Complexity of root-taking in power series fields & related problems

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Root-taking in Puiseux Series

Let K be an algebraically closed field of characteristic 0.

Definition

A *Puiseux series* over K has the form

$$s = \sum_{l \leq i \in \mathbb{Z}} a_i t^{\frac{i}{m}} \text{ for some } m \in \mathbb{N},\ l \in \mathbb{Z},\ a_i \in K.$$

The *support* of s is $Supp(s) = \{\frac{i}{m} \mid l \leq i \in \mathbb{Z} \ \&\ a_i \neq 0\}$.

Let $K\{\{t\}\}$ denote the field of Puiseux series over K.

Example $s = 3t^{-\frac{1}{2}} + \pi t^0 + 2t^{\frac{1}{2}} + -t^1 + \ldots$ with $Supp(s) = \{-\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots\}$.

Newton-Puiseux Theorem

If K is an algebraically closed field, then $K\{\{t\}\}$ is algebraically closed as well.

Generalizing Puiseux Series

Let K be an algebraically closed field of characteristic 0.
Let G be a divisible ordered abelian group.

Definition

A *Hahn series* over K and G has the form

$$s = \sum_{g \in S} a_g t^g \text{ for a well-ordered } S \subset G \text{ and } a_g \in K^{\neq 0}.$$

Let $K((G))$ be the field of Hahn series.

Example $s = \pi t^0 + t^3 + -t^{3.1} + t^{3.14} + t^{3.141} + \ldots + t^4$ with $Supp(s) = \{0, 3, 3.1, 3.14, 3.141, \ldots, 4\}$.

Theorem (Mac Lane '39)

If K is an algebraically closed field and G is a divisible ordered abelian group, then $K((G))$ is algebraically closed as well.

Complexity of the root-taking process

Let

$$p(x) = A_0 + A_1 x + \ldots + A_n x^n,$$

where the A_i are all in $K\{\{t\}\}$ or all in $K((G))$.

Goal

Describe the complexity of the roots of $p(x)$ in terms of the A_i's, K, and G.

Turns out to be related to the complexity of natural problems about well-ordered subsets of G.

Valuation on Puiseux series

Definition

A *Puiseux series* over K has the form

$$\sum_{l \leq i \in \mathbb{Z}} a_i t^{\frac{i}{m}} \text{ for some } m \in \mathbb{N},\ l \in \mathbb{Z},\ a_i \in K.$$

Example $s = 3t^{-\frac{1}{2}} + \pi t^0 + 2t^{\frac{1}{2}} + -t^1 + \ldots$ with

$$Supp(s) = \{-\tfrac{1}{2}, 0, \tfrac{1}{2}, 1, \ldots\}.$$

$K\{\{t\}\}$ has a natural valuation $w : K\{\{t\}\} \to \mathbb{Q} \bigcup \{\infty\}$ s.t.

$$w(s) := \begin{cases} \min(Supp(s)) & \text{if } s \neq 0 \\ \infty & \text{if } s = 0 \end{cases}$$

Think of t as infinitesimal, so t^q infinitesimal if $q > 0$ and t^q infinite if $q < 0$.

Newton-Pusieux Method in $K\{\{t\}\}$

Let $p(x) = A_0 + A_1 x + \ldots + A_n x^n$ be a nonconstant polynomial over $K\{\{t\}\}$.

- $A_0 = 0$ implies 0 is a root of $p(x)$

Suppose $A_0 \neq 0$.

Construct **Newton Polygon** to compute a root r of $p(x)$.

- Calculate leading term $r = bt^{\nu} + \ldots$ to make terms cancel.

Newton-Pusieux Method in $K\{\{t\}\}$

Let $p(x) = A_0 + A_1 x + \ldots + A_n x^n$ be a nonconstant polynomial over $K\{\{t\}\}$ with $A_0 \neq 0$.

Example

$$p(x) = \underbrace{-t^2}_{A_0} + \underbrace{(t + 2t^{3/2})}_{A_1} x + \underbrace{-(2t^{1/2} + t)}_{A_2} x^2 + \underbrace{1}_{A_3} x^3.$$

Roots are t and $t^{1/2}$ (with multiplicity 2).

