Deeply ramified fields and their relatives

Franz-Viktor Kuhlmann

joint work with Anna Rzepka (formerly Blaszczok)

Valuation Theory Seminar
MSRI, September 30, 2020

Two deep open problems

Longstanding open problems in positive characteristic

Two deep open problems

Longstanding open problems in positive characteristic
(their characteristic 0 counterparts were solve in the mid 1960's

Two deep open problems

Longstanding open problems in positive characteristic
(their characteristic 0 counterparts were solve in the mid 1960's
by Hironaka and

Two deep open problems

Longstanding open problems in positive characteristic
(their characteristic 0 counterparts were solve in the mid 1960's
by Hironaka and Ax & Kochen):

Two deep open problems

Longstanding open problems in positive characteristic
(their characteristic 0 counterparts were solve in the mid 1960′s
by Hironaka and Ax & Kochen):

• resolution of singularities

Two deep open problems

Longstanding open problems in positive characteristic
(their characteristic 0 counterparts were solve in the mid 1960's by Hironaka and Ax & Kochen):

• resolution of singularities and its local form,

Two deep open problems

Longstanding open problems in positive characteristic (their characteristic 0 counterparts were solve in the mid 1960's by Hironaka and Ax & Kochen):

• resolution of singularities and its local form, local uniformization,

Two deep open problems

Longstanding open problems in positive characteristic (their characteristic 0 counterparts were solve in the mid 1960's by Hironaka and Ax & Kochen):

• resolution of singularities and its local form, local uniformization,

• decidability of Laurent Series Fields over finite fields.

Two deep open problems

Longstanding open problems in positive characteristic (their characteristic 0 counterparts were solve in the mid 1960's by Hironaka and Ax & Kochen):

• resolution of singularities and its local form, local uniformization,

• decidability of Laurent Series Fields over finite fields.

Closest approximations to the first problem to date:

Two deep open problems

Longstanding open problems in positive characteristic (their characteristic 0 counterparts were solve in the mid 1960's by Hironaka and Ax & Kochen):

• resolution of singularities and its local form, local uniformization,

• decidability of Laurent Series Fields over finite fields.

Closest approximations to the first problem to date: Abhyankar,

Two deep open problems

Longstanding open problems in positive characteristic (their characteristic 0 counterparts were solve in the mid 1960's by Hironaka and Ax & Kochen):

• resolution of singularities and its local form, local uniformization,

• decidability of Laurent Series Fields over finite fields.

Closest approximations to the first problem to date: Abhyankar, de Jong,

Two deep open problems

Longstanding open problems in positive characteristic (their characteristic 0 counterparts were solve in the mid 1960's by Hironaka and Ax & Kochen):

• resolution of singularities and its local form, local uniformization,

• decidability of Laurent Series Fields over finite fields.

Closest approximations to the first problem to date: Abhyankar, de Jong, Knaf & K,

Two deep open problems

Longstanding open problems in positive characteristic (their characteristic 0 counterparts were solve in the mid 1960's by Hironaka and Ax & Kochen):

• resolution of singularities and its local form, local uniformization,

• decidability of Laurent Series Fields over finite fields.

Closest approximations to the first problem to date: Abhyankar, de Jong, Knaf & K, Temkin,

Two deep open problems

Longstanding open problems in positive characteristic (their characteristic 0 counterparts were solve in the mid 1960's by Hironaka and Ax & Kochen):

• resolution of singularities and its local form, local uniformization,

• decidability of Laurent Series Fields over finite fields.

Closest approximations to the first problem to date: Abhyankar, de Jong, Knaf & K, Temkin, Cossart & Piltant.

Two deep open problems

Longstanding open problems in positive characteristic (their characteristic 0 counterparts were solve in the mid 1960's by Hironaka and Ax & Kochen):

• resolution of singularities and its local form, local uniformization,

• decidability of Laurent Series Fields over finite fields.

Closest approximations to the first problem to date: Abhyankar, de Jong, Knaf & K, Temkin, Cossart & Piltant.

Closest approximations to the second problem to date:

Two deep open problems

Longstanding open problems in positive characteristic (their characteristic 0 counterparts were solve in the mid 1960's by Hironaka and Ax & Kochen):

• resolution of singularities and its local form, local uniformization,

• decidability of Laurent Series Fields over finite fields.

Closest approximations to the first problem to date: Abhyankar, de Jong, Knaf & K, Temkin, Cossart & Piltant.

Closest approximations to the second problem to date: model theory of tame valued fields (K),

Two deep open problems

Longstanding open problems in positive characteristic (their characteristic 0 counterparts were solve in the mid 1960's by Hironaka and Ax & Kochen):

• resolution of singularities and its local form, local uniformization,

• decidability of Laurent Series Fields over finite fields.

Closest approximations to the first problem to date: Abhyankar, de Jong, Knaf & K, Temkin, Cossart & Piltant.

Closest approximations to the second problem to date: model theory of tame valued fields (K), and of separably tame fields (K & Pal).

Valued fields, equal and mixed characteristic

Given a valued field (K, v), we denote by vK its value group,

Valued fields, equal and mixed characteristic

Given a valued field (K, v), we denote by vK its value group, by Kv its residue field,

Valued fields, equal and mixed characteristic

Given a valued field (K, v), we denote by vK its value group, by Kv its residue field, and by $\mathcal{O}_K$ its valuation ring with maximal ideal $\mathcal{M}_K$.

Valued fields, equal and mixed characteristic

Given a valued field (K, v), we denote by vK its value group, by Kv its residue field, and by $\mathcal{O}_K$ its valuation ring with maximal ideal $\mathcal{M}_K$.

We are interested in valued fields (K, v) in the following two cases:

Valued fields, equal and mixed characteristic

Given a valued field (K, v), we denote by vK its value group, by Kv its residue field, and by $\mathcal{O}_K$ its valuation ring with maximal ideal $\mathcal{M}_K$.

We are interested in valued fields (K, v) in the following two cases:

positive equal characteristic: $\mathrm{char}\, K = \mathrm{char}\, Kv = p > 0$,

Valued fields, equal and mixed characteristic

Given a valued field (K, v), we denote by vK its value group, by Kv its residue field, and by $\mathcal{O}_K$ its valuation ring with maximal ideal $\mathcal{M}_K$.

We are interested in valued fields (K, v) in the following two cases:

positive equal characteristic: $\mathrm{char}\, K = \mathrm{char}\, Kv = p > 0$,

mixed characteristic: $\mathrm{char}\, K = 0$, $\mathrm{char}\, Kv = p > 0$.

Valued fields, equal and mixed characteristic

Given a valued field (K, v), we denote by vK its value group, by Kv its residue field, and by $\mathcal{O}_K$ its valuation ring with maximal ideal $\mathcal{M}_K$.

We are interested in valued fields (K, v) in the following two cases:

positive equal characteristic: $\mathrm{char}\, K = \mathrm{char}\, Kv = p > 0$,

mixed characteristic: $\mathrm{char}\, K = 0$, $\mathrm{char}\, Kv = p > 0$.

In the following, p will always be the characteristic of the residue field.

Our enemy: the defect

By $(L|K, v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v.

Our enemy: the defect

By $(L|K, v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v.

If $(L|K, v)$ is a finite extension of valued fields and the valuation v of K admits a unique extension to the field L,

Our enemy: the defect

By $(L|K, v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v.

If $(L|K, v)$ is a finite extension of valued fields and the valuation v of K admits a unique extension to the field L, then by the Lemma of Ostrowski,

Our enemy: the defect

By $(L|K,v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v.

If $(L|K,v)$ is a finite extension of valued fields and the valuation v of K admits a unique extension to the field L, then by the Lemma of Ostrowski,

$$[L:K] \;=\; p^{\nu}\cdot(vL:vK)[Lv:Kv]\,,$$

Our enemy: the defect

By $(L|K,v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v.

