Totally Geodesic Surfaces in Twist Knot Complements

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joint with Rebekah Palmer

Results

Theorem (L., Palmer)

There exists infinitely many non-commensurable hyperbolic 3-manifolds that contain exactly k totally geodesic surfaces for any positive integer k.

- we prove the theorem by finding infinitely many non-commensurable hyperbolic 3-manifolds each containing exactly 1 totally geodesic surface
- ...and by using covering tricks to produce examples for all $k > 1$.

Results

Theorem (L., Palmer)

For any odd prime j, the complement $M_j := S^3 \setminus K_j$ contains a unique totally geodesic surface.

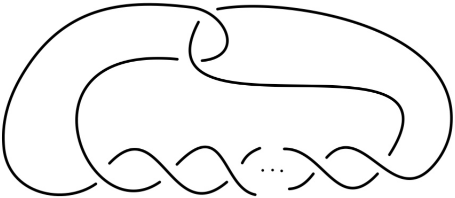

Figure: The twist knot K_j with j half-twists

Main Characters

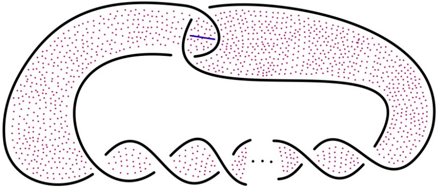

Figure: The 3-punctured sphere N in M_j

- $\Gamma_j := \pi_1(M_j) = \langle a, b | w_j a w_j^{-1} = b \rangle$

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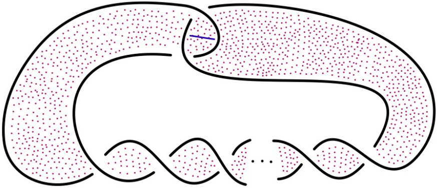

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- $\Gamma_j := \pi_1(M_j) = \langle a, b | w_j a w_j^{-1} = b \rangle$
- M_j admits a complete finite-volume hyperbolic metric for $j \geq 2$ given by $\rho : \Gamma_j \to \mathrm{SL}_2(\mathbb{C})$:

$$a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 0 \\ z_j & 1 \end{pmatrix}$$

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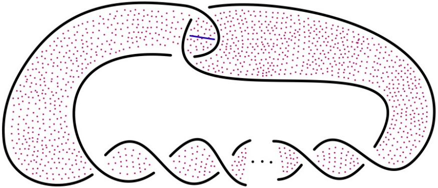

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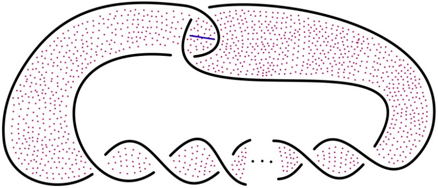

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- (Adams): N is a totally geodesic surface.
- (Hoste – Shanahan): $[\mathbb{Q}(z_j) : \mathbb{Q}] = j$

Arithmetic Constraints

Proposition (Reid)

Let Γ be a non-cocompact Kleinian group of finite covolume that satisfies the following two conditions:

- $\mathbb{Q}(\operatorname{tr}\Gamma)$ *is of odd degree over* $\mathbb{Q}$ *and contains no proper real subfield other than* $\mathbb{Q}$*.*
- Γ *has integral traces.*

Then Γ contains no cocompact Fuchsian subgroups.

Corollary (Reid)

M_j does not contain any closed totally geodesic surface for any odd prime j.

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- *In fact, we prove uniqueness of N in M_j under these two conditions.*
- *Using Magma, we verify that the first condition holds for all $2 \leq j \leq 99$.*
- *This proposition gives us a convenient way to rule out closed surfaces.*

Basic idea

Assuming j is an odd prime and Σ is a totally geodesic surface in M_j:

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- $\mathrm{tr}(\pi_1(\Sigma)) \subset \mathbb{R} \cap \mathbb{Q}(z_j) = \mathbb{Q}$. Integral traces $\Rightarrow \mathrm{tr}(\pi_1(\Sigma)) \subset \mathbb{Z}$.