Draw Newton Polygon

Let $p(x) = A_0 + A_1 x + \ldots + A_n x^n$ be a nonconstant, $A_0 \neq 0$.

Example

$$p(x) = \underbrace{-t^2}_{A_0} + \underbrace{(t + 2t^{3/2})}_{A_1} x + \underbrace{-(2t^{1/2} + t)}_{A_2} x^2 + \underbrace{1}_{A_3} x^3.$$

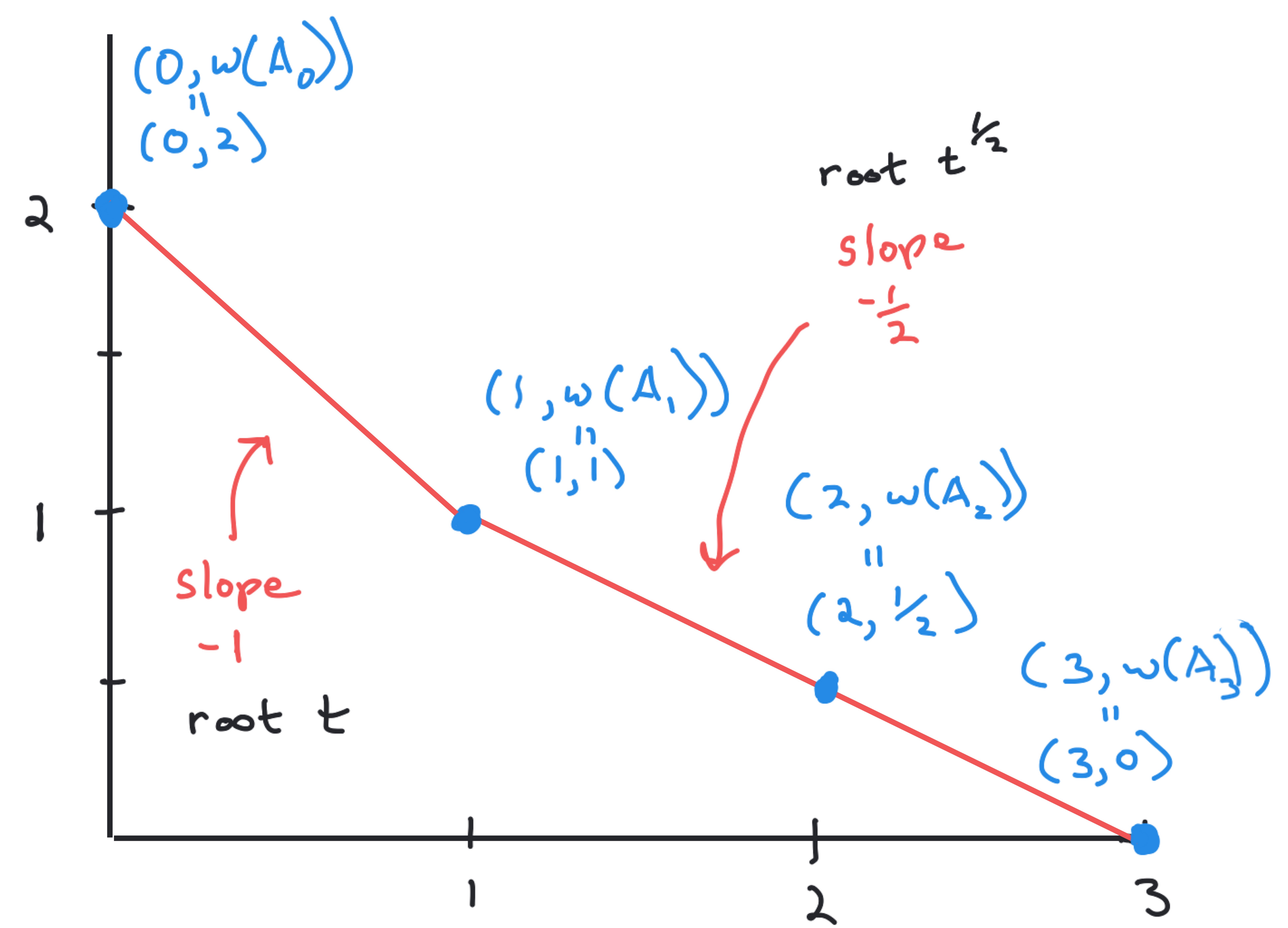
Roots are t and $t^{1/2}$ (with multiplicity 2).