If $(L|K,v)$ is a finite extension of valued fields and the valuation v of K admits a unique extension to the field L, then by the Lemma of Ostrowski,

$$[L:K] \;=\; p^{\nu} \cdot (vL:vK)[Lv:Kv]\,,$$

where $\nu \geq 0$ is an integer.

Our enemy: the defect

By $(L|K,v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v.

If $(L|K,v)$ is a finite extension of valued fields and the valuation v of K admits a unique extension to the field L, then by the Lemma of Ostrowski,

$$[L:K] \;=\; p^{\nu}\cdot(vL:vK)[Lv:Kv]\,,$$

where $\nu\geq 0$ is an integer. (If the characteristic of the residue fields is 0, the formula remains true if we set $p=1$.)

Our enemy: the defect

By $(L|K,v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v.

If $(L|K,v)$ is a finite extension of valued fields and the valuation v of K admits a unique extension to the field L, then by the Lemma of Ostrowski,

$$[L:K] \;=\; p^{\nu}\cdot(vL:vK)[Lv:Kv]\,,$$

where $\nu \geq 0$ is an integer. (If the characteristic of the residue fields is 0, the formula remains true if we set $p=1$.)
The factor $d(L|K,v)=p^{\nu}$ is called the defect of the extension $(L|K,v)$.

Our enemy: the defect

By $(L|K,v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v.

If $(L|K,v)$ is a finite extension of valued fields and the valuation v of K admits a unique extension to the field L, then by the Lemma of Ostrowski,

$$[L:K] \;=\; p^{\nu}\cdot(vL:vK)[Lv:Kv]\,,$$

where $\nu \geq 0$ is an integer. (If the characteristic of the residue fields is 0, the formula remains true if we set $p=1$.)
The factor $d(L|K,v)=p^{\nu}$ is called the defect of the extension $(L|K,v)$. If $p^{\nu}>1$, then $(L|K,v)$ is called a defect extension.

Our enemy: the defect

By $(L|K,v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v.

If $(L|K,v)$ is a finite extension of valued fields and the valuation v of K admits a unique extension to the field L, then by the Lemma of Ostrowski,

$$[L:K] \;=\; p^{\nu}\cdot(vL:vK)[Lv:Kv]\,,$$

where $\nu \geq 0$ is an integer. (If the characteristic of the residue fields is 0, the formula remains true if we set $p=1$.)
The factor $d(L|K,v)=p^{\nu}$ is called the defect of the extension $(L|K,v)$. If $p^{\nu}>1$, then $(L|K,v)$ is called a defect extension. If $p^{\nu}=1$, then we call $(L|K,v)$ a defectless extension.

Tame extensions

An algebraic extension $(L|K,v)$ of henselian fields is called tame

Tame extensions

An algebraic extension $(L|K, v)$ of henselian fields is called tame if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

Tame extensions

An algebraic extension $(L|K, v)$ of henselian fields is called tame if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

(T1) the ramification index $(vE : vK)$ is not divisible by char Kv,

Tame extensions

An algebraic extension $(L|K, v)$ of henselian fields is called tame if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

(T1) the ramification index $(vE : vK)$ is not divisible by char Kv,

(T2) the residue field extension $Ev|Kv$ is separable,

Tame extensions

An algebraic extension $(L|K,v)$ of henselian fields is called tame if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

(T1) the ramification index $(vE : vK)$ is not divisible by char Kv,

(T2) the residue field extension $Ev|Kv$ is separable,

(T3) $(E|K,v)$ is defectless.

Tame extensions

An algebraic extension $(L|K,v)$ of henselian fields is called tame if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

(T1) the ramification index $(vE : vK)$ is not divisible by char Kv,

(T2) the residue field extension $Ev|Kv$ is separable,

(T3) $(E|K,v)$ is defectless.

A henselian valued field (K, v) is called a tame field

Tame and separably tame fields

A henselian valued field (K, v) is called a tame field if every algebraic extension is tame,

Tame and separably tame fields

A henselian valued field (K, v) is called a tame field if every algebraic extension is tame, and a separably tame field if this holds for all separable-algebraic extensions.

Tame and separably tame fields

A henselian valued field (K, v) is called a tame field if every algebraic extension is tame, and a separably tame field if this holds for all separable-algebraic extensions.

All tame fields are perfect,

Tame and separably tame fields

A henselian valued field (K, v) is called a tame field if every algebraic extension is tame, and a separably tame field if this holds for all separable-algebraic extensions.

All tame fields are perfect, with perfect residue field,

Tame and separably tame fields

A henselian valued field (K, v) is called a tame field if every algebraic extension is tame, and a separably tame field if this holds for all separable-algebraic extensions.

All tame fields are perfect, with perfect residue field, and their value group is divisible by the characteristic of the residue field (if it is > 0).

Tame and separably tame fields

A henselian valued field (K, v) is called a tame field if every algebraic extension is tame, and a separably tame field if this holds for all separable-algebraic extensions.

All tame fields are perfect, with perfect residue field, and their value group is divisible by the characteristic of the residue field (if it is > 0). It follows from condition (T3) that tame fields do not admit *any* defect extensions.

Tame and separably tame fields

A henselian valued field (K, v) is called a tame field if every algebraic extension is tame, and a separably tame field if this holds for all separable-algebraic extensions.

All tame fields are perfect, with perfect residue field, and their value group is divisible by the characteristic of the residue field (if it is > 0). It follows from condition (T3) that tame fields do not admit *any* defect extensions.

Remark: the absolute ramification field of a henselian field is its largest tame extension;

A henselian valued field (K, v) is called a tame field if every algebraic extension is tame, and a separably tame field if this holds for all separable-algebraic extensions.

All tame fields are perfect, with perfect residue field, and their value group is divisible by the characteristic of the residue field (if it is > 0). It follows from condition (T3) that tame fields do not admit *any* defect extensions.

Remark: the absolute ramification field of a henselian field is its largest tame extension; hence the field is a tame field if and only if its absolute ramification field is algebraically closed.

Properties of tame fields

Tame fields have many good properties. For example:

Properties of tame fields

Tame fields have many good properties. For example:

Theorem (K)

Tame fields (K, v) satisfy model completeness and decidability relative to the elementary theories of their value groups vK and their residue fields Kv.

Properties of tame fields

Tame fields have many good properties. For example:

Theorem (K)

Tame fields (K, v) satisfy model completeness and decidability relative to the elementary theories of their value groups vK and their residue fields Kv. In the equal characteristic case, also relative completeness holds.

Properties of tame fields

Tame fields have many good properties. For example:

Theorem (K)

Tame fields (K, v) satisfy model completeness and decidability relative to the elementary theories of their value groups vK and their residue fields Kv. In the equal characteristic case, also relative completeness holds.

(But QE is an unsolved problem for tame fields.)

Properties of tame fields

Tame fields have many good properties. For example:

Theorem (K)

Tame fields (K, v) satisfy model completeness and decidability relative to the elementary theories of their value groups vK and their residue fields Kv. In the equal characteristic case, also relative completeness holds.

(But QE is an unsolved problem for tame fields.)

Similar results hold for separably tame fields

Properties of tame fields

Tame fields have many good properties. For example:

Theorem (K)

Tame fields (K, v) satisfy model completeness and decidability relative to the elementary theories of their value groups vK and their residue fields Kv. In the equal characteristic case, also relative completeness holds.

(But QE is an unsolved problem for tame fields.)

Similar results hold for separably tame fields (which are very close to tame fields).

Properties of tame fields

Tame fields have many good properties. For example:

Theorem (K)

Tame fields (K, v) satisfy model completeness and decidability relative to the elementary theories of their value groups vK and their residue fields Kv. In the equal characteristic case, also relative completeness holds.

(But QE is an unsolved problem for tame fields.)

Similar results hold for separably tame fields (which are very close to tame fields).

Valued function fields over tame fields have a relatively good structure theory.