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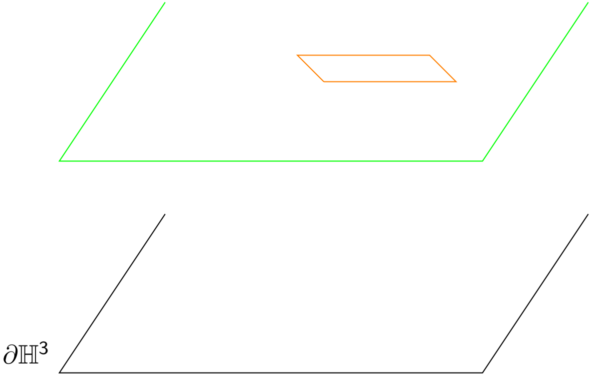

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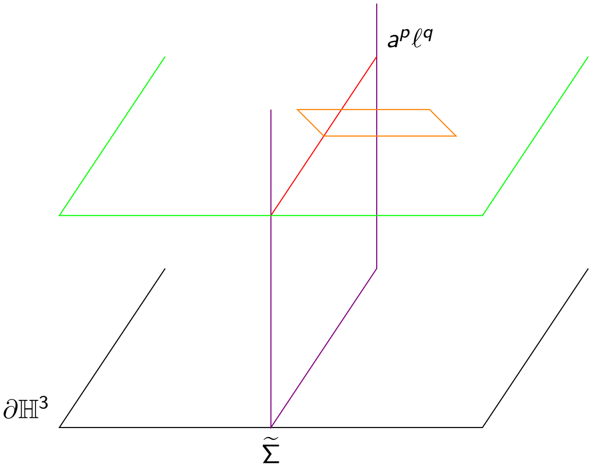

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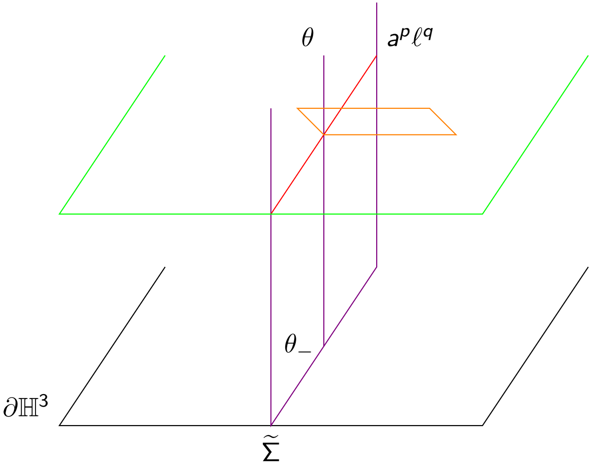

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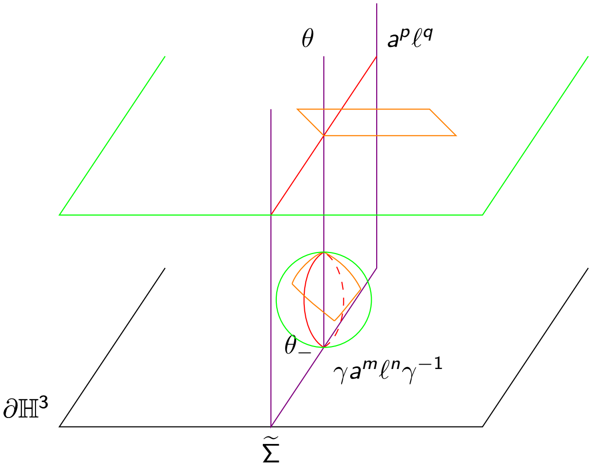

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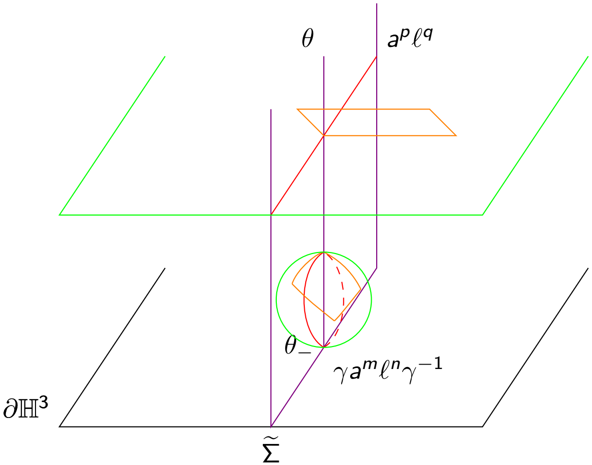