Steps

1. Plot $(i, w(A_i))$ for $i = 0, \ldots, n$.
2. Draw convex Newton Polygon.

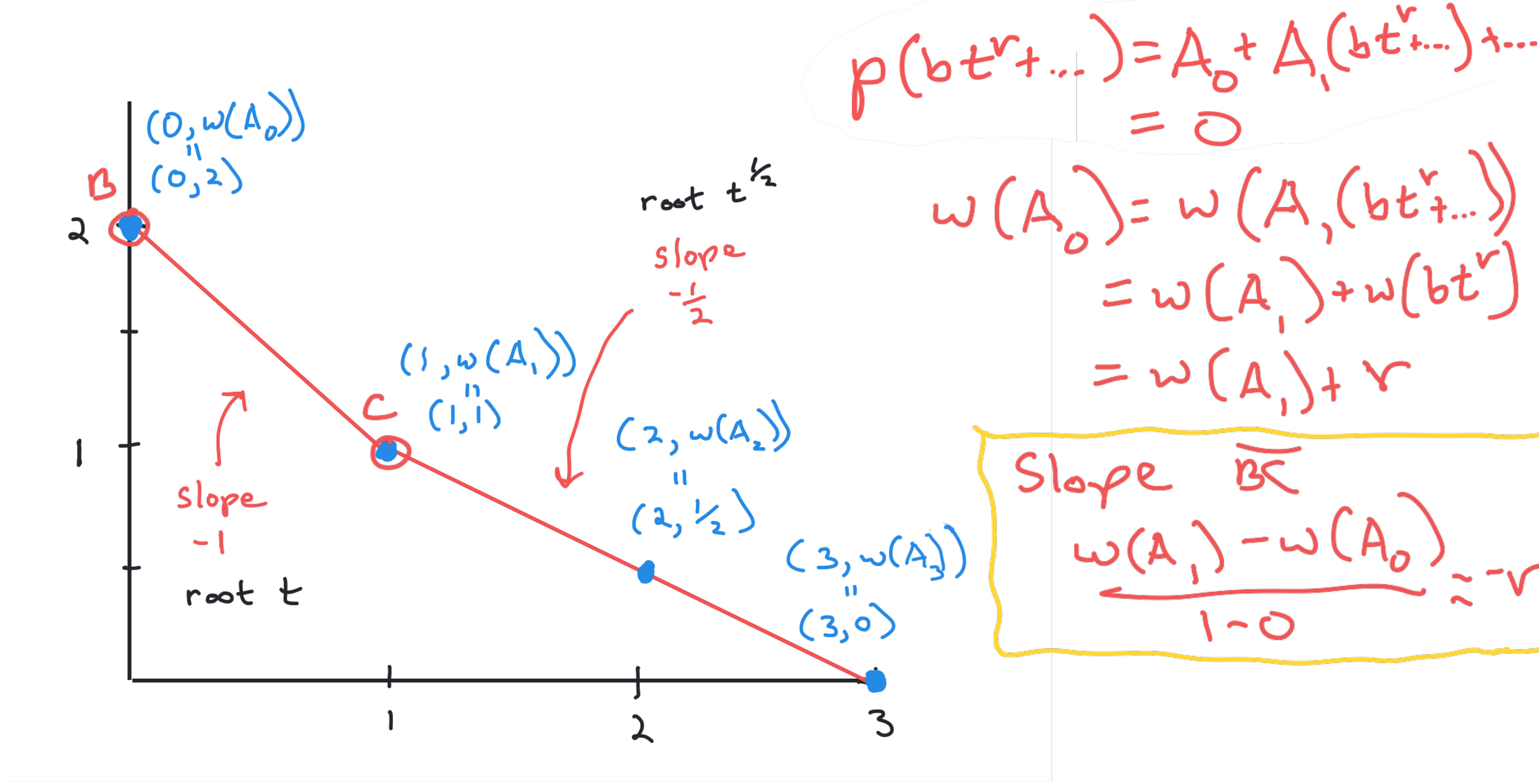
Newton Polygon Example

$$p(x) = \underbrace{-t^2}_{A_0} + \underbrace{(t + 2t^{3/2})}_{A_1} x + \underbrace{-(2t^{1/2} + t)}_{A_2} x^2 + \underbrace{1}_{A_3} x^3.$$



