Totally geodesic surfaces in twist knot complements

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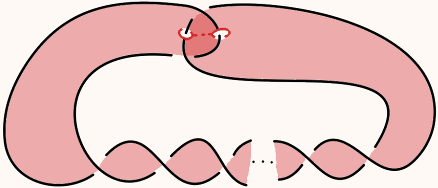
MSRI Postdoc Seminar
17 Sep 2020

Roadmap

- Motivation
- Main results
- History
- Twist knots
- Boundary slopes
- Corollaries



General motivation

A *totally geodesic surface* is

surface Σ	geodesics coincide	3-manifold M
maximal Fuchsian $\pi_1(\Sigma)$	finite coarea $\subset$	Kleinian $\pi_1(M)$

Such surfaces lift to hyperplanes in $\mathbb{H}^3$. Use this to find surfaces!

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Thrice-punctured sphere in twist knot complement is totally geodesic (Adams, '85)

Question: Is this unique?

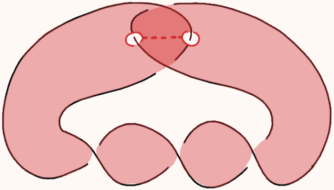

Figure: Thrice-puncture sphere N for 5_2

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Answer: Yes, for infinitely many! (Le-P)

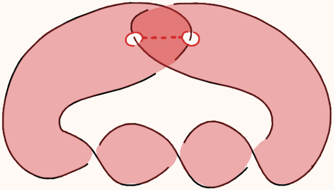


Figure: Thrice-puncture sphere N for 5_2

Theorem (Le–P)

There are infinitely many twist knot complements containing a unique totally geodesic surface.

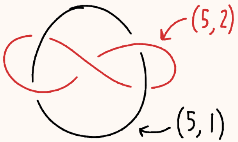
which combines with covering space theory to get

Theorem (Le–P)

There exist infinitely many non-commensurable hyperbolic 3-manifolds that contain exactly k totally geodesic surfaces for any positive integer k.

$$\frac{\text{Weeks manifold}}{\text{Dehn surgery on Whitehead Link}}$$

arithmetic



$$\frac{\text{Knot complement } 8_{20}}{\text{(Calegari, '06)}}$$

knot complement

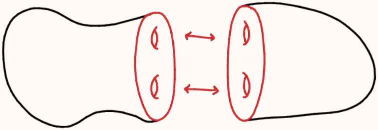
nonarithmetic

Theorem (Fisher–Lafont–Miller–Stover, '18)

There exist infinitely many non-commensurable hyperbolic 3-manifolds with finitely many totally geodesic surfaces.

Gromov–Piatetski-Shapiro construction

part of arithmetic M_1 part of arithmetic M_2



glue along shared totally geodesic surface

Number of totally geodesic surfaces is nonzero and bounded, but not explicitly known.

Theorem (Reid, '91)

An arithmetic hyperbolic 3-manifold admiting one totally geodesic surface admits infinitely many.

Fact: the figure-8 is the *only* arithmetic knot.

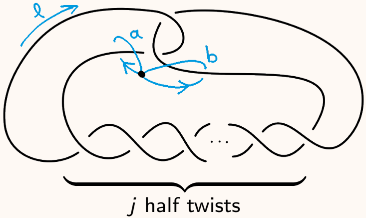
Theorem (Bader–Fisher–Miller–Stover, '19)

If M is a hyperbolic 3-manifold containing infinitely many totally geodesic surfaces, then M is arithmetic.

To find finitely many totally geodesic surfaces,
look to nonarithmetic manifolds!

Motivation for twist knots

Figure-8 is a *twist knot* with $j = 2$ half twists. General K_j:



Longitude ℓ vanishes in homology with respect to meridian a

Twist knot group is generated by two generators and a conjugation relation (Riley, '72):

$$\Gamma_j = \pi_1\left(S^3 \setminus K_j\right) = \langle a, b \mid aw_j = w_j b \rangle$$

There's a representation $\Gamma \hookrightarrow \mathrm{PSL}_2(\mathbb{C})$ by mapping two meridians to parabolics (Riley, '72):

$$a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$$

where z is a root of a polynomial from satisfying the group relation.

