Recovering algebraic curves from L-functions

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MSRI

November 2020

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Abstract

We discuss how to recover an algebraic curve over a finite field from L-functions associated with it. We look at the problem from a number of different angles, including the input coming from model theory, and pose some open questions. Joint work with J. Booher.

Joint work with J. Booher

Introduction

K global field

ζ_K Dedekind zeta function of K

Does ζ_K determine K?

No. For number fields: Gassmann (1926).

For function fields: isogenous elliptic curves.

How about using Artin L-functions?

Artin L-functions

E/K Galois with group G.

$\rho : G \to V$, a linear representation of G.

$$L(s, \rho) = \prod_{v} \det \left[\left(I - N(v)^{-s} \rho(\Phi_v) \right) | V^{I_v} \right]^{-1}$$

$V^{I_v} = V$ if v is unramified and $\Phi_v \in G$ is a Frobenius at v.

Abelian L-functions correspond to $\dim V = 1$.

A general result

Cornelissen, de Smit, Li, Marcolli and Smit proved:

Theorem

The set of (all) abelian L-functions characterizes global fields.

More precisely, an isomorphism between the abelianizations of the absolute Galois group of the fields inducing equality of abelian L-functions comes from an isomorphism of fields.

Question: Which L-functions are necessary?

Function fields

Let $K/\mathbb{F}_q$ be a function field of genus at least two.

Theorem 1

The set of unramified abelian L-functions of $K\mathbb{F}_{q^n}$ for all n characterizes K.

More precisely, we need isomorphism of Jacobians.

These L-functions can be viewed as (non-abelian) L-functions of K

via induced representations.

False for genus one.

Question: Is there an a priori bound for the necessary n?

Sketch of proof

The field K is the function field of some curve $C/\mathbb{F}_q$. Let J_C be its Jacobian.

"Fourier analysis" of these L-functions describes the set $C(\mathbb{F}_{q^n})$ as a subset of $J_C(\mathbb{F}_{q^n})$ and a theorem of Zilber then provides the result.

Remark: From this we give a new proof of a theorem of Mochizuki-Tamagawa that C can be recovered from $\pi_1(C)$.

Zilber's theorem

Theorem

Let $C, D/\mathbb{F}_q$ be curves of genus at least two. If $\psi : J_D(\bar{\mathbb{F}}_q) \to J_C(\bar{\mathbb{F}}_q)$ is an isomorphism of groups such that $\psi(D(\bar{\mathbb{F}}_q)) = \psi(C(\bar{\mathbb{F}}_q))$, then ψ is a morphism of curves composed with a limit of Frobenius maps.

Question: If the Jacobian is replaced by a generalized Jacobian of dimension > 1, does a similar result hold?

We have partial results for tori by reversing the above approach.

Recovering equations

Theorem 2

Let $U \subset C$ be given by an affine equation $F(x, y) = 0$. We can recover the coefficients of F from certain abelian L-functions associated to Artin-Schreier extensions of K given by $z^p - z = f$ (p characteristic of $\mathbb{F}_q$) for some set of "universal" $f \in \mathbb{F}_q[x, y]$.

Proof uses that the exponential sum:

$$S(f) = \sum_{P \in U(\mathbb{F}_q)} \exp(2\pi i \operatorname{Tr}(f(P))/p)$$

(where Tr is the absolute trace to $\mathbb{F}_p$) can be recovered from the L-function.

A special case

We can recover $\sum_{P\in U(\mathbb{F}_q)} \mathrm{Tr}(f(P))$ from $S(f) \pmod{\varpi^2}$, where $\varpi = 1 - e^{2\pi i/p}$ in odd characteristic p.

Elliptic curve $y^2 = x(x-1)(x-\lambda)$.

Using $f = \alpha x(1 - y^{q-1})$ with α running through a basis of $\mathbb{F}_q/\mathbb{F}_p$, we obtain

$$\sum_{P\in U(\mathbb{F}_q), y(P)=0} x(P) = \lambda + 1.$$

THANK YOU