Properties of tame fields

Tame fields have many good properties. For example:

Theorem (K)

Tame fields (K, v) satisfy model completeness and decidability relative to the elementary theories of their value groups vK and their residue fields Kv. In the equal characteristic case, also relative completeness holds.

(But QE is an unsolved problem for tame fields.)

Similar results hold for separably tame fields (which are very close to tame fields).

Valued function fields over tame fields have a relatively good structure theory. This is used to prove the above theorem,

Properties of tame fields

Tame fields have many good properties. For example:

Theorem (K)

Tame fields (K, v) satisfy model completeness and decidability relative to the elementary theories of their value groups vK and their residue fields Kv. In the equal characteristic case, also relative completeness holds.

(But QE is an unsolved problem for tame fields.)

Similar results hold for separably tame fields (which are very close to tame fields).

Valued function fields over tame fields have a relatively good structure theory. This is used to prove the above theorem, and it also has been applied to the problem of local uniformization (Knaf & K).

Beyond tame fields

Can the condition of tameness be relaxed

Beyond tame fields

Can the condition of tameness be relaxed while preserving some parts of the structure theory

Beyond tame fields

Can the condition of tameness be relaxed while preserving some parts of the structure theory that still could lead to good model theoretic results?

Beyond tame fields

Can the condition of tameness be relaxed while preserving some parts of the structure theory that still could lead to good model theoretic results? Could we allow some defect?

Beyond tame fields

Can the condition of tameness be relaxed while preserving some parts of the structure theory that still could lead to good model theoretic results? Could we allow some defect? At least some harmless type of defect?

Beyond tame fields

Can the condition of tameness be relaxed while preserving some parts of the structure theory that still could lead to good model theoretic results? Could we allow some defect? At least some harmless type of defect?

What will follow now contains no new model theoretic results

Beyond tame fields

Can the condition of tameness be relaxed while preserving some parts of the structure theory that still could lead to good model theoretic results? Could we allow some defect? At least some harmless type of defect?

What will follow now contains no new model theoretic results (so far).

Beyond tame fields

Can the condition of tameness be relaxed while preserving some parts of the structure theory that still could lead to good model theoretic results? Could we allow some defect? At least some harmless type of defect?

What will follow now contains no new model theoretic results (so far). The aim is to build a structure theory that then,

Beyond tame fields

Can the condition of tameness be relaxed while preserving some parts of the structure theory that still could lead to good model theoretic results? Could we allow some defect? At least some harmless type of defect?

What will follow now contains no new model theoretic results (so far). The aim is to build a structure theory that then, hopefully,

Beyond tame fields

Can the condition of tameness be relaxed while preserving some parts of the structure theory that still could lead to good model theoretic results? Could we allow some defect? At least some harmless type of defect?

What will follow now contains no new model theoretic results (so far). The aim is to build a structure theory that then, hopefully, can be used by model theorists to push beyond tame fields.

A classification of the defect in equal characteristic

Originally a classification of defect extensions was only introduced in the positive equal characteristic case.

A classification of the defect in equal characteristic

Originally a classification of defect extensions was only introduced in the positive equal characteristic case. There, a Galois extension of prime degree was said to have dependent defect

A classification of the defect in equal characteristic

Originally a classification of defect extensions was only introduced in the positive equal characteristic case. There, a Galois extension of prime degree was said to have dependent defect if in a certain way it is dependent on a purely inseparable defect extension of prime degree.

A classification of the defect in equal characteristic

Originally a classification of defect extensions was only introduced in the positive equal characteristic case. There, a Galois extension of prime degree was said to have dependent defect if in a certain way it is dependent on a purely inseparable defect extension of prime degree. Otherwise, we spoke of independent defect.

A classification of the defect, generalized

Recently, we have been able to generalize this definition to the mixed characteristic case

A classification of the defect, generalized

Recently, we have been able to generalize this definition to the mixed characteristic case by a unified definition that works simultaneously for both cases.

A classification of the defect, generalized

Recently, we have been able to generalize this definition to the mixed characteristic case by a unified definition that works simultaneously for both cases. Take a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p.

A classification of the defect, generalized

Recently, we have been able to generalize this definition to the mixed characteristic case by a unified definition that works simultaneously for both cases. Take a Galois defect extension $\mathcal{E} = (L|K,v)$ of prime degree p. Then the set

$$\Sigma_\sigma := \left\{ v\left(\frac{\sigma f - f}{f}\right) \middle| \, f \in L^\times \right\}$$

A classification of the defect, generalized

Recently, we have been able to generalize this definition to the mixed characteristic case by a unified definition that works simultaneously for both cases. Take a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p. Then the set

$$\Sigma_\sigma := \left\{ v\left(\frac{\sigma f - f}{f}\right) \,\middle|\, f \in L^\times \right\}$$

is independent of the choice of a generator σ of Gal $(L|K)$,

A classification of the defect, generalized

Recently, we have been able to generalize this definition to the mixed characteristic case by a unified definition that works simultaneously for both cases. Take a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p. Then the set

$$\Sigma_\sigma \ := \ \left\{ v\left(\frac{\sigma f - f}{f}\right) \,\middle|\, f \in L^\times \right\}$$

is independent of the choice of a generator σ of $\mathrm{Gal}\,(L|K)$, and we denote it by $\Sigma_{\mathcal{E}}$.

A classification of the defect, generalized

Recently, we have been able to generalize this definition to the mixed characteristic case by a unified definition that works simultaneously for both cases. Take a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p. Then the set

$$\Sigma_\sigma \;:=\; \left\{ v\left(\frac{\sigma f - f}{f}\right) \,\middle|\, f \in L^\times \right\}$$

is independent of the choice of a generator σ of $\mathrm{Gal}\,(L|K)$, and we denote it by $\Sigma_{\mathcal{E}}$. We say that $\mathcal{E}$ has independent defect if

A classification of the defect, generalized

Recently, we have been able to generalize this definition to the mixed characteristic case by a unified definition that works simultaneously for both cases. Take a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p. Then the set

$$\Sigma_\sigma := \left\{ v\left(\frac{\sigma f - f}{f}\right) \middle| f \in L^\times \right\}$$

is independent of the choice of a generator σ of $\mathrm{Gal}\,(L|K)$, and we denote it by $\Sigma_\mathcal{E}$. We say that $\mathcal{E}$ has independent defect if

$$\Sigma_\mathcal{E} = \{\alpha \in vK \mid \alpha > H_\mathcal{E}\}$$

A classification of the defect, generalized

Recently, we have been able to generalize this definition to the mixed characteristic case by a unified definition that works simultaneously for both cases. Take a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p. Then the set

$$\Sigma_\sigma := \left\{ v\left(\frac{\sigma f - f}{f}\right) \middle| f \in L^\times \right\}$$

is independent of the choice of a generator σ of $\mathrm{Gal}\,(L|K)$, and we denote it by $\Sigma_\mathcal{E}$. We say that $\mathcal{E}$ has independent defect if

$$\Sigma_\mathcal{E} = \{\alpha \in vK \mid \alpha > H_\mathcal{E}\}$$

for some proper convex subgroup $H_\mathcal{E}$ of vK;

A classification of the defect, generalized

Recently, we have been able to generalize this definition to the mixed characteristic case by a unified definition that works simultaneously for both cases. Take a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p. Then the set

$$\Sigma_\sigma := \left\{ v\left(\frac{\sigma f - f}{f}\right) \middle| \, f \in L^\times \right\}$$

is independent of the choice of a generator σ of $\mathrm{Gal}\,(L|K)$, and we denote it by $\Sigma_\mathcal{E}$. We say that $\mathcal{E}$ has independent defect if

$$\Sigma_\mathcal{E} = \{\alpha \in vK \mid \alpha > H_\mathcal{E}\}$$

for some proper convex subgroup $H_\mathcal{E}$ of vK; otherwise we say that $\mathcal{E}$ has dependent defect.

