Cohesive powers of linear orders

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Cohesive sets

Let

$$\vec{A} = (A_0, A_1, A_2, \dots)$$

be a countable sequence of subsets of $\mathbb{N}$.

Then there is an **infinite** set $C \subseteq \mathbb{N}$ such that, for every i:

$$\text{either } C \subseteq^* A_i \quad \text{or } C \subseteq^* \mathbb{N} \setminus A_i.$$

C is called **cohesive** for $\vec{A}$, or simply $\vec{A}$-**cohesive**.

If $\vec{A}$ is the sequence of recursive sets, then C is called **r-cohesive**.

If $\vec{A}$ is the sequence of r.e. sets, then C is called **cohesive**.

Skolem's countable non-standard model of true arithmetic

Skolem (1934):
Let C be cohesive for the sequence of arithmetical sets.

Consider arithmetical functions $f, g\colon \mathbb{N} \to \mathbb{N}$. Define:

$$
\begin{array}{lll}
f =_C g & \text{if} & C \subseteq^* \{n : f(n) = g(n)\} \\
f < g & \text{if} & C \subseteq^* \{n : f(n) < g(n)\} \\
(f+g)(n) & = & f(n) + g(n) \\
(f \times g)(n) & = & f(n) \times g(n)
\end{array}
$$

Let $[f] = \{g : g =_C f\}$ denote the $=_C$-equivalence class of f.

Form a structure $\mathcal{M}$ with domain $\{[f] : f \text{ arithmetical}\}$ and

$$[f] < [g] \text{ if } f < g; \qquad [f] + [g] = [f+g]; \qquad [f] \times [g] = [f \times g].$$

Then $\mathcal{M}$ models true arithmetic!

- The **standard** elements of $\mathcal{M}$ are those represented by constant functions.
- The **non-standard** part of $\mathcal{M}$ is everything else. Note $[\mathrm{id}]$ is non-standard.

Effectivizing Skolem's construction

Tennenbaum wanted to know:
What if we did Skolem's construction, but

- used recursive functions $f\colon \mathbb{N} \to \mathbb{N}$ in place of arithmetical functions;
- only assumed that C is r-cohesive?

Do we still get models of true arithmetic?

Feferman-Scott-Tennenbaum (1959):
It is not even possible to get models of Peano arithmetic in this way.

Let $\Pi_C(\mathbb{N}; <, +, \times)$ denote the **cohesive power** of the standard model of arithmetic by the (r-)cohesive set C.

Theorem (Lerman 1970)

Let C and D be co-r.e. cohesive sets. Then $\Pi_C(\mathbb{N}; <, +, \times)$ and $\Pi_D(\mathbb{N}; <, +, \times)$ are elementarily equivalent if and only if $C \equiv_{\mathrm{m}} D$.

Cohesive powers

Dimitrov (2009):
Let $\mathcal{A}$ be a computable structure.
(i.e., $\mathcal{A}$ has domain $\mathbb{N}$ and recursive functions and relations.)

Let C be cohesive. Form the **cohesive power** $\Pi_C\mathcal{A}$ of $\mathcal{A}$ by C:

Consider partial recursive $\varphi, \psi \colon \mathbb{N} \to \mathbb{N}$ with $C \subseteq^* \mathrm{dom}(\varphi)$. Define:

$$
\begin{array}{lcl}
\varphi =_C \psi & \text{if} & C \subseteq^* \{n : \varphi(n) = \psi(n)\} \\
R(\psi_0, \dots, \psi_{k-1}) & \text{if} & C \subseteq^* \{n : R(\psi_0(n), \dots, \psi_{k-1}(n))\} \\
F(\psi_0, \dots, \psi_{k-1})(n) & = & F(\psi_0(n), \dots, \psi_{k-1}(n))
\end{array}
$$

Let $[\varphi]$ denote the $=_C$-equivalence class of φ.