- There exists $\gamma \in \Gamma$ such that $\gamma(\infty) = \theta_-$
- $\mathrm{tr}(\gamma a^m \ell^n \gamma^{-1} a^p \ell^q) \in \mathbb{Z}$

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Assuming j is an odd prime and Σ is a totally geodesic surface in M_j:

- Σ must be a cusped surface.
- $\mathrm{tr}(\pi_1(\Sigma)) \subset \mathbb{R} \cap \mathbb{Q}(z_j) = \mathbb{Q}$. Integral traces $\Rightarrow$ $\mathrm{tr}(\pi_1(\Sigma)) \subset \mathbb{Z}$.
- The trace condition holds:

$$\mathrm{tr}(\gamma a^m \ell^n \gamma^{-1} a^p \ell^q) = \delta^2(nq\tau^2 + (mq + np)\tau + mp) \in \mathbb{Z}$$

  where τ is the cusp shape and

$$\gamma = \begin{pmatrix} \alpha & \beta \\ \delta & \eta \end{pmatrix}$$

To execute this idea, we need...

- ...to find cusp-to-cusp geodesics θ.
- ...to find $\gamma \in \Gamma_j$ such that $\gamma(\infty) = \theta_-$.

Geometric Observation

Lemma (Fisher, Lafont, Miller, Stover)

Let M be a complete finite-volume hyperbolic 3-manifold. Let Σ_1, Σ_2 are two distinct properly immersed totally geodesic surfaces with non-empty intersection. Then each component of $\Sigma_1 \cap \Sigma_2$ is either a closed geodesic or a cusp-to-cusp geodesic.

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The key is that:

- The intersections are all transverse.
- All components in the intersection must be properly immersed, complete, 1 dimensional submanifolds.
- By looking at intersections of lifts of totally geodesic surfaces to $\mathbb{H}^3$, we can find cusp-to-cusp geodesics.

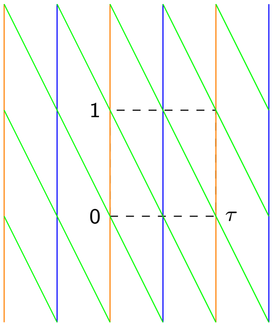
Boundary Slope Restrictions

Lemma

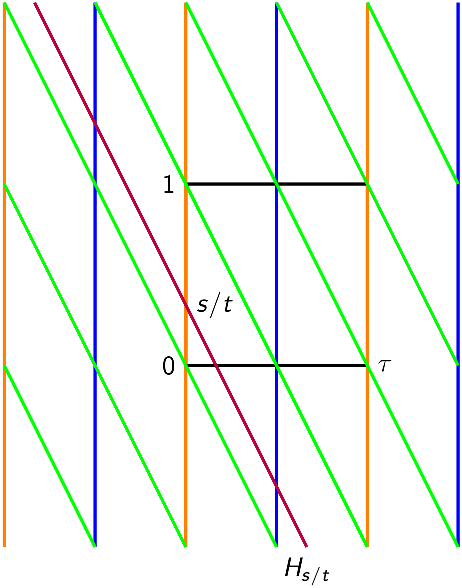
Let j be an odd prime and Σ be any (cusped) totally geodesics surface in M_j. Then the complete set of boundary slopes of Σ is $\{1/0, -2\}$ (up to multiplicity).

- $\Sigma = N$: Knot diagrams $\Rightarrow \rho(\pi_1(N)) = \langle a, y \rangle$ where

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} -1 & 0 \\ -4 & -1 \end{pmatrix}.$$



Proof of Uniqueness of N (Sketch)



- The boundary slopes restriction implies that any totally geodesic surface Σ has a lift $H_{s/t}$.
- In fact, the slope at s/t of $H_{s/t}$ is $1/0$.
- Using $\pi_1(N)$, we can find $\zeta \in \Gamma_j$ mapping s/t to ∞.