For a twist knot K_j, this polynomial is irreducible of degree j and has root z_j (Hoste–Shanahan, '01).

Trace field

With this Kleinian representation of Γ_j, look at the *trace field*

$$\mathbb{Q}(\{\operatorname{tr}\gamma \mid \gamma \in \Gamma\}) = \mathbb{Q}(z_j)$$

The degree of this extension is precisely j because the polynomial is irreducible.

Trace field coincides here with *cusp set*: the points in $\hat{\mathbb{C}}$ that identify to the cusp under Γ_j are precisely $\mathbb{Q}(z_j) \cup \{\infty\}$.

Helpful conditions for Γ:

- Γ has integral traces
- $\mathbb{Q}(\operatorname{tr}\Gamma)$ is of odd degree over $\mathbb{Q}$ and contains no proper real subfield other than $\mathbb{Q}$

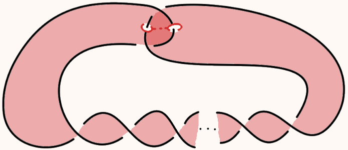
Proposition (Reid, '91)

Let Γ satisfy the above two conditions. Then Γ contains no cocompact Fuchsian groups, and $\operatorname{tr}(\Delta) \subset \mathbb{Z}$ for any Fuchsian subgroup Δ.

This is immediately satisfed by j odd prime, so takeaway:

- traces are integers
- no closed surfaces — look to cusps!

(for convenience, no more j)



Thrice-punctured sphere N is totally geodesic. We examine any other totally geodesic surface Σ by how it intersects N.

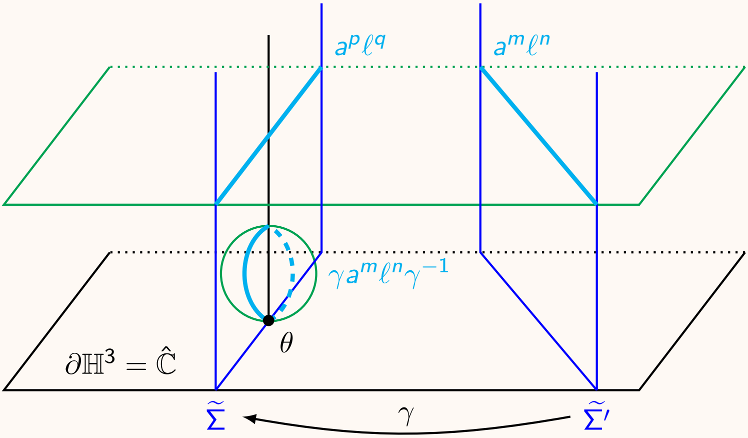
Lemma (Fisher–Lafont–Miller–Stover, '18)

Any connected component of any two distinct properly immersed totally geodesic surfaces in M is either a closed geodesic or a cusp-to-cusp geodesic.

So if we start at a cusp point, we want to end at a cusp point.

Approaching the surface Σ

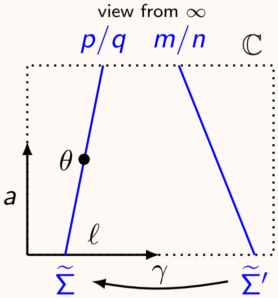
Consider a totally geodesic surface Σ in M and choose a lift $\widetilde{\Sigma}$.
Look at two parabolic elements of $\mathrm{Stab}(\widetilde{\Sigma})$.