Let $\Pi_C\mathcal{A}$ be the structure with domain $\{[\varphi] : C \subseteq^* \mathrm{dom}(\varphi)\}$ and

$$
\begin{array}{lcl}
R([\psi_0], \dots, [\psi_{k-1}]) & \text{if} & R(\psi_0, \dots, \psi_{k-1}) \\
F([\psi_0], \dots, [\psi_{k-1}]) & = & [F(\psi_0, \dots, \psi_{k-1})].
\end{array}
$$

A little Łoś

For cohesive powers:

1. Łoś's theorem holds for Σ_2 sentences and Π_2 sentences.
2. A one-way Łoś's theorem holds for Π_3 sentences.

('Sentence' means a sentence in the language of the structure under consideration.)

Theorem (Łoś's theorem for cohesive powers; Dimitrov)

Let $\mathcal{A}$ be a computable structure, and let C be cohesive. Then

1. *If θ is a Σ_2 sentence or a Π_2 sentence, then*

$$\Pi_C\mathcal{A} \models \theta \quad \textit{if and only if} \quad \mathcal{A} \models \theta.$$

2. *If θ is a Π_3 sentence, then*

$$\Pi_C\mathcal{A} \models \theta \quad \textit{implies} \quad \mathcal{A} \models \theta.$$

A dual investigation to Lerman's

Lerman's investigation:

- Fix a computable presentation $\mathcal{A}$ of a computably-presentable structure.
- Vary the cohesive set C.
- See what structures $\Pi_C\mathcal{A}$ arise as cohesive powers.
- **Result**: For co-r.e. cohesive sets C and D, $\Pi_C(\mathbb{N}; <, +, \times) \equiv \Pi_D(\mathbb{N}; <, +, \times)$ if and only if $C \equiv_{\mathrm{m}} D$.

Our investigation:

- Fix a co-r.e. cohesive set C.
- Fix a computably-presentable structure.
- Vary the computable presentation $\mathcal{A}$ of the structure.
- See what structures $\Pi_C\mathcal{A}$ arise as cohesive powers.

Moving the setting from arithmetic to linear orders

Lerman's structure is $(\mathbb{N}; <, +, \times)$.

- Only one presentation of $(\mathbb{N}; <, +, \times)$, up to recursive isomorphism.
- I do not know names for models of PA^- (besides the standard one).
- I.e., you can say whether or not two cohesive powers $\Pi_C(\mathbb{N}; <, +, \times)$ and $\Pi_D(\mathbb{N}; <, +, \times)$ are the same, but it is hard to identify a cohesive power $\Pi_C(\mathbb{N}; <, +, \times)$ as some previously-known structure.

We study **linear orders** instead of arithmetic.

- Lots of linear orders have names!
  - ω is the order-type of $(\mathbb{N}, <)$.
  - ζ is the order-type of $(\mathbb{Z}, <)$.
  - η is the order-type of $(\mathbb{Q}, <)$.
- Even more names: $\omega + \omega$ ω^* $\omega + \eta$ $\omega + \zeta\eta$ etc.
- The linear order ω has lots of computable presentations.

Cohesive powers of linear orders: Basic results

(D H M Sh So V)
Let $\mathcal{L}$ and $\mathcal{M}$ be computable linear orders, and let C be cohesive. Then:

- $\Pi_C(\mathcal{L} + \mathcal{M}) \;\cong\; \Pi_C\mathcal{L} + \Pi_C\mathcal{M}$
- $\Pi_C(\mathcal{L}\mathcal{M}) \;\cong\; (\Pi_C\mathcal{L})(\Pi_C\mathcal{M})$
- $\Pi_C\mathcal{L}^* \;\cong\; (\Pi_C\mathcal{L})^*$.

Let $\mathbb{N} = (\mathbb{N}, <)$ denote the **standard presentation** of ω, and let C be cohesive. Then:

$$\Pi_C\mathbb{N} \;\cong\; \omega + \zeta\eta.$$

(Recall that $\omega + \zeta\eta$ is the order-type of countable non-standard models of PA.)

There is a computable copy $\mathcal{L}$ of ω that is **not** recursively isomorphic to the standard presentation such that for every cohesive set C:

$$\Pi_C\mathcal{L} \;\cong\; \omega + \zeta\eta.$$

Are there other cohesive powers of copies of ω?

