Big Ramsey degrees of the Rationals and the Rado Graph and Computability Theory

Peter Cholak

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Joint work with Paul-Elliot Anglès d'Auriac, Damir Dzhafarov, Benoît Monin, and Ludovic Patey.

https://arxiv.org/abs/2007.09739

https://www.nd.edu/~cholak/papers/msri2020.pdf

Some Applications of Milliken's Tree Theorem

The structures, countable discrete set, the rationals, and the Rado graph, have finite big Ramsey degree. For all of these structures, Milliken's Tree Theorem can be used to prove this.

Coloring Structures

- Let $\mathcal{B}$ be an infinite structure and $\mathcal{A}$ is any finite substructure of $\mathcal{B}$.
- $\binom{\mathcal{B}}{\mathcal{A}}$ are copies of $\mathcal{A}$ in $\mathcal{B}$.
- For $\ell \leq k$, the notation

$$\mathcal{B} \to (\mathcal{B})^{\mathcal{A}}_{k,\ell}$$

means that for every coloring $f : \binom{\mathcal{B}}{\mathcal{A}} \to k$ there exists an isomorphic substructure $\mathcal{B}'$ of $\mathcal{B}$ such that $|f^{-1}\binom{\mathcal{B}'}{\mathcal{A}}| \leq \ell$.

Countable Discrete Set

Examples of $\mathcal{B}$ and $\mathcal{A}$

ω is the infinite but countable set. No language. The finite substructures are finite sets of size n. Color sets of size n. Ramsey's Theorem is the statement that, for all n, for all k, $\omega \to (\omega)^n_{k,1}$.

The Rationals

Examples of $\mathcal{B}$ and $\mathcal{A}$

$\mathbb{Q}$ is the rationals with the ordering $<_{\mathbb{Q}}$. A finite substructure is just a linear ordered set of size n. Color sets of linear ordered (under $<_{\mathbb{Q}}$) sets of size n.

The Rado Graph

Examples of $\mathcal{B}$ and $\mathcal{A}$

$\mathcal{R} = (\mathbb{N}, E)$ will be the Rado graph. For every two disjoint finite sets of vertices $F_0, F_1 \subseteq \mathbb{N}$, there exists $x \in \mathbb{N}$ such that xEy, for all $y \in F_0$, and $\neg xEy$, for all $y \in F_1$. E is included in the language. Any finite graph $\mathcal{G}$ is a substructure of $\mathcal{R}$. Color copies of $\mathcal{G}$. (Normally we use V not $\mathbb{N}$.)

Examples of $\mathcal{B}$ and $\mathcal{A}$

All these infinite structures are computable and computably categorical. Also we will use the binary branching tree $2^{<\omega}$ for Miliken Tree Theorem.

Finite Big Ramsey Degree

- For a finite substructure $\mathcal{A}$ of $\mathcal{B}$, the *big Ramsey degree of $\mathcal{A}$ in $\mathcal{B}$* is the least number $\ell \in \omega$, if it exists, such that $\mathcal{B} \to (\mathcal{B})^{\mathcal{A}}_{k,\ell}$ for all $k \in \omega$, in which case we say that the big Ramsey degree of $\mathcal{A}$ is *finite*.
- We say that a structure $\mathcal{B}$ *has finite big Ramsey degrees* if, for every finite substructure $\mathcal{A}$ of $\mathcal{B}$ has finite big Ramsey degree.

All 3 of our examples have finite big Ramsey degree.