Facts about the Newton Polygon

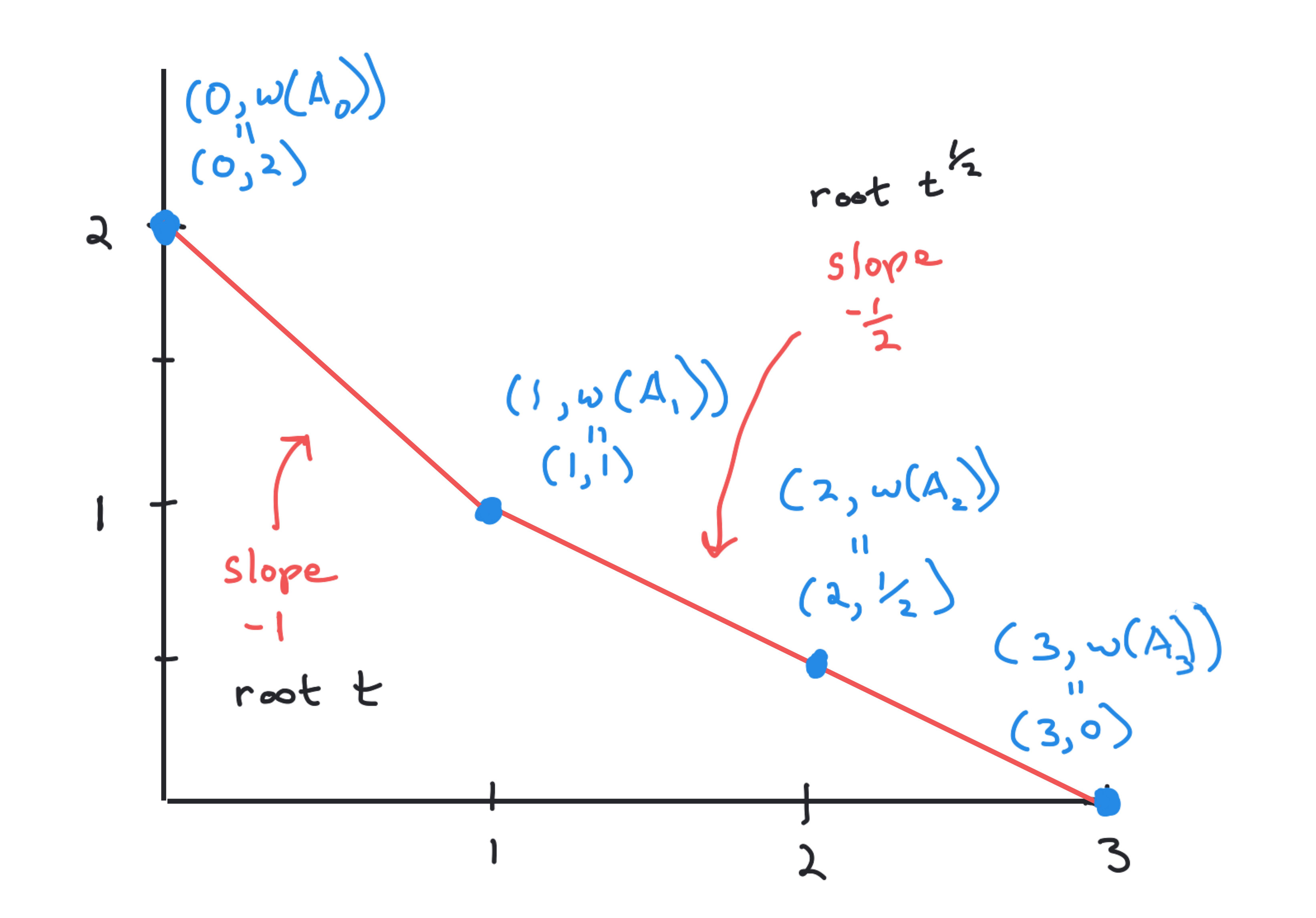
Example $p(x) = \underbrace{-t^2}_{A_0} + \underbrace{(t + 2t^{3/2})}_{A_1} x + \underbrace{-(2t^{1/2} + t)}_{A_2} x^2 + \underbrace{1}_{A_3} x^3.$