Question: Is it possible to achieve linear orders other than $\omega + \zeta\eta$ as cohesive powers of computable copies of ω?

Previous slide: To make $\Pi_C\mathcal{L} \ncong \omega + \zeta\eta$, you would have to do more than make $\mathcal{L}$ not recursively isomorphic to the standard presentation of ω.

We show:

Theorem (D H M Sh So V)

For every co-r.e. cohesive set C, there is a computable copy $\mathcal{L}$ of ω such that

$$\Pi_C\mathcal{L} \cong \omega + \eta.$$

Notice: The Π_3 sentence "every point has an immediate successor" is satisfied by ω but not by $\omega + \eta$. So Łoś's theorem for cohesive powers is **optimal**.

More generally: We show that it is possible to achieve

$$\omega + \text{various shuffles}$$

as cohesive powers of computable copies of ω.

Shuffles, blocks, and condensations

Recall the **shuffle** $\boldsymbol{\sigma}(X)$ of an at-most-countable collection X of linear orders:

- List X as $(\mathcal{L}_i : i < |X|)$.
- Paint $\mathbb{Q}$ with $|X|$ many colors so that each color occurs densely.
- Replace each point in $\mathbb{Q}$ of color i with a copy of $\mathcal{L}_i$.

For example, $\boldsymbol{\sigma}(\{\mathbf{2}, \mathbf{3}\})$ consists of dense copies of $\mathbf{2}$ and dense copies of $\mathbf{3}$.

Let $\mathcal{L} = (L, \prec_L)$ be a linear order.

- The **finite condensation** $\mathbf{c}_{\mathrm{F}}(x)$ of an $x \in L$ is

$$\mathbf{c}_{\mathrm{F}}(x) = \{y \in L : \text{there are only finitely many points between } x \text{ and } y\}.$$

- Also call $\mathbf{c}_{\mathrm{F}}(x)$ a **block** of $\mathcal{L}$. Blocks are maximal discrete intervals.

- The **finite condensation** of $\mathbf{c}_{\mathrm{F}}(\mathcal{L})$ of $\mathcal{L}$ is

$$\mathbf{c}_{\mathrm{F}}(\mathcal{L}) = \{\mathbf{c}_{\mathrm{F}}(x) : x \in L\}.$$

For example, $\mathbf{c}_{\mathrm{F}}(\boldsymbol{\sigma}(\{\mathbf{2}, \mathbf{3}\})) \cong \eta$.

A few more facts about cohesive powers of linear orders

A computable linear order $\mathcal{L}$ **canonically embeds** into its cohesive powers $\Pi_C \mathcal{L}$ by

$$x \mapsto [\text{the constant function with value } x].$$

Let $\mathcal{L}$ be a computable copy of ω, and let C be a cohesive set.

- The range of the canonical embedding is an initial segment of $\Pi_C \mathcal{L}$.
- So $\Pi_C \mathcal{L} \;\cong\; \omega + \mathcal{M}$ for some linear order $\mathcal{M}$.
- Call the ω-part of $\Pi_C \mathcal{L}$ the **standard** part.
- Call the $\mathcal{M}$-part of $\Pi_C \mathcal{L}$ the **non-standard** part. Note $[\mathrm{id}] \in \mathcal{M}$.

Theorem (D H M Sh So V)

Let $\mathcal{L}$ be a computable linear order, and let C be a co-r.e. cohesive set.

- *The condensation $\mathbf{c}_{\mathrm{F}}(\Pi_C \mathcal{L})$ is dense.*
- *If $\mathcal{L} \cong \omega$, then $\mathbf{c}_{\mathrm{F}}(\Pi_C \mathcal{L}) \;\cong\; \mathbf{1} + \eta$.*

Upshot: Aim to make $\Pi_C \mathcal{L} \;\cong\; \omega +$ a shuffle of discrete linear orders.