$\ell_{\mathcal{A}}$

Theorem

There is a copy of $\mathcal{B}$ such that, for all $\mathcal{A}$, there is a $\ell_{\mathcal{A}}$ and computable coloring, $C_{\ell_{\mathcal{A}}}$, of $\mathcal{B}$ with $\ell_{\mathcal{A}}$ colors such that

- *(Existence) $\mathcal{B} \to (\mathcal{B})^{\mathcal{A}}_{k,\ell_{\mathcal{A}}}$.*
- *(Tightness) In all the copies $\mathcal{B}'$ of $\mathcal{B}$, for all $j \leq \ell_{\mathcal{A}}$, there is a copy of $\mathcal{A}$ with color j from $C_{\ell_{\mathcal{A}}}$ appearing.*

Milliken's Tree Theorem is used to show $\ell_{\mathcal{A}}$ exists. Tightness is related to the (model theoretic) types of $\mathcal{A}$ coded into a yet to be determined superstructure of $\mathcal{B}$ allowing for a larger language. A color for each type. We will later discuss how these copies of $\mathcal{A}$ and the types or colors of $\mathcal{A}$ occur within $\mathcal{B}'$.

For ω, $\ell_n = 1$ for all n. The superstructure is just ω.

When $\mathcal{A}$ is a singleton

Here we are just coloring the domain. $\mathcal{B}'$ can be created as the limit of finite substructures $\mathcal{B}'_s$ of $\mathcal{B}$. Run the greedy algorithm to find a $\mathcal{B}'_{s+1}$ for all colors. It that fails we get a copy of $\mathcal{B}$ with fewer colors. Repeat till it works or we are down to a single color. So nonuniformly computable. Hence provable in RCA_0.

Coding the halting set, K, into $\omega \to (\omega)^3_{2,1}$

Carl Jockusch's Coding

Let $C_J(x,s,t)$ be RED iff $x < s < t$ and $K_s \upharpoonright x = K_t \upharpoonright x$.

Assume $H \subseteq \omega$ is infinite and $|C_J^{-1}(H)| = 1$. Pick z. Find a *large* $x > z$ in H. There is a large $s > x$ in H such that $K_s \upharpoonright x = K \upharpoonright x$. There is a large $t > s$ in H. So this triple is colored RED.

If the triple $\{x,s,t\}$ is a substructure of H then there is a large t', a large s'', and a large t'' such that $\{x,s,t'\}$ and $\{x,s'',t''\}$ are substructures of H. The triple *largely extendable* in H. The new triples are *large companions* of the original triple.

Hence all triples in H are colored RED and $K_s(z) = K_t(z) = K(z)$.

Largely Extendable

- All our structures are computable. Hence there is a one-to-one computable enumeration of the domains. Say d_i for $i \in \mathbb{N}$. Sometimes we need these enumerations to have some special properties related to the structure.
- We say d_i in the domain of $\mathcal{A}$ is *large* if i is large.
- When $\mathcal{A}$ has size 3, we can use this notion of large to define when a copy $\mathcal{A}$ in $\mathcal{B}'$ is *largely extendable* and *large companions*.

Theorem (Largeness)

For all our $\mathcal{B}$, for all our $\mathcal{A}$, each appearance of $\mathcal{A}$ in $\mathcal{B}'$ is largely extendable. Moreover, we can find large companions with the same color as our copy of $\mathcal{A}$ w.r.t. $C_{\ell_{\mathcal{A}}}$.

Coding K when $\mathcal{A}$ has size 3

$C_{\ell_\mathcal{A}}$ is a computable coloring witnesses that all $\ell_\mathcal{A}$ colors must be realized by some copy of $\mathcal{A}$ in any copy $\mathcal{B}'$ of $\mathcal{B}$ inside $\mathcal{B}$. Each of these copies of $\mathcal{A}$ is largely extendable preserving the color.

Use the product coloring, $C_{\ell_\mathcal{A}} \times C_J$ and repeat Jockusch's proof for any color $j \leq \ell_\mathcal{A}$.

$\mathbb{Q}$ and sets of size 2

Code $\mathbb{Q}$ into $2^{<\omega}$ by $q_\sigma = \sum_{i \leq |\sigma|}(\sigma(i) - \frac{1}{2})2^{-i}$. $<_\mathbb{Q}$ is clear in this setting. Let $C(\sigma, \tau)$ be 1 when $|\sigma| < |\tau|$ iff $q_\sigma < q_\tau$. Any subcopy of $\mathbb{Q}$ in our fixed copy must have pairs $\{\sigma, \tau\}$ colored 0 and 1 with large gaps between $|\sigma|$ and $|\tau|$. This a precursor to the coloring C_{ℓ_2} for $\mathbb{Q}$.