A classification of the defect, generalized

Recently, we have been able to generalize this definition to the mixed characteristic case by a unified definition that works simultaneously for both cases. Take a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p. Then the set

$$\Sigma_\sigma := \left\{ v\left(\frac{\sigma f - f}{f}\right) \,\middle|\, f \in L^\times \right\}$$

is independent of the choice of a generator σ of $\mathrm{Gal}\,(L|K)$, and we denote it by $\Sigma_{\mathcal{E}}$. We say that $\mathcal{E}$ has independent defect if

$$\Sigma_{\mathcal{E}} = \{\alpha \in vK \mid \alpha > H_{\mathcal{E}}\}$$

for some proper convex subgroup $H_{\mathcal{E}}$ of vK; otherwise we say that $\mathcal{E}$ has dependent defect. (Note: rank 1 implies $H_{\mathcal{E}} = \{0\}$.)

The importance of this classification

There are several results

The importance of this classification

There are several results (Temkin,

The importance of this classification

There are several results (Temkin, Cutkosky & Piltant in conjunction with ElHitti & Ghezzi)

The importance of this classification

There are several results (Temkin, Cutkosky & Piltant in conjunction with ElHitti & Ghezzi) which indicate that the dependent defect is more harmful than the independent defect

The importance of this classification

There are several results (Temkin, Cutkosky & Piltant in conjunction with ElHitti & Ghezzi) which indicate that the dependent defect is more harmful than the independent defect for the solution of the above mentioned open problems.

Perfectoid fields and their shortcomings

In the equal characteristic case, a perfect valued field

Perfectoid fields and their shortcomings

In the equal characteristic case, a perfect valued field (such as $\mathbb{F}_p((t))^{1/p^\infty}$)

Perfectoid fields and their shortcomings

In the equal characteristic case, a perfect valued field (such as $\mathbb{F}_p((t))^{1/p^\infty}$) has no dependent defect extensions.

Perfectoid fields and their shortcomings

In the equal characteristic case, a perfect valued field (such as $\mathbb{F}_p((t))^{1/p^\infty}$) has no dependent defect extensions. The same holds if we pass to the completion of our example,

Perfectoid fields and their shortcomings

In the equal characteristic case, a perfect valued field (such as $\mathbb{F}_p((t))^{1/p^\infty}$) has no dependent defect extensions. The same holds if we pass to the completion of our example, which then is one of those famous perfectoid fields.

Perfectoid fields and their shortcomings

In the equal characteristic case, a perfect valued field (such as $\mathbb{F}_p((t))^{1/p^\infty}$) has no dependent defect extensions. The same holds if we pass to the completion of our example, which then is one of those famous perfectoid fields. What about perfectoid fields in mixed characteristic?

Perfectoid fields and their shortcomings

In the equal characteristic case, a perfect valued field (such as $\mathbb{F}_p((t))^{1/p^\infty}$) has no dependent defect extensions. The same holds if we pass to the completion of our example, which then is one of those famous perfectoid fields. What about perfectoid fields in mixed characteristic?

It turns out that perfectoid fields are too special for our purposes.

Perfectoid fields and their shortcomings

In the equal characteristic case, a perfect valued field (such as $\mathbb{F}_p((t))^{1/p^\infty}$) has no dependent defect extensions. The same holds if we pass to the completion of our example, which then is one of those famous perfectoid fields. What about perfectoid fields in mixed characteristic?

It turns out that perfectoid fields are too special for our purposes. By definition they are complete,

Perfectoid fields and their shortcomings

In the equal characteristic case, a perfect valued field (such as $\mathbb{F}_p((t))^{1/p^\infty}$) has no dependent defect extensions. The same holds if we pass to the completion of our example, which then is one of those famous perfectoid fields. What about perfectoid fields in mixed characteristic?

It turns out that perfectoid fields are too special for our purposes. By definition they are complete, with value group embeddable in $\mathbb{R}$

Perfectoid fields and their shortcomings

In the equal characteristic case, a perfect valued field (such as $\mathbb{F}_p((t))^{1/p^\infty}$) has no dependent defect extensions. The same holds if we pass to the completion of our example, which then is one of those famous perfectoid fields. What about perfectoid fields in mixed characteristic?

It turns out that perfectoid fields are too special for our purposes. By definition they are complete, with value group embeddable in $\mathbb{R}$ (then we speak of rank 1).

Perfectoid fields and their shortcomings

In the equal characteristic case, a perfect valued field (such as $\mathbb{F}_p((t))^{1/p^\infty}$) has no dependent defect extensions. The same holds if we pass to the completion of our example, which then is one of those famous perfectoid fields. What about perfectoid fields in mixed characteristic?

It turns out that perfectoid fields are too special for our purposes. By definition they are complete, with value group embeddable in $\mathbb{R}$ (then we speak of rank 1). Both conditions are not first order axiomatizable.

Perfectoid fields and their shortcomings

In the equal characteristic case, a perfect valued field (such as $\mathbb{F}_p((t))^{1/p^\infty}$) has no dependent defect extensions. The same holds if we pass to the completion of our example, which then is one of those famous perfectoid fields. What about perfectoid fields in mixed characteristic?

It turns out that perfectoid fields are too special for our purposes. By definition they are complete, with value group embeddable in $\mathbb{R}$ (then we speak of rank 1). Both conditions are not first order axiomatizable. It is better to work with deeply ramified fields

Perfectoid fields and their shortcomings

In the equal characteristic case, a perfect valued field (such as $\mathbb{F}_p((t))^{1/p^\infty}$) has no dependent defect extensions. The same holds if we pass to the completion of our example, which then is one of those famous perfectoid fields. What about perfectoid fields in mixed characteristic?

It turns out that perfectoid fields are too special for our purposes. By definition they are complete, with value group embeddable in $\mathbb{R}$ (then we speak of rank 1). Both conditions are not first order axiomatizable. It is better to work with deeply ramified fields in the sense of the book "Almost ring theory"

Perfectoid fields and their shortcomings

In the equal characteristic case, a perfect valued field (such as $\mathbb{F}_p((t))^{1/p^\infty}$) has no dependent defect extensions. The same holds if we pass to the completion of our example, which then is one of those famous perfectoid fields. What about perfectoid fields in mixed characteristic?

It turns out that perfectoid fields are too special for our purposes. By definition they are complete, with value group embeddable in $\mathbb{R}$ (then we speak of rank 1). Both conditions are not first order axiomatizable. It is better to work with deeply ramified fields in the sense of the book "Almost ring theory" by Gabber and Ramero.

Following Gabber and Ramero,

Deeply ramified fields

Following Gabber and Ramero, a valued field (K, v) is deeply ramified if

Deeply ramified fields

Following Gabber and Ramero, a valued field (K,v) is deeply ramified if

$$\Omega_{\mathcal{O}_{K^{\mathrm{sep}}}|\mathcal{O}_K} = 0\,, \tag{1}$$

Deeply ramified fields

Following Gabber and Ramero, a valued field (K, v) is deeply ramified if

$$\Omega_{\mathcal{O}_{K^{\mathrm{sep}}}|\mathcal{O}_K} = 0\,, \tag{1}$$

where $\mathcal{O}_K$ is the valuation ring of K,

Deeply ramified fields

Following Gabber and Ramero, a valued field (K, v) is deeply ramified if

$$\Omega_{\mathcal{O}_{K^{\mathrm{sep}}}|\mathcal{O}_K} = 0\,, \tag{1}$$

where $\mathcal{O}_K$ is the valuation ring of K, $\mathcal{O}_{K^{\mathrm{sep}}}$ is the valuation ring of the separable-algebraic closure of K,

Deeply ramified fields

Following Gabber and Ramero, a valued field (K,v) is deeply ramified if

$$\Omega_{\mathcal{O}_{K^{\mathrm{sep}}}|\mathcal{O}_K} = 0\,, \tag{1}$$

where $\mathcal{O}_K$ is the valuation ring of K, $\mathcal{O}_{K^{\mathrm{sep}}}$ is the valuation ring of the separable-algebraic closure of K, and $\Omega_{B|A}$ denotes the module of relative differentials when A is a ring and B is an A-algebra.