Achieving $\Pi_C\mathcal{L} \;\cong\; \omega + \eta$

Theorem (D H M Sh So V)

For every co-r.e. cohesive set C, there is a computable copy $\mathcal{L}$ of ω such that

$$\Pi_C\mathcal{L} \;\cong\; \omega + \eta.$$

(Observe that $\eta \cong \boldsymbol{\sigma}(\{\mathbf{1}\})$.)

Goal: Compute $\mathcal{L} \cong \omega$ so as to make the non-standard points of $\Pi_C\mathcal{L}$ dense.

That is, compute $\mathcal{L}$ so that if ψ and φ represent **non-standard** points of $\Pi_C\mathcal{L}$ with

$$[\psi] \prec_{\Pi_C\mathcal{L}} [\varphi],$$

then there is a θ with

$$[\psi] \prec_{\Pi_C\mathcal{L}} [\theta] \prec_{\Pi_C\mathcal{L}} [\varphi].$$

Observations and lemmas

If C is co-r.e. and φ is partial recursive, then there is a total recursive f with $f =_C \varphi$.

So every point $[\varphi] \in \Pi_C \mathcal{A}$ in a cohesive power by a **co-r.e.** cohesive set has a total representative $[f] = [\varphi]$.

Let $\mathcal{L}$ be a computable linear order, and let C be cohesive.

If $\mathcal{L} \cong \omega$, then an $[\varphi] \in \Pi_C \mathcal{L}$ is non-standard if and only if $\lim_{n \in C} \varphi(n) = \infty$.

Also, TFAE:

- $[\varphi]$ is the $\prec_{\Pi_C \mathcal{L}}$-immediate successor of $[\psi]$.
- $\varphi(n)$ is the $\prec_{\mathcal{L}}$-immediate successor of $\psi(n)$ for a.e. $n \in C$.
- $\varphi(n)$ is the $\prec_{\mathcal{L}}$-immediate successor of $\psi(n)$ for infinitely many $n \in C$.

Unpacking the requirements

Fix a co-r.e. cohesive set C.

We want to compute a copy $\mathcal{L} = (L, \prec_{\mathcal{L}})$ of ω so that

$$[\varphi] \text{ is } \textbf{not} \text{ the } \prec_{\Pi_C \mathcal{L}}\text{-immediate successor of } [\psi]$$

whenever $[\psi]$ and $[\varphi]$ are non-standard points of $\Pi_C \mathcal{L}$.

So we want to achieve

$$\psi(n)\downarrow \prec_{\mathcal{L}} \varphi(n)\downarrow \quad \Rightarrow \quad \exists k \in L \left[\psi(n) \prec_{\mathcal{L}} k \prec_{\mathcal{L}} \varphi(n)\right]$$

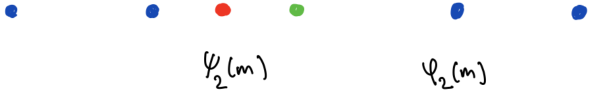
for a.e. $n \in C$ whenever $\lim_{n \in C} \psi(n) = \lim_{n \in C} \varphi(n) = \infty$.
(It suffices to achieve this for total ψ and φ.)

However:
For every x we put in L, we must only put finitely many points $\prec_{\mathcal{L}}$-below it!

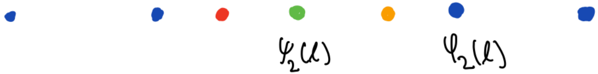
Ugly pictures of what to avoid

Suppose we are building a linear order point-by-point.

At some stage, $\varphi_9(n)$ looks like the immediate successor of $\psi_9(n)$.
So add the red point in between.



Later, $\varphi_2(m)$ looks like the immediate successor of $\psi_2(n)$ = the red point.
So add the green point in between.



Even later, $\varphi_2(\ell)$ looks like the immediate successor of $\psi_2(\ell)$ = the green point.
So add the orange point in between.

Preparing to compute $\mathcal{L}$

Order all pairs $(\langle \psi_p, \varphi_p \rangle : p \in \mathbb{N})$ of partial recursive functions.