- The valuation ν of at least one root $r = bt^{\nu} + \cdots$ is the negative of the slope of a side.

Facts about the Newton Polygon

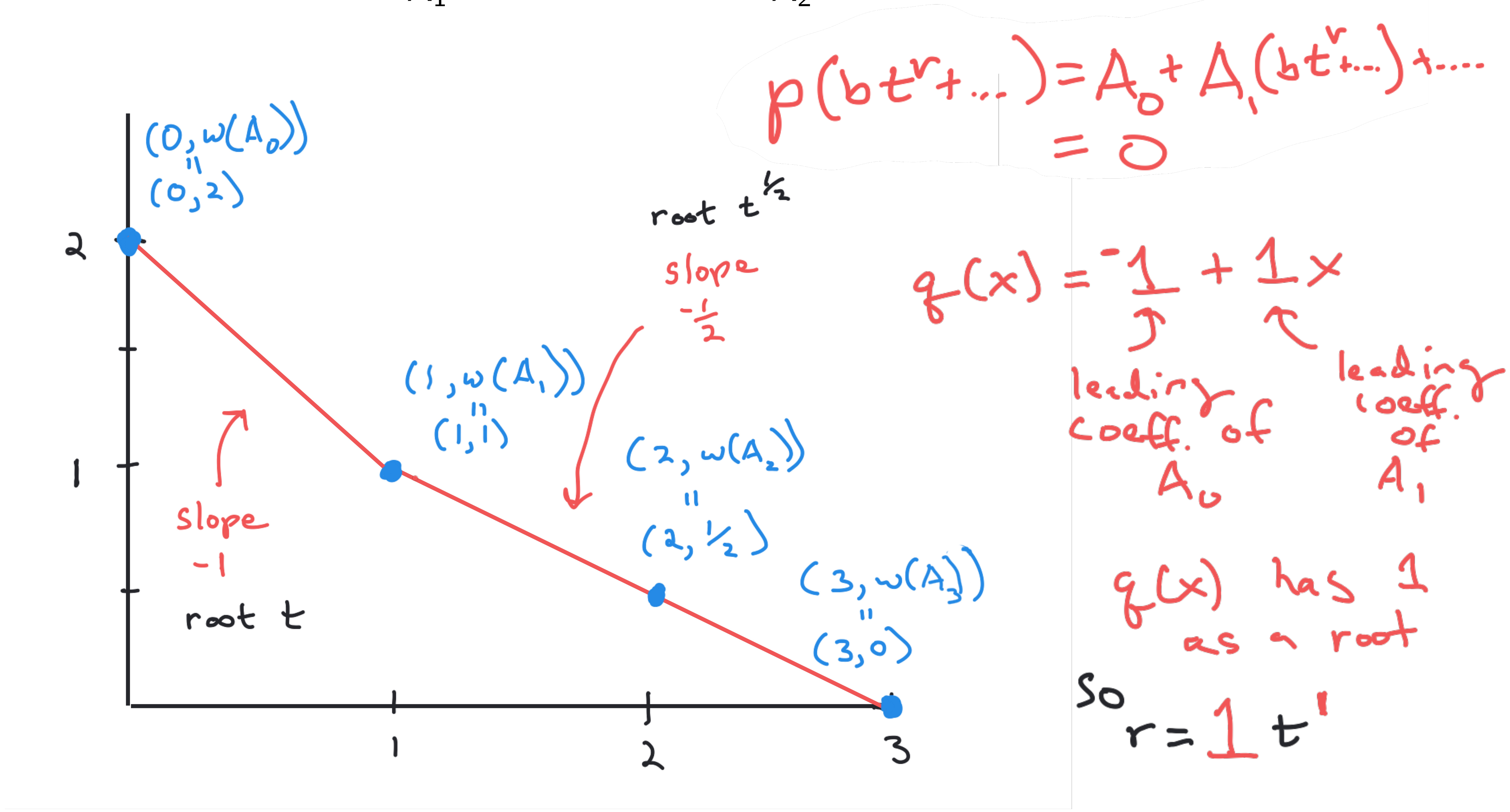
Example $p(x) = \underbrace{-t^2}_{A_0} + \underbrace{(t + 2t^{3/2})}_{A_1} x + \underbrace{-(2t^{1/2} + t)}_{A_2} x^2 + \underbrace{1}_{A_3} x^3.$