Theorem (Gabber & Ramero)

Take a valued field (K, v) of rank 1.

Characterization of rank 1 deeply ramified fields

Theorem (Gabber & Ramero)

Take a valued field (K, v) of rank 1. In the positive equal characteristic case, (K, v) is deeply ramified

Characterization of rank 1 deeply ramified fields

Theorem (Gabber & Ramero)

Take a valued field (K, v) of rank 1. In the positive equal characteristic case, (K, v) is deeply ramified if and only if its completion is perfect.

Characterization of rank 1 deeply ramified fields

Theorem (Gabber & Ramero)

Take a valued field (K, v) of rank 1. In the positive equal characteristic case, (K, v) is deeply ramified if and only if its completion is perfect. In the mixed characteristic case, (K, v) is deeply ramified

Characterization of rank 1 deeply ramified fields

Theorem (Gabber & Ramero)

Take a valued field (K, v) of rank 1. In the positive equal characteristic case, (K, v) is deeply ramified if and only if its completion is perfect. In the mixed characteristic case, (K, v) is deeply ramified if and only if the value vp is not the smallest positive value in vK

Characterization of rank 1 deeply ramified fields

Theorem (Gabber & Ramero)

Take a valued field (K,v) of rank 1. In the positive equal characteristic case, (K,v) is deeply ramified if and only if its completion is perfect. In the mixed characteristic case, (K,v) is deeply ramified if and only if the value vp is not the smallest positive value in vK and

$$\mathcal{O}_K/p\mathcal{O}_K \ni x \mapsto x^p \in \mathcal{O}_K/p\mathcal{O}_K$$

is surjective

Characterization of rank 1 deeply ramified fields

Theorem (Gabber & Ramero)

Take a valued field (K, v) of rank 1. In the positive equal characteristic case, (K, v) is deeply ramified if and only if its completion is perfect. In the mixed characteristic case, (K, v) is deeply ramified if and only if the value vp is not the smallest positive value in vK and

$$\mathcal{O}_K / p\mathcal{O}_K \ni x \mapsto x^p \in \mathcal{O}_K / p\mathcal{O}_K$$

is surjective ("the Frobenius on $\mathcal{O}_K$ is surjective modulo p").

Shortcomings of deeply ramified fields

For valued fields of higher rank,

Shortcomings of deeply ramified fields

For valued fields of higher rank, Gabber & Ramero's characterization of deeply ramified fields is more complicated

Shortcomings of deeply ramified fields

For valued fields of higher rank, Gabber & Ramero's characterization of deeply ramified fields is more complicated and involves an additional property of the value groups

Shortcomings of deeply ramified fields

For valued fields of higher rank, Gabber & Ramero's characterization of deeply ramified fields is more complicated and involves an additional property of the value groups that makes no sense for our purposes

Shortcomings of deeply ramified fields

For valued fields of higher rank, Gabber & Ramero's characterization of deeply ramified fields is more complicated and involves an additional property of the value groups that makes no sense for our purposes (namely, no archimedean component is discrete).

Shortcomings of deeply ramified fields

For valued fields of higher rank, Gabber & Ramero's characterization of deeply ramified fields is more complicated and involves an additional property of the value groups that makes no sense for our purposes (namely, no archimedean component is discrete). Therefore, we have introduced the larger class of generalized deeply ramified fields (gdr fields)

Shortcomings of deeply ramified fields

For valued fields of higher rank, Gabber & Ramero's characterization of deeply ramified fields is more complicated and involves an additional property of the value groups that makes no sense for our purposes (namely, no archimedean component is discrete). Therefore, we have introduced the larger class of generalized deeply ramified fields (gdr fields) by taking the characterizations of the theorem of Gabber & Ramero

Shortcomings of deeply ramified fields

For valued fields of higher rank, Gabber & Ramero's characterization of deeply ramified fields is more complicated and involves an additional property of the value groups that makes no sense for our purposes (namely, no archimedean component is discrete). Therefore, we have introduced the larger class of generalized deeply ramified fields (gdr fields) by taking the characterizations of the theorem of Gabber & Ramero as a definition in arbitrary rank.

Gdr fields and defect

We denote by $(vK)_{vp}$ the smallest convex subgroup of vK that contains vp if char $K = 0$,

Gdr fields and defect

We denote by $(vK)_{vp}$ the smallest convex subgroup of vK that contains vp if char $K = 0$, and set $(vK)_{vp} = vK$ otherwise.

Gdr fields and defect

We denote by $(vK)_{vp}$ the smallest convex subgroup of vK that contains vp if char $K = 0$, and set $(vK)_{vp} = vK$ otherwise.

Theorem (Rzepka & K)

Take a valued field (K, v) *with* char $Kv = p > 0$.

Gdr fields and defect

We denote by $(vK)_{vp}$ the smallest convex subgroup of vK that contains vp if $\mathrm{char}\, K = 0$, and set $(vK)_{vp} = vK$ otherwise.

Theorem (Rzepka & K)

Take a valued field (K, v) with $\mathrm{char}\, Kv = p > 0$. Then (K, v) is a gdr field if and only if

Gdr fields and defect

We denote by $(vK)_{vp}$ the smallest convex subgroup of vK that contains vp if char $K = 0$, and set $(vK)_{vp} = vK$ otherwise.

Theorem (Rzepka & K)

Take a valued field (K,v) with char $Kv = p > 0$. Then (K,v) is a gdr field if and only if $(vK)_{vp}$ is p-divisible,

Gdr fields and defect

We denote by $(vK)_{vp}$ the smallest convex subgroup of vK that contains vp if char $K = 0$, and set $(vK)_{vp} = vK$ otherwise.

Theorem (Rzepka & K)

Take a valued field (K, v) with char $Kv = p > 0$*. Then (K, v) is a gdr field if and only if $(vK)_{vp}$ is p-divisible, Kv is perfect,*

Gdr fields and defect

We denote by $(vK)_{vp}$ the smallest convex subgroup of vK that contains vp if $\mathrm{char}\, K = 0$, and set $(vK)_{vp} = vK$ otherwise.

Theorem (Rzepka & K)

Take a valued field (K, v) with $\mathrm{char}\, Kv = p > 0$. Then (K, v) is a gdr field if and only if $(vK)_{vp}$ is p-divisible, Kv is perfect, and every Galois defect extension of prime degree has independent defect.

Gdr fields and defect

We denote by $(vK)_{vp}$ the smallest convex subgroup of vK that contains vp if char $K = 0$, and set $(vK)_{vp} = vK$ otherwise.

Theorem (Rzepka & K)

Take a valued field (K, v) with char $Kv = p > 0$. *Then (K, v) is a gdr field if and only if $(vK)_{vp}$ is p-divisible, Kv is perfect, and every Galois defect extension of prime degree has independent defect.*

We call a valued field an independent defect field

Gdr fields and defect

We denote by $(vK)_{vp}$ the smallest convex subgroup of vK that contains vp if char $K = 0$, and set $(vK)_{vp} = vK$ otherwise.

Theorem (Rzepka & K)

Take a valued field (K, v) with char $Kv = p > 0$*. Then (K, v) is a gdr field if and only if $(vK)_{vp}$ is p-divisible, Kv is perfect, and every Galois defect extension of prime degree has independent defect.*

We call a valued field an independent defect field if every Galois defect extension of prime degree has independent defect.

Beyond tame fields, more concretely

The important question arises whether structure results on tame fields

Beyond tame fields, more concretely

The important question arises whether structure results on tame fields that are important for our open problems

Beyond tame fields, more concretely

The important question arises whether structure results on tame fields that are important for our open problems can be generalized to deeply ramified or even gdr fields.

Beyond tame fields, more concretely

The important question arises whether structure results on tame fields that are important for our open problems can be generalized to deeply ramified or even gdr fields.