Main idea:

- Eventually we have to add a k between some $\psi_p(n)$ and $\varphi_p(n)$.
- Try to choose k to avoid $\psi_q(C)$ and $\varphi_q(C)$ for higher priority $\langle \psi_q, \varphi_q \rangle$.

Uniformly compute a sequence $(\langle A^{i,0}, A^{i,1} \rangle : i \in \mathbb{N})$ of pairs of sets such that

- $A^{i,0}$ and $A^{i,1}$ partition $\mathbb{N}$ into two parts;
- $\bigcap_{i<n} A^{i,\sigma(i)}$ is infinite for every n and every $\sigma \in \{0,1\}^n$.

Assign $\langle A^{2p,0}, A^{2p,1} \rangle$ to ψ_p and assign $\langle A^{2p+1,0}, A^{2p+1,1} \rangle$ to φ_p.

By cohesiveness, either $\psi_p(C) \subseteq^* A^{2p,0}$ or $\psi_p(C) \subseteq^* A^{2p,1}$. **Try to learn which!**

Similarly, either $\varphi_p(C) \subseteq^* A^{2p+1,0}$ or $\varphi_p(C) \subseteq^* A^{2p+1,1}$. **Try to learn which!**

Sketch of the algorithm

Fix a recursive enumeration $W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots$ of $W = \mathbb{N} \setminus C$.

At the start of stage s, we have $\prec_{\mathcal{L}}$ defined on a finite $X_{s-1} \supseteq \{0, \ldots, s-1\}$.

Start with $X_s = X_{s-1}$, and add more elements to X_s.
If $s \notin X_s$, add it and make it $\prec_{\mathcal{L}}$-maximum of what we have so far.

Consider each $\langle p, N \rangle < s$ in order.

Here N is a guess of where the cohesiveness of ψ_p and φ_p begin with respect to the partitions $\langle A^{2p,0}, A^{2p,1} \rangle$ and $\langle A^{2p+1,0}, A^{2p+1,1} \rangle$.

That is, we guess that either

$$\forall m \geq N \ \big(m \in C \to \psi_p(m) \in A^{2p,0}\big)$$

or

$$\forall m \geq N \ \big(m \in C \to \psi_p(m) \in A^{2p,1}\big)$$

and similarly for φ_p.

Sketch of the algorithm

For $\langle p, N \rangle$, think of an $(a, b, n) \in \{0,1\} \times \{0,1\} \times \{N, N+1, \ldots, s\}$ as guessing

- $\psi_p(m) \in A^{2p,a}$ for all $m \in C$ with $m \geq N$;
- $\varphi_p(m) \in A^{2p+1,b}$ for all $m \in C$ with $m \geq N$;
- $n \in C$.

For each $\langle p, N \rangle < s$, check for an (a, b, n) such that the following hold.

- For all $m \leq n$, $\psi_{p,s}(m){\downarrow}$ and $\varphi_{p,s}(m){\downarrow}$.
- Both $\psi_p(n) \in A^{2p,a}$ and $\varphi_p(n) \in A^{2p+1,b}$.
- For all m with $N \leq m \leq n$,
  - $\psi_p(m) \in A^{2p,1-a} \rightarrow m \in W_s$, and
  - $\varphi_p(m) \in A^{2p+1,1-b} \rightarrow m \in W_s$.
- Currently $\varphi_p(n)$ is the $\prec_{\mathcal{L}}$-immediate successor of $\psi_p(n)$ in X_s.
- The element $\psi_p(n)$ is not $\preceq_{\mathcal{L}}$-below any of $0, 1, \ldots, \langle p, N \rangle$.

If there is such an (a, b, n), then we need to act for $\langle p, N \rangle$.

Sketch of the algorithm

We found an (a, b, n), so we need to act for $\langle p, N\rangle$. Call (a, b) the **action sides**.

- Let r be greatest such that $\langle r, M\rangle \leq \langle p, N\rangle$ for some M.
- For each $q \leq r$, let (a_q, b_q) be the most recently used action sides by any $\langle q, M\rangle \leq \langle p, N\rangle$.
- Choose the first k in the infinite set

$$\bigcap_{q \leq r} \left(A^{2q,1-a_q} \cap A^{2q+1,1-b_q}\right) \setminus X_s.$$

  Essentially, we are guessing that k is **not** in $\psi_q(C)$ or $\varphi_q(C)$ for any $q \leq r$.

