Groups definable in difference-differential fields

(Joint work in progress with Ronald Bustamante and Samaria Montenegro, U. of Costa-Rica)

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(These slides are not the original slides, they have been slightly modified. The corrections appear in blue)

Origins of our study and question

All fields are of characteristic 0

The algebra of fields with m commuting derivations was developed in particular by J. Ritt and E. Kolchin. The theory of fields of characteristic 0 with a set Δ of m commuting derivations has a model completion, DCF_m. The theory DCF_m is ω-stable, eliminates quantifiers and imaginaries. In particular it has prime models, the so called *differential closures*.

DCF_m was first studied by A. Robinson and L. Blum for $m = 1$, and later by T. McGrail and O. León Sánchez in the general case. $\mathcal{U}\Delta$ denotes the Lie algebra of linear combinations of elements of Δ. Definable subfields of $\mathcal{U}$ correspond to subspaces of $\mathcal{U}\Delta$ generated by commuting derivations.

The starting point of our research was the result by Phyllis Cassidy on groups definable in differentially closed fields:

Theorem (Cassidy, 1989, J. of Alg.). *Let $\mathcal{U}$ be a differentially closed field, H be a simple algebraic group, and G a definable connected Zariski dense subgroup of $H(\mathcal{U})$ which is definably simple. Then there is a definable subfield L of $\mathcal{U}$, which is the field of constants of a finite subset of $\mathcal{U}\Delta$, such that G is conjugate to $H(L)$.*

A stronger version. *Let $\mathcal{U}$ be a differentially closed field, G a group definable in $\mathcal{U}$ which is definably simple. Then there are a simple algebraic group H defined and split over $\mathbb{Q}$, a definable subfield L of $\mathcal{U}$, and a definable isomorphism $\varphi : G \to H(L)$.*

The theory DCF_mA

A version of this result exists for ACFA, the model-companion of the theory of fields with an automorphism (C-Hrushovski-Peterzil, 2002).

One can also mix derivations and automorphisms. The theory of differential fields with m commuting derivations and one automorphism admits a model-companion, DCF_mA. This was shown by Bustamante in 2006 for $m = 1$, and recently (2016) by León Sánchez in the general case. The theory DCF_mA behaves very much like ACFA, but the derivations make it more complicated.

It stops there: with two commuting automorphisms, there is no model-companion. Without the commutativity hypothesis on the automorphisms, the model-companion exists but very little is known of the interactions between definable sets.

The result

Theorem 1

Let $\mathcal{U}$ be a model of DCF_mA, let H be a simple algebraic group defined and split over $\mathbb{Q}$, and let $G \leq H(\mathcal{U})$ be definable, definably quasi-simple, and Zariski dense in H. Then G has a definable subgroup G_0 of finite index, which is conjugate to a subgroup of $H(K)$, where K is either a field of constants L as in Cassidy's result, or a subfield of such an L of the form $\mathrm{Fix}(\sigma^\ell) \cap L$, for some integer $\ell \geq 1$.

The stronger result: *If $\mathcal{U}$ is as above, and G is a group definable in $\mathcal{U}$, which is definably quasi-simple, then there are a definable subgroup G_0 of finite index in G, a simple algebraic group H as above, and a definable homormorphism $\varphi : G_0 \to H(\mathcal{U})$ with finite kernel and Zariski dense image in H.*

Some ingredients of the proof

In fact the stronger result is a direct consequence of a result by Blossier, Martin-Pizarro and Wagner (2015): $\mathrm{DCF}_m\mathrm{A}$ is what they call *one-based over* the (ω-stable) theory DCF_m, and they show the existence of a definable subgroup G_0 of finite index, a Δ-algebraic group H, and a definable homomorphism $\varphi : G_0 \to H(\mathcal{U})$ with finite kernel.
As you might expect, definably quasi-simple has something to do with simple: a definable group G is *definably quasi-simple* if whenever V is a definable infinite subgroup of G and of infinite index in G, then $N_G(V)$ has infinite index in G. Note that this property is stable under going to subgroups of finite index.
So, if $1 \neq V$ is a connected normal Δ-algebraic subgroup of H, then $\varphi^{-1}(V(\mathcal{U})) \cap G_0$ is a normal subgroup of G_0, hence must be finite, and we may compose φ with the natural projection $H(\mathcal{U}) \to (H/V)(\mathcal{U})$.

The proof of Theorem 1 uses Cassidy's result in a major way. First one replaces G by a definable subgroup of finite index G_0 which is the intersection of G with the connected component of the closure of G for the σ-Δ-topology. One first assumes H centerless. Then one defines the prolongations: for each $n \geq 1$, let $p_n : H \to H^{n+1}$ be defined by $g \mapsto (g, \sigma(g), \dots, \sigma^n(g))$, and let $G_{(n)}$ be the closure for the Δ-topology of $p_n(G)$ in $H^{n+1}(\mathcal{U})$. So $G_{(0)}$ is of the form $H(L)$, with L a definable subfield of the differential field $\mathcal{U}$, and if n is minimal such that $G_{(n)} \neq \prod_{i=1}^{n} \sigma^i(H(L))$, then one shows that $G_{(n)}$ defines an isomorphism $\psi : H(L) \to \sigma^n(H(L))$. By a result of Sonat Suer (2007), distinct definable subfields of the differential field $\mathcal{U}$ are orthogonal, so we must have $L = \sigma^n(L)$, i.e., ψ defines an automorphism of $H(L)$. A little more work gives that G_0 is conjugate to a subgroup of $H(\mathrm{Fix}(\sigma^\ell) \cap L)$.