Another possible direction of generalization is offered by the class of extremal fields

Beyond tame fields, more concretely

The important question arises whether structure results on tame fields that are important for our open problems can be generalized to deeply ramified or even gdr fields.

Another possible direction of generalization is offered by the class of extremal fields —studied in joint work with

Beyond tame fields, more concretely

The important question arises whether structure results on tame fields that are important for our open problems can be generalized to deeply ramified or even gdr fields.

Another possible direction of generalization is offered by the class of extremal fields —studied in joint work with Durhan (formerly Azgin),

Beyond tame fields, more concretely

The important question arises whether structure results on tame fields that are important for our open problems can be generalized to deeply ramified or even gdr fields.

Another possible direction of generalization is offered by the class of extremal fields —studied in joint work with Durhan (formerly Azgin), Pop,

Beyond tame fields, more concretely

The important question arises whether structure results on tame fields that are important for our open problems can be generalized to deeply ramified or even gdr fields.

Another possible direction of generalization is offered by the class of extremal fields —studied in joint work with Durhan (formerly Azgin), Pop, Anscombe—

The important question arises whether structure results on tame fields that are important for our open problems can be generalized to deeply ramified or even gdr fields.

Another possible direction of generalization is offered by the class of extremal fields —studied in joint work with Durhan (formerly Azgin), Pop, Anscombe— but this is another story.

To answer this question, we may want to stay, To answer this question, we may want to stay, at least initially, To answer this question, we may want to stay, at least initially, relatively close to the tame fields, To answer this question, we may want to stay, at least initially, relatively close to the tame fields, meaning that we only relax our conditions by allowing independent defect extensions.

To answer this question, we may want to stay, at least initially, relatively close to the tame fields, meaning that we only relax our conditions by allowing independent defect extensions. We call (K, v) a semitame field

Semitame fields

To answer this question, we may want to stay, at least initially, relatively close to the tame fields, meaning that we only relax our conditions by allowing independent defect extensions. We call (K, v) a semitame field if it is a gdr field

Semitame fields

To answer this question, we may want to stay, at least initially, relatively close to the tame fields, meaning that we only relax our conditions by allowing independent defect extensions. We call (K, v) a semitame field if it is a gdr field and its value group is p-divisible (if $\mathrm{char}\, Kv = p > 0$).

Semitame fields

To answer this question, we may want to stay, at least initially, relatively close to the tame fields, meaning that we only relax our conditions by allowing independent defect extensions. We call (K, v) a semitame field if it is a gdr field and its value group is p-divisible (if $\mathrm{char}\, Kv = p > 0$).

Semitame fields are our best bet when it comes to generalizing the results we have proved in the past for tame fields.

Fields with finitely many Artin-Schreier extensions

Chernikov, Kaplan and Simon showed that an infinite field of positive characteristic that is definable in an NTP_2 theory

Fields with finitely many Artin-Schreier extensions

Chernikov, Kaplan and Simon showed that an infinite field of positive characteristic that is definable in an NTP_2 theory has only finitely many Artin-Schreier extensions.

Fields with finitely many Artin-Schreier extensions

Chernikov, Kaplan and Simon showed that an infinite field of positive characteristic that is definable in an NTP_2 theory has only finitely many Artin-Schreier extensions. The following result was inspired by discussions at the Münster conference in 2019:

Fields with finitely many Artin-Schreier extensions

Chernikov, Kaplan and Simon showed that an infinite field of positive characteristic that is definable in an NTP_2 theory has only finitely many Artin-Schreier extensions. The following result was inspired by discussions at the Münster conference in 2019:

Theorem (K)

A nontrivially valued field of characteristic $p > 0$ which has only finitely many Artin-Schreier extensions

Fields with finitely many Artin-Schreier extensions

Chernikov, Kaplan and Simon showed that an infinite field of positive characteristic that is definable in an NTP_2 theory has only finitely many Artin-Schreier extensions. The following result was inspired by discussions at the Münster conference in 2019:

Theorem (K)

A nontrivially valued field of characteristic $p > 0$ which has only finitely many Artin-Schreier extensions is dense in its perfect hull.

Fields with finitely many Artin-Schreier extensions

Chernikov, Kaplan and Simon showed that an infinite field of positive characteristic that is definable in an NTP_2 theory has only finitely many Artin-Schreier extensions. The following result was inspired by discussions at the Münster conference in 2019:

Theorem (K)

A nontrivially valued field of characteristic $p > 0$ which has only finitely many Artin-Schreier extensions is dense in its perfect hull. Consequently, it is a semitame field.

A hierarchy of fields

Theorem (Rzepka & K)

1) If (K, v) is a nontrivially valued field with char $Kv = p > 0$,

A hierarchy of fields

Theorem (Rzepka & K)

1) If (K, v) is a nontrivially valued field with $\operatorname{char} Kv = p > 0$*, then the following logical relations between its properties hold:*

A hierarchy of fields

Theorem (Rzepka & K)

1) If (K, v) is a nontrivially valued field with $\mathrm{char}\, Kv = p > 0$*, then the following logical relations between its properties hold:*

tame field

A hierarchy of fields

Theorem (Rzepka & K)

1) If (K, v) is a nontrivially valued field with $\operatorname{char} Kv = p > 0$, *then the following logical relations between its properties hold:*

tame field $\Rightarrow$ *separably tame field*

A hierarchy of fields

Theorem (Rzepka & K)

1) If (K, v) is a nontrivially valued field with $\mathrm{char}\, Kv = p > 0$*, then the following logical relations between its properties hold:*

$$\textit{tame field} \Rightarrow \textit{separably tame field} \Rightarrow \textit{semitame field}$$

A hierarchy of fields

Theorem (Rzepka & K)

1) If (K, v) is a nontrivially valued field with $\mathrm{char}\, Kv = p > 0$, *then the following logical relations between its properties hold:*

tame field $\Rightarrow$ *separably tame field* $\Rightarrow$ *semitame field* $\Rightarrow$ *deeply ramified field*

A hierarchy of fields

Theorem (Rzepka & K)

1) If (K, v) is a nontrivially valued field with char $Kv = p > 0$*, then the following logical relations between its properties hold:*

tame field $\Rightarrow$ separably tame field $\Rightarrow$ semitame field $\Rightarrow$ deeply ramified field $\Rightarrow$ gdr field

A hierarchy of fields

Theorem (Rzepka & K)

1) If (K, v) *is a nontrivially valued field with* $\mathrm{char}\, Kv = p > 0$*, then the following logical relations between its properties hold:*

tame field $\Rightarrow$ *separably tame field* $\Rightarrow$ *semitame field* $\Rightarrow$
deeply ramified field $\Rightarrow$ *gdr field* $\Rightarrow$ *independent defect field.*

A hierarchy of fields

Theorem (Rzepka & K)

1) If (K, v) is a nontrivially valued field with char $Kv = p > 0$, *then the following logical relations between its properties hold:*

tame field $\Rightarrow$ *separably tame field* $\Rightarrow$ *semitame field* $\Rightarrow$
deeply ramified field $\Rightarrow$ *gdr field* $\Rightarrow$ *independent defect field.*

Note that in positive equal characteristic,

A hierarchy of fields

Theorem (Rzepka & K)

1) If (K, v) is a nontrivially valued field with char $Kv = p > 0$*, then the following logical relations between its properties hold:*

tame field ⇒ separably tame field ⇒ semitame field ⇒
deeply ramified field ⇒ gdr field ⇒ independent defect field.

Note that in positive equal characteristic, semitame, deeply ramified and gdr fields coincide

A hierarchy of fields

Theorem (Rzepka & K)

1) If (K, v) is a nontrivially valued field with $\mathrm{char}\, Kv = p > 0$*, then the following logical relations between its properties hold:*

tame field $\Rightarrow$ separably tame field $\Rightarrow$ semitame field $\Rightarrow$ deeply ramified field $\Rightarrow$ gdr field $\Rightarrow$ independent defect field.

Note that in positive equal characteristic, semitame, deeply ramified and gdr fields coincide and are exactly those that are dense in their perfect hull.