A Sketch of these Types

The tree structure gives us the meet, $\sigma \wedge \tau$, of σ and τ. The type of pair $\{\sigma, \tau\}$ must include the pair, the color of the pair, and the meet of the pair. These are just partial types.

Pairs inside an Extension of $\mathbb{Q}$

We will consider the structure $\mathbb{Q}^* = (2^{<\omega}, <_{\mathbb{Q}}, \wedge)$. Let U be a subset of $2^{<\omega}$ such that $(U, <_{\mathbb{Q}})$ is a copy of $\mathbb{Q}$. Let $U^{\wedge}$ be the meet closure of U. Let $\mathcal{U}^* = (U^{\wedge}, <_{\mathbb{Q}}, \wedge)$. Fixed an enumeration e of $2^{<\omega}$ such that if $|e(i)| < |e(j)|$ then $i < j$.

Theorem (Selective Largeness)

For all z, we can computably find an copy of $\{\sigma \wedge \tau, \sigma, \tau\}$, where $|\sigma| < |\tau|$ iff $q_\sigma < q_\tau$ (so the pair $\{\sigma, \tau\}$ is colored 1), and $|\sigma \wedge \tau| > z$, in $\mathcal{U}^*$ which is largely extendable. Moreover, there are large companions with the same color. Similarly for $\{\sigma \wedge \tau, \sigma, \tau\}$, where is not the case that $|\sigma| < |\tau|$ iff $q_\sigma < q_\tau$ (so the pair σ, τ is colored 0),

The proof makes several critical uses of the fact that $(U, <_{\mathbb{Q}})$ is a copy of $\mathbb{Q}$.

Let $C_J(\sigma, \tau) = C_J(\sigma \wedge \tau, \sigma, \tau)$. We can show the product coloring codes K.

When $|\mathcal{A}| \geq_{\mathbb{N}} 3$ or $\mathcal{B} = \mathbb{Q}$

Assume that Miliken Tree Theorem implies the existence of $\ell_{\mathcal{A}}$. Then, as we unlikely recall from the Dzhafarov's talk, Miliken Tree Theorem follows from ACA, arithmetic comprehension, over RCA_0. So everything here is equivalent to ACA over RCA_0. We explore these equivalencies more finely in the paper. There are many results and many open questions.

$\mathcal{R} \rightarrow (\mathcal{R})^{\mathcal{K}_2}_{k,2}$

Take a coloring of all copies of $\mathcal{K}_2$ in $\mathcal{R}$. Fix C not computable. Then there is a subcopy $\mathcal{R}'$ of $\mathcal{R}$ where the copies of $\mathcal{K}_2$ need just 2 colors and $\mathcal{R}'$ does not compute C. This is cone avoidance and done via forcing. So weaker than ACA_0.

We need to explore why $\mathcal{K}_2$ in $\mathcal{R}$ is different from 2 in $\mathbb{Q}$.

It is partially open how the statement $\mathcal{R} \rightarrow (\mathcal{R})^{\mathcal{K}_2}_{k,2}$ is related to other statements weaker than ACA. A number of results appear in our paper.

Strong subtrees

Definition

A *tree* is a subset T of $\omega^{<\omega}$ as follows:

- there exists a *root* $\rho \in T$ such that $\rho \preceq \sigma$ for all $\sigma \in T$;
- if $\sigma, \tau \in T$ then also $\sigma \wedge \tau \in T$;
- every $\sigma \in T$ there are finitely many $\tau \in T$ such that $\sigma \prec \tau$ and there is no τ' such that $\sigma \prec \tau' \prec \tau$.