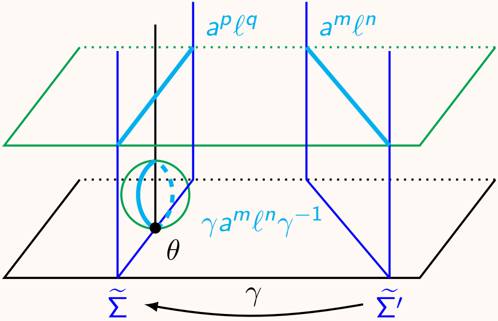


Trace condition

Requirements:

- find cusp point θ
- determine conjugating word $\gamma \in \Gamma$
- check *trace condition* $\mathrm{tr}(a^p \ell^q \gamma a^m \ell^n \gamma^{-1}) \in \mathbb{Z}$





Lemma

Let Σ be any cusped totally geodesic surface in a twist knot complement with appropriate number of half twists. Then the complete set of boundary slopes of Σ is $\{1/0, -2\}$.

Idea: look at every possible way $\widetilde{\Sigma}$ might intersect the lifts of N. Boundary slopes of N are $1/0$ and -2. Let $\widetilde{\Sigma}$ have slope p/q.

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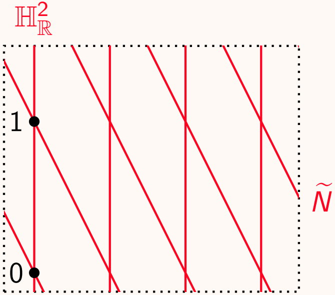

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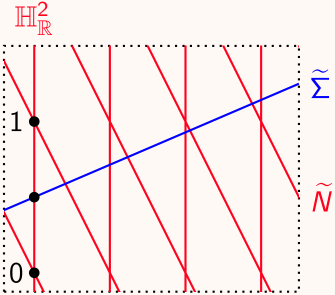

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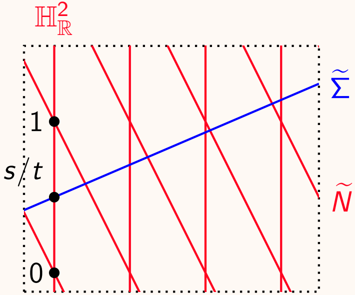

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- $\widetilde{\Sigma} \cap \mathbb{H}^2_{\mathbb{R}}$ is cusp-to-cusp geodesic
- $\theta \in \mathbb{Q}(z) \cap \mathbb{R} = \mathbb{Q}$
- either no γ and $\widetilde{\Sigma}'$ exists
  OR $nq = (2n+m)(2q+p) = 0$

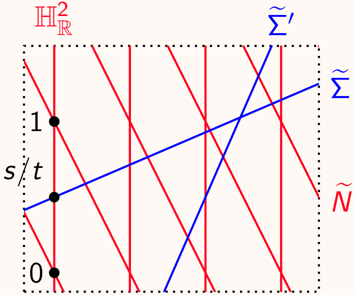
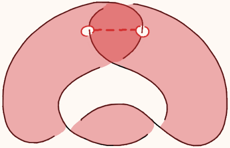
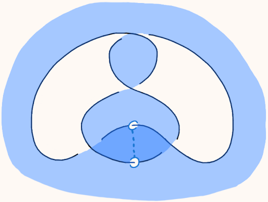


Figure-8 knot complement

Diagram of figure-8 gives a second totally geodesic surface:



N has slopes $1/0$, -2



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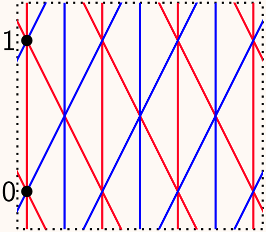

View from ∞ of lifts

Furthermore: Every rational number occurs as a boundary slope for some totally geodesic surface in the figure-8 knot complement.

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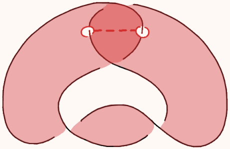

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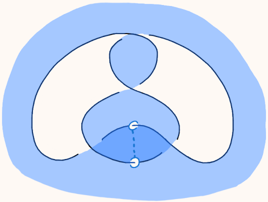

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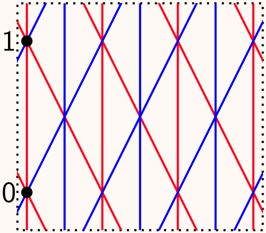


View from ∞ of lifts

Furthermore: Every rational number occurs as a boundary slope for some totally geodesic surface in the figure-8 knot complement.