Good news for model theorists

Theorem (Rzepka & K)

The classes of semitame, deeply ramified and gdr fields

Good news for model theorists

Theorem (Rzepka & K)

The classes of semitame, deeply ramified and gdr fields of fixed characteristic and residue characteristic

Good news for model theorists

Theorem (Rzepka & K)

The classes of semitame, deeply ramified and gdr fields of fixed characteristic and residue characteristic are first order axiomatizable in the language of valued fields.

From Gabber & Ramero's definition (1) of deeply ramified fields the following result is easy to deduce:

Algebraic extensions of deeply ramified fields

From Gabber & Ramero's definition (1) of deeply ramified fields the following result is easy to deduce:

Theorem (Gabber & Ramero)

Algebraic extensions of deeply ramified fields

Algebraic extensions of deeply ramified fields

From Gabber & Ramero's definition (1) of deeply ramified fields the following result is easy to deduce:

Theorem (Gabber & Ramero)

Algebraic extensions of deeply ramified fields are again deeply ramified fields.

Algebraic extensions of deeply ramified fields

From Gabber & Ramero's definition (1) of deeply ramified fields the following result is easy to deduce:

Theorem (Gabber & Ramero)

Algebraic extensions of deeply ramified fields are again deeply ramified fields.

From this theorem, the same follows for gdr fields

Algebraic extensions of deeply ramified fields

From Gabber & Ramero's definition (1) of deeply ramified fields the following result is easy to deduce:

Theorem (Gabber & Ramero)

Algebraic extensions of deeply ramified fields are again deeply ramified fields.

From this theorem, the same follows for gdr fields via the characterization theorem of Gabber & Ramero.

Algebraic extensions of deeply ramified fields

From Gabber & Ramero's definition (1) of deeply ramified fields the following result is easy to deduce:

Theorem (Gabber & Ramero)

Algebraic extensions of deeply ramified fields are again deeply ramified fields.

From this theorem, the same follows for gdr fields via the characterization theorem of Gabber & Ramero. However, the proof of that theorem is quite involved.

Proof of the extension theorem

In our work we have shown that in order to prove the above extension theorem

Proof of the extension theorem

In our work we have shown that in order to prove the above extension theorem without referring to Gabber & Ramero's definition (1) of deeply ramified fields,

Proof of the extension theorem

In our work we have shown that in order to prove the above extension theorem without referring to Gabber & Ramero's definition (1) of deeply ramified fields, it suffices to prove that every Galois defect extension of prime degree of a deeply ramified field

Proof of the extension theorem

In our work we have shown that in order to prove the above extension theorem without referring to Gabber & Ramero's definition (1) of deeply ramified fields, it suffices to prove that every Galois defect extension of prime degree of a deeply ramified field (which can be assumed equal to its absolute ramification field)

Proof of the extension theorem

In our work we have shown that in order to prove the above extension theorem without referring to Gabber & Ramero's definition (1) of deeply ramified fields, it suffices to prove that every Galois defect extension of prime degree of a deeply ramified field (which can be assumed equal to its absolute ramification field) is again a deeply ramified field.

Proof of the extension theorem

In our work we have shown that in order to prove the above extension theorem without referring to Gabber & Ramero's definition (1) of deeply ramified fields, it suffices to prove that every Galois defect extension of prime degree of a deeply ramified field (which can be assumed equal to its absolute ramification field) is again a deeply ramified field.

Challenge:

Proof of the extension theorem

In our work we have shown that in order to prove the above extension theorem without referring to Gabber & Ramero's definition (1) of deeply ramified fields, it suffices to prove that every Galois defect extension of prime degree of a deeply ramified field (which can be assumed equal to its absolute ramification field) is again a deeply ramified field.

Challenge: prove this *directly in a purely valuation theoretical approach*

Proof of the extension theorem

In our work we have shown that in order to prove the above extension theorem without referring to Gabber & Ramero's definition (1) of deeply ramified fields, it suffices to prove that every Galois defect extension of prime degree of a deeply ramified field (which can be assumed equal to its absolute ramification field) is again a deeply ramified field.

Challenge: prove this *directly in a purely valuation theoretical approach* by studying the behaviour of valuation rings

Proof of the extension theorem

In our work we have shown that in order to prove the above extension theorem without referring to Gabber & Ramero's definition (1) of deeply ramified fields, it suffices to prove that every Galois defect extension of prime degree of a deeply ramified field (which can be assumed equal to its absolute ramification field) is again a deeply ramified field.

Challenge: prove this *directly in a purely valuation theoretical approach* by studying the behaviour of valuation rings under such defect extensions.

Extensions in the absolute ramification field

Our reduction goes by the following theorem, Our reduction goes by the following theorem, for which we give a constructive proof:

Extensions in the absolute ramification field

Our reduction goes by the following theorem, for which we give a constructive proof:

Theorem (Rzepka & K)

If (L, v) is contained in the absolute ramification field of (K, v),

Extensions in the absolute ramification field

Our reduction goes by the following theorem, for which we give a constructive proof:

Theorem (Rzepka & K)

If (L, v) is contained in the absolute ramification field of (K, v), then (K, v) is a gdr field if and only if (L, v) is.

Our reduction goes by the following theorem, for which we give a constructive proof:

Theorem (Rzepka & K)

If (L, v) is contained in the absolute ramification field of (K, v), then (K, v) is a gdr field if and only if (L, v) is.

Note that if (K, v) is henselian, then the condition on (L, v) just means that it is a tame extension of (K, v).

The second argument in the reduction

For the reduction, we then also use the fact that the absolute Galois group of an absolute ramification field

The second argument in the reduction

For the reduction, we then also use the fact that the absolute Galois group of an absolute ramification field of residue characteristic $p > 0$

The second argument in the reduction

For the reduction, we then also use the fact that the absolute Galois group of an absolute ramification field of residue characteristic $p > 0$ is a p-group, which implies that

The second argument in the reduction

For the reduction, we then also use the fact that the absolute Galois group of an absolute ramification field of residue characteristic $p > 0$ is a p-group, which implies that every finite extension of this field is a tower of normal extensions of degree p.

The second argument in the reduction

For the reduction, we then also use the fact that the absolute Galois group of an absolute ramification field of residue characteristic $p > 0$ is a p-group, which implies that every finite extension of this field is a tower of normal extensions of degree p.

By the way, this argument had already been used by Abhyankar

The second argument in the reduction

For the reduction, we then also use the fact that the absolute Galois group of an absolute ramification field of residue characteristic $p > 0$ is a p-group, which implies that every finite extension of this field is a tower of normal extensions of degree p.

By the way, this argument had already been used by Abhyankar in his work on resolution of singularities in positive characteristic.

Independent defect fields

By our earlier theorem, every gdr field is an independent defect field.

Independent defect fields

By our earlier theorem, every gdr field is an independent defect field.

There are many open problems about independent defect fields.

Independent defect fields

By our earlier theorem, every gdr field is an independent defect field.

There are many open problems about independent defect fields. For instance, if (K, v) is an independent defect field,

Independent defect fields

By our earlier theorem, every gdr field is an independent defect field.

There are many open problems about independent defect fields. For instance, if (K, v) is an independent defect field, does the same hold for its absolute ramification field?

Independent defect fields

By our earlier theorem, every gdr field is an independent defect field.

There are many open problems about independent defect fields. For instance, if (K, v) is an independent defect field, does the same hold for its absolute ramification field? (The converse holds.)

Independent defect fields

By our earlier theorem, every gdr field is an independent defect field.

There are many open problems about independent defect fields. For instance, if (K, v) is an independent defect field, does the same hold for its absolute ramification field? (The converse holds.)

Questions of this type are very hard to study when the valued field under consideration

Independent defect fields

By our earlier theorem, every gdr field is an independent defect field.

There are many open problems about independent defect fields. For instance, if (K, v) is an independent defect field, does the same hold for its absolute ramification field? (The converse holds.)