- Add k to X_s and put it $\prec_{\mathcal{L}}$-between $\psi_p(n)$ and $\varphi_p(n)$.

This does it!

- Can show that $\mathcal{L} \cong \omega$ because the actions of $\langle p, N\rangle$ eventually do not interfere with the actions of higher-priority $\langle q, M\rangle$.
- Can show that $[\varphi_p]$ is not the $\prec_{\Pi_C \mathcal{L}}$-immediate successor of $[\psi_p]$ (if non-std).

More cohesive powers of computable copies of ω

For $k \geq 1$, let $\boldsymbol{k}$ denote the k-element linear order. Notice that $\boldsymbol{k}\omega \cong \omega$.

Fix a co-r.e. cohesive set C.

Let $\mathcal{L}$ be a computable copy of ω such that $\Pi_C \mathcal{L} \ \cong\ \omega + \eta$.

Then $\boldsymbol{k}\mathcal{L} \cong \omega$, and

$$\Pi_C(\boldsymbol{k}\mathcal{L}) \ \cong\ (\Pi_C \boldsymbol{k})(\Pi_C \mathcal{L}) \ \cong\ \boldsymbol{k}(\omega + \eta) \ \cong\ \boldsymbol{k}\omega + \boldsymbol{k}\eta \ \cong\ \omega + \boldsymbol{k}\eta.$$

So $\mathbf{1}\mathcal{L}, \mathbf{2}\mathcal{L}, \mathbf{3}\mathcal{L}, \cdots$ are computable copies of ω giving rise to a sequence of pairwise **non-elementarily equivalent** cohesive powers:

$$\Pi_C(\mathbf{1}\mathcal{L}), \quad \Pi_C(\mathbf{2}\mathcal{L}), \quad \Pi_C(\mathbf{3}\mathcal{L}), \quad \ldots$$

If $k \neq m$, then $\Pi_C(\boldsymbol{k}\mathcal{L})$ satisfies the Σ_3 sentence "there is a block of size k," but $\Pi_C(\boldsymbol{m}\mathcal{L})$ does not.

Shuffling finite linear orders into cohesive powers of ω

Fix a co-r.e. cohesive set C.

Suppose we want a computable copy $\mathcal{M}$ of ω with $\Pi_C \mathcal{M} \cong \omega + \boldsymbol{\sigma}(\{2,3\})$.

Strategy:
Compute a copy $\mathcal{L}$ of ω with $\Pi_C \mathcal{L} \cong \omega + \eta$ along with a **coloring** $F \colon \mathcal{L} \to \{0,1\}$.

F induces a coloring $\widehat{F} : \Pi_C \mathcal{L} \to \{0,1\}$ by

$$\widehat{F}([\varphi]) = \begin{cases} 0 & \text{if } F(\varphi(n)) = 0 \text{ for a.e. } n \in C \\ 1 & \text{if } F(\varphi(n)) = 1 \text{ for a.e. } n \in C. \end{cases}$$

Compute F so that colors 0 and 1 are dense in the **non-standard** part of $\Pi_C \mathcal{L}$.

To compute $\mathcal{M}$:

- Start with $\mathcal{L}$ and its coloring F.
- Replace points in $\mathcal{L}$ with color 0 by copies of $\mathbf{2}$.
- Replace points in $\mathcal{L}$ with color 1 by copies of $\mathbf{3}$.

Then $\Pi_C \mathcal{M} \cong \omega + \boldsymbol{\sigma}(\{2,3\})$.

Inducing colorings of cohesive powers

More generally:

- Let C be cohesive, and let $\mathcal{L}$ be a computable linear order.
- Let $F\colon \mathcal{L} \to \mathbb{N}$ be a computable coloring.