Very different from the $\{1/0, -2\}$ in our nonarithmetic case!

A manifold is *right-angled* if it can be constructed by gluing together a set of polyhedra whose dihedral angles are all $\pi/2$.

Conjecture (Champanerkar–Kofman–Purcell)

There does not exist a right-angled knot.

True for all knots up to 11 crossings by volume techniques.

Lemma (Champanerkar–Kofman–Purcell, '19)

The faces of the right-angled polyhedra give rise to immersed totally geodesic surfaces.

Corollary

No twist knot with odd prime half twists is right-angled.

Let H be a hyperplane with boundary slope -2 at ∞.

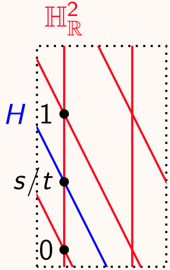
$\mathrm{Stab}(H)$ is a maximal Fuchsian subgroup containing $a^{-2}\ell$ and a conjugate of b. These two parabolics generate a *non-elementary* subgroup; that is, no point in the boundary has finite orbit.

Proposition

The subgroup $\mathrm{Stab}(H)$ *in* Γ *is non-elementary maximal Fuchsian.*

Corollary

The subgroup $\mathrm{Stab}(H)$ *in* Γ *has infinite coarea for all but finitely many* H *(up to conjugacy).*



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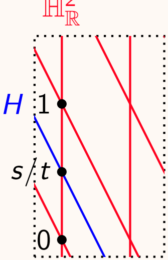

Comparison: In the arithmetic case, every non-elementary maximal Fuchsian subgroup has finite coarea.

Corollary: non-elementary maximal Fuchsian subgroups

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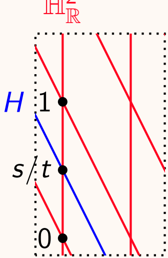

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Stay tuned: for a family of twist knots, finite coarea occurs only when H is a lift of our favorite thrice-punctured sphere!

Surfaces seem to appear by looking at consecutive tangent horospheres:

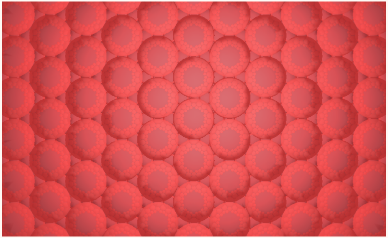
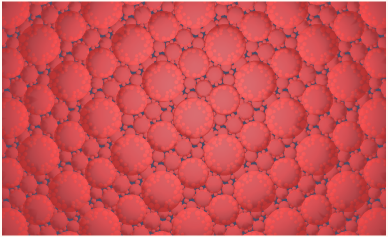


figure-8 knot complement



5_2 knot complement

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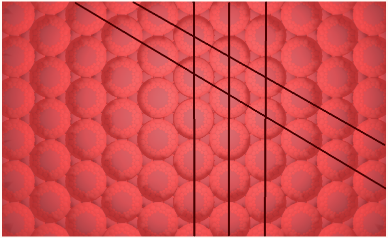

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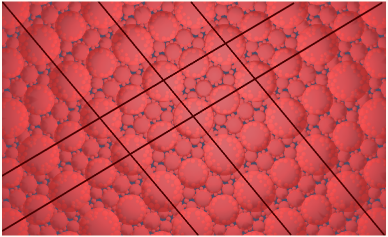

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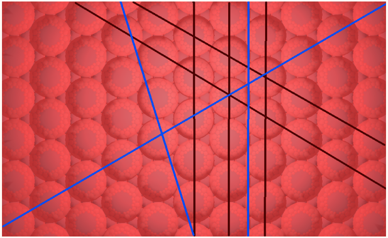

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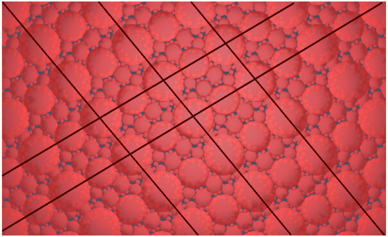

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