- Convexity means slopes increasing, so root of greatest valuation associated with leftmost side.

Facts about the Newton Polygon

Example $p(x) = \underbrace{-t^2}_{A_0} + \underbrace{(t + 2t^{3/2})}_{A_1} x + \underbrace{-(2t^{1/2} + t)}_{A_2} x^2 + \underbrace{1}_{A_3} x^3.$



- ▶ Calculate $b \in K$ by finding a root of poly. in $K[x]$ determined by leading coefficients of terms lying on corresponding side of Newton polygon.

Continuing to approximate r

Let $p(x) = A_0 + A_1 x + \ldots + A_n x^n$ be a nonconstant, $A_0 \neq 0$.

To find the next term in root $r = bt^{\nu} + \cdots$ having calculated $r_1 = bt^{\nu}$,

Consider $q(x) = p(r_1 + x) = B_0 + B_1 x + \cdots + B_n x^n$.

If $B_0 = 0$, then r_1 is a root.

If $B_0 \neq 0$, then repeat this process.

Representing Puiseux series

Suppose K has universe ω.
Fix a computable copy of $\mathbb{Q}$ with universe ω.

Consider the *Puiseux series*

$$s = \sum_{I \leq i \in \mathbb{Z}} a_i t^{\frac{i}{m}} \text{ for some } m \in \mathbb{N},\ I \in \mathbb{Z},\ a_i \in K.$$

Represent s by a function $f : \omega \to K \times \mathbb{Q}$ s.t.
if $f(n) = (a_n, q_n)$, then

$$s = \sum_{n \in \omega} a_n t^{q_n}.$$

and

- q_n increases with n, so
- there is a uniform bound on the denominators of the q_n terms, so $\lim_{n \to \infty} q_n = \infty$.

Complexity of basic operations in $K\{\{t\}\}$

Lemma

Let K and $s, s' \in K\{\{t\}\}$ be given.

1. *We can effectively compute $s + s'$ and $s \cdot s'$.*
2. *It is Π^0_1, but not computable, to say that $s = 0$.*
   - *Given that $s \neq 0$, we can effectively find $w(s)$.*
   - *Regardless of whether $s \neq 0$, we can effectively order $w(s)$ and any $q \in \mathbb{Q}$.*

Complexity of root-taking over $K\{\{t\}\}$

Theorem (Knight, L., Solomon)

There is a uniform effective procedure that, given K and the sequence of coefficients for a non-constant polynomial over $K\{\{t\}\}$, yields a root.

Corollary

Let $p(x) = A_0 + A_1x + \ldots + A_nx^n$ be a polynomial over $K\{\{t\}\}$. Then all roots of $p(x)$ are computable in K and the coefficients A_i.

