \cong (\Pi_C \mathbb{N})^{rev} \cong (\mathbb{N} + (\mathbb{Q} \times \mathbb{Z}))^{rev} \cong (\mathbb{Q} \times \mathbb{Z}) + \mathbb{N}^{rev}$$

- Similarly,

  $$\Pi_C \mathbb{Z} \cong \Pi_C (\mathbb{N}^{rev} + \mathbb{N}) \cong \mathbb{Q} \times \mathbb{Z}$$

- Let $\mathcal{L}$ be a computable dense linear order without endpoints. Then $\Pi_C \mathcal{L} \cong \mathcal{L}$.

- **Proof**. The theory of dense linear orders without endpoints is $\forall\exists$-axiomatizable and countably categorical (has only one countable model, up to isomorphism).

  $\Pi_C \mathcal{L}$ is countable, so $\Pi_C \mathcal{L} \cong \mathcal{L}$.

When the successor function is computable

- Let $\mathcal{M}$ be a computable linear order of order type ω, with a *computable* successor function. Then for every c.e.-indecomposable C, we have $\Pi_C\mathcal{M} \cong \mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$.

  $\mathcal{M}$ is computably isomorphic to the standard model $\mathbb{N}$.

- Having a computable successor function is not necessary for this order type of an effective ultrapower.

- There is a computable linear order $\mathcal{A}$ of order type ω, with a *non-computable* successor function, such that for every c.e.-indecomposable C, we have $\Pi_C\mathcal{A} \cong \mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$.

When $\Pi_C \mathcal{L} \not\equiv \mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$

- Let C be a c.e.-indecomposable set. There is a computable linear order $\mathcal{L}$ of order type ω such $\Pi_C \mathcal{L}$ and $\mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$ are not elementarily equivalent.

- **Proof**. Construct a computable linear order $\mathcal{L} = (X, <_{\mathcal{L}})$ of order type ω.

  Assure that if φ is a partial computable function such that

  $[\mathrm{id}] <_{\Pi_C \mathcal{L}} [\varphi]$, then $[\varphi]$ is not the $<_{\Pi_C \mathcal{L}}$-immediate successor of $[\mathrm{id}]$.

- *Construction*

- Fix an infinite computable set $R \subseteq \overline{C}$.

- $X_0 = \{0\}$

- At stage $s > 0$, we have that $<_{\mathcal{L}}$ is defined on some finite $X_{s-1} \supseteq \{0, 1, \ldots, s-1\}$.

- If $s \notin X_{s-1}$, then put s in X_s and extend $<_{\mathcal{L}}$ to make s the $<_{\mathcal{L}}$-greatest element.

Consider each $\langle e, n \rangle < s$ in order. *If*

- $\varphi_{e,s}(n) \downarrow \in X_s$,
- $\varphi_e(n)$ is currently the $<_{\mathcal{L}}$-immediate successor of n in X_s,
- $n \notin R$, and
- n is not $<_{\mathcal{L}}$-below any of $0, 1, \ldots, e$.

Then let m be the least element of $R - X_s$.

- Add m to X_s and extend $<_{\mathcal{L}}$ so that $n <_{\mathcal{L}} m <_{\mathcal{L}} \varphi_e(n)$.

- It follows that $\Pi_C \mathcal{L}$ and $\mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$ are not elementarily equivalent because every element of $\mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$ has an immediate successor, but $[\text{id}] \in \Pi_C \mathcal{L}$ does not have an immediate successor.

- The sentence σ that states that every element has an immediate successor is $\forall\exists\forall$. Then for the computable linear order $\mathcal{L}$ of type ω, constructed above, we have $\mathcal{L} \models \sigma$ but $\Pi_C \mathcal{L} \models \neg\sigma$.

When c.e.-indecomposable sets are co-maximal

- A set $M \subseteq \omega$ is *maximal* if M is c.e. and its complement $\overline{M} = C$ is c.e.-indecomposable.

  Equivalently, M is c.e., $\overline{M}$ is infinite, and for every c.e. set W with $M \subseteq W \subseteq \omega$, either $\omega - W$ or $W - M$ is finite.

- For every $[\varphi] \in \Pi_C \mathcal{A}$, there is a (total) computable function f such that $[f] = [\varphi]$.

- **Proof**. Define $\widehat{f}(n) = \begin{cases} \varphi(n) & \text{if } \varphi(n)\downarrow \text{ first,} \\ 0 & \text{if } n \text{ is enumerated into } M \text{ first.} \end{cases}$

  $\widehat{f}(n)$ is defined for all but finitely many n.

- Let C be a co-maximal set. Then there is a computable linear order $\mathcal{L}$ of order type ω such that $\Pi_C \mathcal{L} \cong \mathbb{N} + \mathbb{Q}$.

- There is a countable set of computable linear orders of order type ω, which are pairwise non-elementarily equivalent.

- It is possible for non-elementarily equivalent computable linear orders to have isomorphic effective ultrapowers.

- Let $\mathcal{X}$ be a non-empty at most countable set of order types.

  Let $|\mathcal{X}|$ be the size of $\mathcal{X}$.

- The shuffle $sh(\mathcal{X})$ is obtained by densely coloring $\mathbb{Q}$ with $|\mathcal{X}|$ many colors, assigning each order type in $\mathcal{X}$ with a distinct color and replacing each $q \in \mathbb{Q}$ with the order type corresponding to the color of q.

- Let C be a co-maximal set.

- Let $k_0, ..., k_n$ be positive natural numbers, and $\mathbf{k}_0, ..., \mathbf{k}_n$ the corresponding ordered sets.

  $\mathbf{k}$ is $0 < 1 < \cdots < k - 1$

- There is a computable linear order $\mathcal{M}$ of order type ω such that $\Pi_C \mathcal{M}$ has order type $\omega + sh(\mathbf{k}_0, ..., \mathbf{k}_n)$.

- Let $\mathcal{X}$ be a $\forall\exists$ or $\exists\forall$ (possibly infinite) set of finite non-empty order types. Then there is a computable linear order $\mathcal{L}$ of order type ω such that $\Pi_C \mathcal{L}$ has order type $\omega + sh(\mathcal{X} \cup \{\mathbb{N} + (\mathbb{Q} \times \mathbb{Z}) + \mathbb{N}^{rev}\})$.

THANK YOU!