F induces a coloring $\widehat{F}$ on $\Pi_C \mathcal{L}$ by

$$\widehat{F}([\varphi]) = [\![F \circ \varphi]\!].$$

Colors are $=_C$-equivalence classes of partial recursive functions $\delta\colon \mathbb{N} \to \mathbb{N}$.
Write $[\![\delta]\!]$ for δ's equivalence class when thinking of δ as a color.

A **solid color** is a color $[\![\delta]\!]$ where δ is constant. For example, $[\![0]\!]$ is a solid color.

A **striped color** is a color $[\![\delta]\!]$ that is not solid.

Shuffling finite collections of finite linear orders

We can induce a dense colorings of a cohesive powers.

Theorem (D H M Sh So V)

For every co-r.e. cohesive set C, there is a computable copy $\mathcal{L}$ of ω and a computable $F\colon L \to \mathbb{N}$ such that

- $\Pi_C\mathcal{L} \;\cong\; \omega + \eta$.
- *Every solid $\widehat{F}$-color occurs densely in the non-standard part of $\Pi_C\mathcal{L}$*
- *Between any two distinct non-standard points of $\Pi_C\mathcal{L}$, there is a point with a non-standard $\widehat{F}$-color.*

We can shuffle finite collections of finite linear orders into a cohesive power.

Theorem (D H M Sh So V)

Let $0 < k_0 < k_1 < \cdots < k_n$. For every co-r.e. cohesive set C, there is a computable copy $\mathcal{M}$ of ω such that

$$\Pi_C\mathcal{M} \;\cong\; \omega + \boldsymbol{\sigma}(\{\boldsymbol{k}_0, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n\}).$$

Shuffling infinite collections of finite linear orders

We can even shuffle infinite collections of finite linear orders.

Think of a set $X \subseteq \mathbb{N} \setminus \{0\}$ as a set of finite order-types.

Our most general result is the following:

Theorem (D H M Sh So V)

Let $X \subseteq \mathbb{N} \setminus \{0\}$ be either Σ^0_2 or Π^0_2. Let C be a co-r.e. cohesive set. Then there is a computable copy $\mathcal{M}$ of ω such that

$$\Pi_C \mathcal{M} \;\cong\; \omega + \boldsymbol{\sigma}(X \cup \{\omega + \zeta\eta + \omega^*\}).$$

The $\omega + \zeta\eta + \omega^*$ occurs naturally when shuffling infinitely many finite linear orders using the color-and-replace strategy.

This is because a cohesive product of all finite order-types has type $\omega + \zeta\eta + \omega^*$.

Further examples

Let C be cohesive. Observe that

$$\Pi_C\mathbb{Z} \cong \Pi_C(\mathbb{N}^* + \mathbb{N}) \cong (\Pi_C\mathbb{N})^* + \Pi_C\mathbb{N} \cong (\zeta\eta + \omega^*) + (\omega + \zeta\eta) \cong \zeta\eta.$$

A cohesive power can be isomorphic to the original structure:

- $\Pi_C\mathbb{Q} \cong \eta$ (due to Łoś's theorem for cohesive powers)
- $\Pi_C\mathbb{Z}\mathbb{Q} \cong (\Pi_C\mathbb{Z})(\Pi_C\mathbb{Q}) \cong (\zeta\eta)\eta \cong \zeta\eta.$

We saw many examples of isomorphic linear orders having non-elementarily equivalent cohesive powers.

It is also possible for non-elementarily equivalent linear orders to have isomorphic cohesive powers:

- Let C be a co-r.e. cohesive set.
- Let $\mathcal{L}$ be a computable copy of ω with $\Pi_C\mathcal{L} \cong \omega + \eta$.
- Then $\Pi_C(\mathcal{L} + \mathbb{Q}) \cong \Pi_C\mathcal{L} + \Pi_C\mathbb{Q} \cong (\omega + \eta) + \eta \cong \omega + \eta$.
- Thus $\mathcal{L} \not\equiv \mathcal{L} + \mathbb{Q}$, but $\Pi_C\mathcal{L} \cong \Pi_C(\mathcal{L} + \mathbb{Q})$.

Thank you!

Thank you for coming to my talk!
Do you have a question about it?