Complexity of root-taking over $K\{\{t\}\}$: Key Issues

Theorem (Knight, L., Solomon)

There is a uniform effective procedure that, given K and the sequence of coefficients for a non-constant polynomial over $K\{\{t\}\}$, yields a root.

Cannot effectively

- determine if a coefficient $A_i = 0$.

  Hence, can't check if $A_0 = 0$, i.e., 0 is a root.
- determine the valuation $w(A_i)$.

  So cannot uniformly compute Newton Polygon
- tell if the root r is a finite sum.

  But must append terms to r while checking if done.

Definition: Hahn fields $K((G))$

1. Let $K((G))$ be the set of formal sums $s = \sum_{g \in S} a_g t^g$ where
   - $a_g \in K^{\neq 0}$ and
   - S is a well ordered subset of G.

   S is the *support* of s and is denoted $Supp(s)$.
   The *length* of s is the order type of S in G.

2. The *natural valuation* is the function $w : K((G)) \to G \cup \{\infty\}$ such that

$$w(s) = \begin{cases} \min Supp(s) & \text{if } s \neq 0 \\ \infty & \text{if } s = 0 \end{cases}$$

Example $s = \pi t^0 + t^3 + -t^{3.1} + t^{3.14} + t^{3.141} + \ldots + t^4$ with

$$Supp(s) = \{0, 3, 3.1, 3.14, 3.141, \ldots, 4\}.$$

$$length(s) = \omega + 1$$

Representing Hahn series: two approaches

Let $s = \sum_{g \in S} a_g t^g \in K((G))$.

Represent s in two ways as:

1. a function $f : \alpha \to K \times G$ for some ordinal α s.t.

   if $f(\gamma) = (a_\gamma, g_\gamma)$, then $s = \sum_{\gamma < \alpha} a_\gamma t^{g_\gamma}$ and

   $g_\beta < g_\gamma$ for all $\beta < \gamma < \alpha$.

2. a function $\sigma : G \to K$ s.t.

   $S = \{g \in G : \sigma(g) \neq 0\}$ is well ordered and

   $s = \sum_{g \in S} \sigma(g) t^g$.

Admissible Sets

Definition

An *admissible set* is a transitive set that satisfies essentially

- the axioms of ZF but with **no power set** axiom and
- the axioms of Comprehension and Replacement **restricted to** Δ^0_0-**formulas**, finite conjuncts and disjuncts of atomic formulas and their negations.

Example: $L_{\omega_1^{CK}}$, the least admissible set containing ω.

The subsets of ω in $L_{\omega_1^{CK}}$ are exactly the Δ^1_1 sets, i.e., the hyperarithmetical sets.

Advantage of Admissible Sets containing ω

Theorem

Let A be an admissible set containing the field K and group G. Then the generalized Newton-Puiseux Theorem holds in A, i.e.,

any polynomial $p(x)$ over $K((G))$ with coefficients in A has a root r in A.

Can define functions F by induction on the ordinals,

as long as have a Σ_1 formula describing
how to obtain $F(\alpha)$ from $F|\alpha$.

Lengths of roots & other tools

Theorem (Knight & L.)

Let $p(x) = A_0 + \ldots + A_n x^n$ be a polynomial over $K((G))$.

If γ is a a limit ordinal greater than the lengths of all A_i,

then any root of $p(x)$ has length less than ω^{ω^γ}.

Lemma

Let A be an admissible set containing the field K and group G.

- *The function $\alpha \to \omega^\alpha$ is Σ_1-definable on A.*
- *If s, s' are elements of $K((G))$ in A, then*

  $s + s'$, $s \cdot s'$, $Supp(s)$ and the length of s are all in A.

Root-taking in Hahn Fields

Theorem

Let A be an admissible set containing the field K and group G. Then the generalized Newton-Puiseux Theorem holds in A, i.e.,

any polynomial $p(x)$ over $K((G))$ with coefficients in A has a root r in A.

Initial segments of roots

New Procedure

Let $p(x) = A_0 + A_1 x + \ldots + A_n x^n$ be a polynomial over $K((G))$.