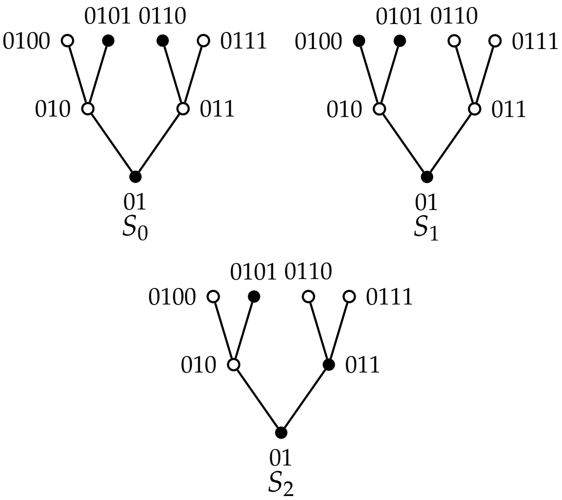
Definition

For each $n \in \mathbb{N}$, let $T(n) = \{\sigma \in T : |\tau \in T : \tau \prec \sigma| = n\}$ and $\text{height}(T) = \sup\{n + 1 \in \mathbb{N} : T(n) \neq \emptyset\}$.

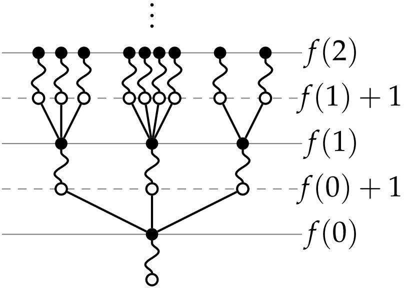
Definition

Let $U \subseteq T$ be trees. U is a *strong subtree* of T if there is a *level function* $f : \text{height}(U) \to T$ such that for all $n < \text{height}(U)$, if $\sigma \in U(n)$ then $\sigma \in T(f(n))$. A node $\sigma \in U$ is k-branching in U if and only if it is k-branching in T.

Examples of subtrees, strong and not strong



Examples of subtrees, strong and not strong



Miliken's tree theorem

Definition

Given a tree T, let $\mathcal{S}_\alpha(T)$ denote the class of all subtrees of T of height $\alpha \leq \omega$.

Theorem (Milliken's tree theorem 1979)

Let T be an infinite tree with no leaves. For all $n, k \geq 1$ and all $c : \mathcal{S}_n(T) \to k$ there is a $U \in \mathcal{S}_\omega(T)$ such that c is contant on $\mathcal{S}_n(U)$.

If we color T by levels we get Ramsey's Theorem. We will use $T = 2^{<\omega}$.

Revisiting ℓ_2 for $\mathbb{Q}$

How do these partial types embed into $\mathcal{S}_3(2^{<\omega})$?

We coded $\mathbb{Q}$ into $2^{<\omega}$ by $q_\sigma = \sum_{i \leq |\sigma|} (\sigma(i) - \frac{1}{2}) 2^{-i}$. Let $C(\sigma, \tau)$ be 1 when $|\sigma| < |\tau|$ iff $q_\sigma < q_\tau$. Must include the meet.

Coding $\mathbb{Q}$ to have just 2 embeddings

We want to code $\mathbb{Q}$ by a set of the nodes U such that U is antichain; every node in $U^\wedge$, the meet closure of U, has a unique length; $U^\wedge - U$ is effective isomorphic via g to $2^{<\omega}$; and, for all $\sigma \in U$, for all τ, if $|\tau| > |\sigma|$, $\sigma(|\tau|) = 0$ (the passing number of τ by σ is 0). Each leaf in U codes a rational as above. Within U, $<_{\mathbb{Q}}$ is just $<_L$. Nodes in $U^\wedge - U$ will be called *meets*. Now pairs of leaves have exactly one of two types, one of each color, 0 or 1.

By the above we know these types appear in every subcopy of U which is a dense linear order under $<_L$ without end points. So we have tightness.