Questions of this type are very hard to study when the valued field under consideration does not have a p-divisible value group

Independent defect fields

By our earlier theorem, every gdr field is an independent defect field.

There are many open problems about independent defect fields. For instance, if (K,v) is an independent defect field, does the same hold for its absolute ramification field? (The converse holds.)

Questions of this type are very hard to study when the valued field under consideration does not have a p-divisible value group or does not have a perfect residue field.

Further results and work in progress

In our joint paper we also characterized independent defect of Galois extensions $(L|K, v)$

Further results and work in progress

In our joint paper we also characterized independent defect of Galois extensions $(L|K, v)$ via ramification jumps and ramification ideals

Further results and work in progress

In our joint paper we also characterized independent defect of Galois extensions $(L|K,v)$ via ramification jumps and ramification ideals connected with higher ramification groups,

Further results and work in progress

In our joint paper we also characterized independent defect of Galois extensions $(L|K, v)$ via ramification jumps and ramification ideals connected with higher ramification groups, as well as via distances from K of suitably chosen generators of the extension.

Further results and work in progress

In our joint paper we also characterized independent defect of Galois extensions $(L|K, v)$ via ramification jumps and ramification ideals connected with higher ramification groups, as well as via distances from K of suitably chosen generators of the extension. Further, we computed the trace of the valuation ideal $\mathcal{M}_L$ of v on L and proved:

Further results and work in progress

In our joint paper we also characterized independent defect of Galois extensions $(L|K,v)$ via ramification jumps and ramification ideals connected with higher ramification groups, as well as via distances from K of suitably chosen generators of the extension. Further, we computed the trace of the valuation ideal $\mathcal{M}_L$ of v on L and proved:

Theorem (Rzepka & K)

Take a Galois defect extension $(L|K,v)$ of prime degree.

Further results and work in progress

In our joint paper we also characterized independent defect of Galois extensions $(L|K,v)$ via ramification jumps and ramification ideals connected with higher ramification groups, as well as via distances from K of suitably chosen generators of the extension. Further, we computed the trace of the valuation ideal $\mathcal{M}_L$ of v on L and proved:

Theorem (Rzepka & K)

Take a Galois defect extension $(L|K,v)$ of prime degree. Then $(L|K,v)$ has independent defect if and only if

Further results and work in progress

In our joint paper we also characterized independent defect of Galois extensions $(L|K,v)$ via ramification jumps and ramification ideals connected with higher ramification groups, as well as via distances from K of suitably chosen generators of the extension. Further, we computed the trace of the valuation ideal $\mathcal{M}_L$ of v on L and proved:

Theorem (Rzepka & K)

Take a Galois defect extension $(L|K,v)$ of prime degree. Then $(L|K,v)$ has independent defect if and only if the trace $\mathrm{Tr}_{L|K}(\mathcal{M}_L)$ is a valuation ideal on K which is contained in $\mathcal{M}_K$.

Further results and work in progress

In our joint paper we also characterized independent defect of Galois extensions $(L|K,v)$ via ramification jumps and ramification ideals connected with higher ramification groups, as well as via distances from K of suitably chosen generators of the extension. Further, we computed the trace of the valuation ideal $\mathcal{M}_L$ of v on L and proved:

Theorem (Rzepka & K)

Take a Galois defect extension $(L|K,v)$ of prime degree. Then $(L|K,v)$ has independent defect if and only if the trace $\mathrm{Tr}_{L|K}(\mathcal{M}_L)$ is a valuation ideal on K which is contained in $\mathcal{M}_K$.
(Rank 1 implies that it can only be equal to $\mathcal{M}_K$.)

Further results and work in progress

In our joint paper we also characterized independent defect of Galois extensions $(L|K, v)$ via ramification jumps and ramification ideals connected with higher ramification groups, as well as via distances from K of suitably chosen generators of the extension. Further, we computed the trace of the valuation ideal $\mathcal{M}_L$ of v on L and proved:

Theorem (Rzepka & K)

Take a Galois defect extension $(L|K, v)$ of prime degree. Then $(L|K, v)$ has independent defect if and only if the trace $\mathrm{Tr}_{L|K}(\mathcal{M}_L)$ is a valuation ideal on K which is contained in $\mathcal{M}_K$.
(Rank 1 implies that it can only be equal to $\mathcal{M}_K$.)

Presently ongoing work is aimed at proving that $(L|K, v)$ has independent defect

Further results and work in progress

In our joint paper we also characterized independent defect of Galois extensions $(L|K, v)$ via ramification jumps and ramification ideals connected with higher ramification groups, as well as via distances from K of suitably chosen generators of the extension. Further, we computed the trace of the valuation ideal $\mathcal{M}_L$ of v on L and proved:

Theorem (Rzepka & K)

Take a Galois defect extension $(L|K, v)$ of prime degree. Then $(L|K, v)$ has independent defect if and only if the trace $\mathrm{Tr}_{L|K}(\mathcal{M}_L)$ is a valuation ideal on K which is contained in $\mathcal{M}_K$.
(Rank 1 implies that it can only be equal to $\mathcal{M}_K$.)

Presently ongoing work is aimed at proving that $(L|K, v)$ has independent defect if and only if $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$.

References

- Anscombe, S. – Kuhlmann, F.-V.: *Notes on extremal and tame valued fields*, J. Symb. Logic **81** (2016), 400–416
- Azgin, S. – Kuhlmann, F.-V. – Pop, F.: *Characterization of Extremal Valued Fields*, Proc. Amer. Math. Soc. **140** (2012), 1535–1547
- Blaszczok, A. – Kuhlmann, F.-V.: *Deeply ramified fields, semitame fields, and the classification of defect extensions*, submitted
- Chernikov, A. – Kaplan, I. – Simon, P.: *Groups and fields with* NTP_2, Proc. Amer. Math. Soc. **143** (2015), 395–406

References

- Cutkosky, D. – Piltant, O.: *Ramification of valuations*, Adv. Math. **183** (2004), 1–79
- ElHitti, S. – Ghezzi, L.: *Dependent Artin-Schreier defect extensions and strong monomialization*, J. Pure Appl. Algebra **220** (2016), 1331–1342
- Gabber, O. – Ramero, L.: *Almost ring theory*, Lecture Notes in Mathematics **1800**, Springer-Verlag, Berlin, 2003
- Knaf, H. – Kuhlmann, F.-V.: *Abhyankar places admit local uniformization in any characteristic*, Ann. Scient. Ec. Norm. Sup. **38** (2005), 833–846.
- Knaf, H. – Kuhlmann, F.-V.: *Every place admits local uniformization in a finite extension of the function field*, Adv. Math. **221** (2009), 428–453.

References

- Kuhlmann, F.-V.: *Elimination of Ramification I: The Generalized Stability Theorem*, Trans. Amer. Math. Soc. **362** (2010), 5697–5727
- Kuhlmann, F.-V.: *A classification of Artin-Schreier defect extensions and a characterization of defectless fields*, Illinois J. Math. **54** (2010), 397–448.
- Kuhlmann, F.-V.: *The algebra and model theory of tame valued fields*, J. reine angew. Math. **719** (2016), 1–43
- Kuhlmann, F.-V.: *Elimination of Ramification II: Henselian Rationality*, Israel J. Math. 234 (2019), 927–958
- Kuhlmann, F.-V.: *Valued fields with finitely many defect extensions of prime degree*, submitted

References

- Kuhlmann, F.-V. – Pal, K.: *The model theory of separably tame fields*, J. Alg. **447** (2016), 74–108
- Temkin, M. : *Inseparable local uniformization*, J. Algebra **373** (2013), 65–119

Preprints and further information

Thank you for your attention!

Preprints and further information

Thank you for your attention!

Preprints and further materials are available at:

Preprints and further information

Thank you for your attention!

Preprints and further materials are available at:

The Valuation Theory Home Page
http://math.usask.ca/fvk/Valth.html