At step α determine an initial segment r_α of a root of $p(x)$, s.t.

$$r_0 = 0 \text{ and for } \alpha > 0,$$

$$\text{either } r_\alpha \text{ has length } \alpha \text{ and extends } r_\beta \text{ for all } \beta < \alpha$$

$$\text{or there is some } \beta < \alpha \text{ s.t. } r_\beta \text{ is already root and } r_\alpha = r_\beta.$$

View r_α as a function $r_\alpha : G \to K$ with well ordered support.

New Goal

Bound complexity of carrying out this procedure to step α when given K, G, and $p(x)$.

Complexity of root-taking procedure in $K((G))$

Proposition

The procedure to carry out step α is $\Delta^0_{f(\alpha)}$ in K, G, and p, where f is defined as:

1. $f(\alpha) = sup_{\beta<\alpha} f(\beta) + 1.$
2. *for* $n \geq 1$, $f(\alpha + n) = f(\alpha) + 1.$

For finite $n \geq 1$, the results below, apart from the last, are sharp.

Step n is Δ^0_2.

Step ω is Δ^0_3.

Step $\omega + n$ is Δ^0_4.

Step $\omega + \omega$ is Δ^0_5, but unknown if sharp.

Complexity of root-taking procedure in $K((G))$

Determining $r_{\omega+\omega}$ as a function is Δ^0_5, but unknown.

But Complexity continues to go up with length.

Proposition

For each computable ordinal α, Step ω^α is $\Pi^0_{2\alpha}$-hard.

Proof: Step ω^α is $\Pi^0_{2\alpha}$-hard

Let S be a $\Pi^0_{2\alpha}$ set.

Key ingredient
There is a uniformly computable sequence of orderings $\mathcal{C}_n$ s.t.
$\mathcal{C}_n \subset \mathbb{Q} \cap (0,1)$ has o.t. ω^α if $n \in S$ and some $\gamma < \omega^\alpha$ otherwise.

Let $B_n = \sum_{q \in \mathcal{C}_n} t^q$.

Consider the polynomial $p_n(x) = B_n - x$, with unique root $r = B_n$.

If $n \in S$, then $r = r_{\omega^\alpha}$.

If $n \notin S$, then $r = r_\gamma$ for some $\gamma < \omega^\alpha$.

So, S is reducible to Step ω^α applied to $(p_n(x))_{n \in \omega}$. $\square$

Bounds on Root-taking procedure in $K((G))$ sharp?

Proposition

The procedure to carry out step α is $\Delta^0_{f(\alpha)}$ in K, G, and p, where f was defined as before.

For finite $n \geq 1$, the results below, apart from the last, are sharp.

Step n is Δ^0_2.

Step ω is Δ^0_3.

Step $\omega + n$ is Δ^0_4.

Step $\omega + \omega$ is Δ^0_5, but unknown if sharp.

But seemingly not using full power of multiplication.

Pivot to simpler setting

Goal

Get better bounds on the root-taking process for $K((G))$.

Let $s \in K((G))$.

- $\mathit{support}(s^2)$ is a well ordered subset of sums of pairs of elements in $\mathit{support}(s) \subset G$.
- Natural to consider complexity of problems associated with well-ordered subsets of G.

Problems associated with well-ordered subsets A, B of G

How hard is it to:

1. Check that A has order type at least α?

   Find the α^{th} element of A?

2. Let $A + B := \{a + b : a \in A \ \& \ b \in B\}$.

   Check $A + B$ has order type at least α?

   Compute initial segments of $A + B$?

3. If $A \subseteq G^{\geq 0}$, the set $[A]$ of finite sums of elements of A is well-ordered.

   Check $[A]$ has order type at least α?

   Compute initial segments of $[A]$?

Takeaways

1. Newton's Method over $K\{\{t\}\}$ is uniformly computable in K and a nonconstant polynomial.

2. Newton's Method over $K((G))$ can be carried out in any admissible set containing the field K and group G.

3. Latter problem naturally involves complexity of problems involving well ordered subsets of G.

Thanks!

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