Using Miliken's Tree Theorem

Using all this we can create an effective isomorphism h from $2^{<\omega}$ to U such that: the image of a strong subcopy of $2^{<\omega}$ is a subcopy of $\mathbb{Q}$; the preimage of every pair of leaves is embedded in a unique copy of $S_3(2^{<\omega})$; and the image of every copy of $S_3(2^{<\omega})$ contains exactly one pair of leaves of each type. The preimage of colorings of a type result in a coloring of $S_3(2^{<\omega})$. Now use Miliken's Tree Theorem. Take the image of the result. Repeat for the other type.

Joyce Trees and ℓ_n for the Rationals

The closure of n leaves under meets inside $U^\wedge$ forms a *Joyce* tree of size n. The number of nonisomorphic trees is nth odd tangent number. All have $2n-1$ nodes.

We can show these types appear in every subcopy of U which is a dense linear order under $<_L$ without end points. (Tightness)

This gives us ℓ_n for the rationals.

An Order Definable from the Rado Graph

Let $\mathcal{R} = (V, E)$ be a computable Rado graph. For $i < j$, let $\sigma_j(i) = 1$ iff there is an edge from j to i and 0 otherwise. Thus $|\sigma_j| = j$. Define $i <_{\mathcal{R}} j$ iff $\sigma_i <_L \sigma_j$. Call $(V, E, <_{\mathcal{R}})$ the *Joyce Rado Graph*. $(V, <_{\mathcal{R}})$ is a copy of $\mathbb{Q}$.

Theorem (ACA)

Let (V', E) be a subcopy of the Rado Graph inside $\mathcal{R}$ then there is subcopy (V'', E) of the Rado graph inside (V', E) where $(V'', E, <_{\mathcal{R}})$ is isomorphic to $(V, E, <_{\mathcal{R}})$.

For finite substructures $\mathcal{A}$ of the ordered Rado Graph, the $\ell_{\mathcal{A}}$ for the Rado graph and for the ordered Rado graph are the same.

$\ell_{\mathcal{A}}$ for the Joyce Rado Graph

The collection of all σ_i is effectively isomorphic to $2^{<\omega}$ under $<_L$. Whether there is an edge between j and i is determined by the passing number of σ_i by σ_j. We can code this, like $\mathcal{Q}$, by antichain U. But we have to remove the restriction there on the passing number. Let $c(\sigma)$ be this coding.

For the completely (dis)connected graph of size n, everything works like for subset of set n in the rationals. The passing number is constant.

If the meet of $c(\sigma)$ and $c(\tau)$ is longer than $c(\rho)$ then the passing number of $c(\sigma)$ and $c(\tau)$ by $c(\rho)$ agreed. Hence a finite graph $\mathcal{G}$ limits the number of Joyce trees which can code $\mathcal{G}$. But they can be counted and appear in every subcopy of a Joyce Rado Graph.

$\mathcal{R} \to (\mathcal{R})^{\mathcal{K}_2}_{k,2}$, Again

How we code $\mathcal{K}_2$ for $\mathcal{R}$ is exactly how we code a pair for $\mathbb{Q}$. But the first satisfies cone avoidance and the second codes K.

Any subcopy of the rationals is a copy of $\mathbb{Q}$ w.r.t. to original ordering. We needed this fact for selective largeness and hence to code K. With a Rado graph we can defined an ordering $<_{\mathcal{R}}$ which gives us copy of $\mathbb{Q}$. Any subcopy of a Rado graph contains a copy of $\mathbb{Q}$ w.r.t. $<_{\mathcal{R}}$, possibly properly. That provides enough room for cone avoidance for $\mathcal{A}$ of size 2. Using ACA we can find a subordering which is a copy of $\mathbb{Q}$ w.r.t. $<_{\mathcal{R}}$.

Thanks!

Paper: `https://arxiv.org/abs/2007.09739`
Slides: `https://www.nd.edu/~cholak/papers/msri2020.pdf`