These results generalize to the case of semi-simple algebraic groups (no infinite normal commutative algebraic subgroup), and to the corresponding notion of definably quasi-semi-simple groups. The statement is a little more complicated in case we allow finite centers, but similar.

While $H(L)$ is simple as an abstract group when H is a simple algebraic group, $H(\mathrm{Fix}(\sigma) \cap L)$ is in general not. Indeed, $\mathrm{Fix}(\sigma)$ (or $\mathrm{Fix}(\sigma^\ell)$) is a pseudo-finite field. Results of Hrushovski-Pillay (1995) show that if there is some algebraic isogeny $f : H' \to H$ defined over a pseudo-finite field F, then $[H(F) : f(H'(F))] = |\mathrm{Ker}(f)(F)|$.

We address two problems:

• Show that a Zariski dense definable subgroup G of $H(L)$ is definably quasi-simple.

• Show their connected component has finite index, i.e. that such a G has a smallest definable subgroup of finite index.

Definable subgroups of algebraic groups

As explained above, the study of groups definable in a model $\mathcal{U}$ of DCF_mA reduces, using the result of Blossier-MartinPizarro-Wagner, up to finite kernel and going to a subgroup of finite index, to the study of definable subgroups of algebraic groups.

If H is an algebraic group, among the definable subgroups of $H(\mathcal{U})$ are of course those which are quantifier-free definable, i.e., more or less defined by difference-differential equations. But there are other ones. One knows (by supersimplicity of the completions of DCF_mA) that if $G \leq H(\mathcal{U})$ is definable, and $\bar{G}$ is the closure of G for the σ-Δ-topology, then $[\bar{G} : G] < \infty$.

The inspiration comes again from the paper of Hrushovski and Pillay. They showed that if the definable subgroup G of $H(F)$ is Zariski dense in H, F a pseudo-finite field, then there is an algebraic group H' and an isogeny $f : H' \to H$, such that $f(H'(F))$ has finite index in G.

Let G be a definable subgroup of $H(\mathcal{U})$, with σ-Δ-closure $\bar{G}$. So, $\bar{G}$ is quantifier-free definable, by the set of difference-differential equations which vanish on G. The result we obtain is the following:

Theorem 2

Let H be an algebraic group, $G \leq H(\mathcal{U})$ a definable subgroup. Then there is a quantifier-free definable group H' (living in some algebraic group), together with a definable map $\pi : H' \to G$ with finite kernel, and such that $\pi(H')$ has finite index in G.

It is known that there is some quantifier-free definable set W, together with a definable projection f, such that $G = f(W)$, and the fibers of W are finite. The difficulty is therefore to replace this W by some quantifier-free definable group H'. This is done using several tools:

Taking three independent generics g_1, g_2, g_3 of some (generic) irreducible component of W, and getting a group configuration.

Replacing the tuples g_1, g_2, g_3 by the infinite tuples obtained by applying all derivations, and σ,σ^{-1} (i.e., $g \mapsto (\sigma^i \delta_1^{i_1} \cdots \delta_m^{i_m}(g))_{i \in \mathbb{Z}, i_j \in \mathbb{N}}$), doing some manipulation to transform the configuration, obtain a projective limit H_ω of algebraic groups, and generics h_1, h_2, h_3 of H_ω which are equi-algebraic with g_1, g_2, g_3.

Get $\pi : H' \to G_0 \leq G$.

Definable subgroups of $H(\mathrm{Fix}(\sigma) \cap L)$

One can show that the only induced structure on $\mathrm{Fix}(\sigma)$ is the differential field structure. In particular, definable subsets are definable with parameters in $\mathrm{Fix}(\sigma)$. Similarly, if $\ell > 1$, then the structure on $\mathrm{Fix}(\sigma^\ell)$ is the structure of the differential field, together with an automorphism of order ℓ. Some work allows to transform Theorem 2 into the following:

Theorem 3

(80%) Let L be a definable subfield of $\mathcal{U}$, H a simple algebraic group defined over $\mathbb{Q}$, and G a definable subgroup of $H(\mathrm{Fix}(\sigma^\ell) \cap L)$ which is Zariski dense in $H$and definably quasi-siomple ($\ell \geq 1$). Then there are a simple algebraic group H', a quantifier-free definable subgroup of $H'(\mathcal{U})$, and an isogeny $\pi : H' \to H$, such that $\pi(G')$ is a subgroup of finite index of G.

Connected component

Theorem 4

Let G be a definably quasi-simple group which is definable in $\mathcal{U}$. Then G has a smallest definable subgroup of finite index.

Sketch of the proof

- Reduce to the case where $G \leq H(L)$, H a simple algebraic group, L a definable subfield of $\mathcal{U}$, G Zariski dense in H.
  We know that there is a definable G_0 of finite index in G, such that $G_0/Z(G_0)$ embeds into such an $H(L)$; know $Z = Z(G_0)$ is finite; if $G_1 \leq G_0$ is such that G_1Z/Z has no definable subgroup of finite index, then any subgroup of finite index of G_1 has index $\leq |Z \cap G_1|$.
- By the above we may assume that G and H are centerless. We may replace G by its σ-Δ-closure $\bar{G}$, and there are two cases to consider: If $\bar{G} = H(L)$, then $H(L)$ has no definable subgroup of finite index.

Proof (ctd)

Assume that $\bar{G} \leq H(L)$ is defined by $\sigma^{\ell}(g) = \varphi(g)$, some algebraic automorphism φ of H, and let $f : \tilde{H} \to H$ be the universal central cover of H. It suffices to show that the connected component (for the σ-Δ-topology) of $f^{-1}(\bar{G})$ has no definable subgroup of finite index. This is done using Thm 2 and the fact that $\tilde{H}$ has no proper finite central cover.

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