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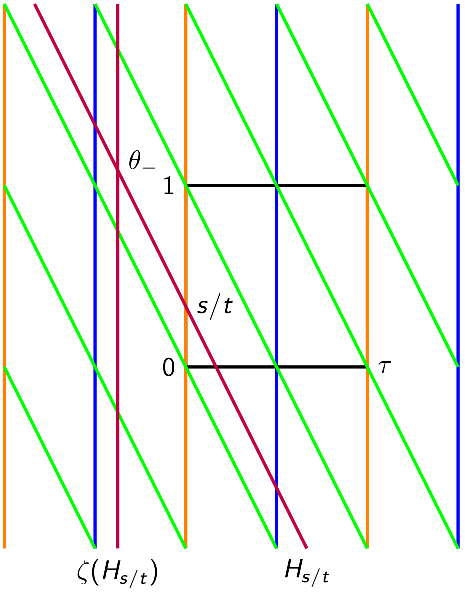

- Moving $H_{s/t}$ by $\zeta \in \Gamma_j$ mapping s/t to ∞.
- We get a cusp-to-cusp geodesic $\theta = (\theta_-, \infty)$ where $\theta_- \in \mathbb{Q}(z_j)$.
- Therefore, there exists

$$\gamma = \begin{pmatrix} \alpha & \beta \\ \delta & \eta \end{pmatrix} \in \Gamma_j \leq \mathrm{SL}_2(\mathbb{Z}[z_j])]$$

that maps θ_- to ∞.

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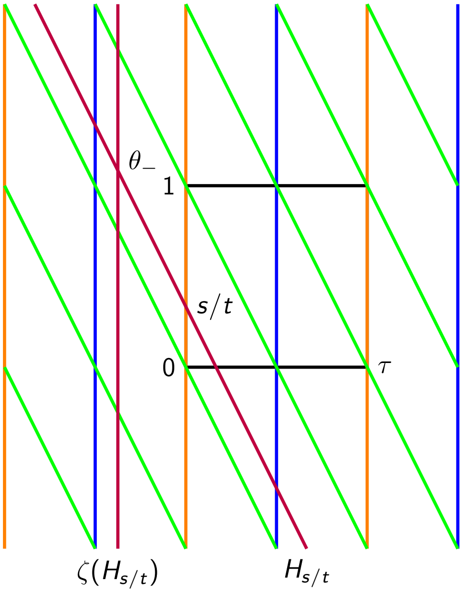

Goal: Finding algebraic conditions on entries of γ and check $\det(\gamma) = 1$.

- We have $\gamma(\theta_-) = \infty$. Therefore,

$$\delta\theta_- + \eta = 0$$

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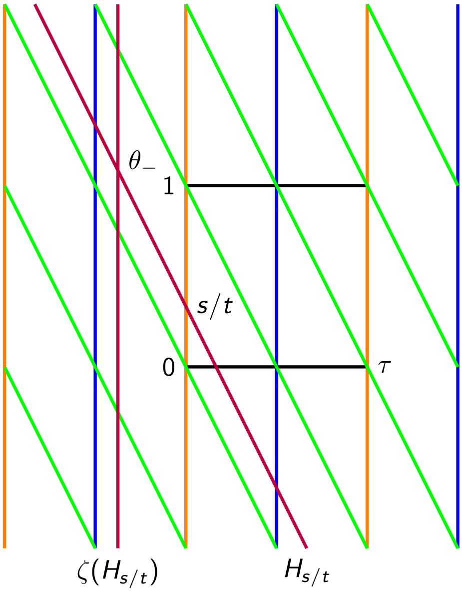

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- The slope of $H_{s/t}$ at θ_- is either $1/0$ or -2. Suppose that it is -2. Applying the trace condition to the cusp-to-cusp geodesic θ, we get:

$$\delta = \pm t z_j \in \mathbb{Z}[z_j]$$

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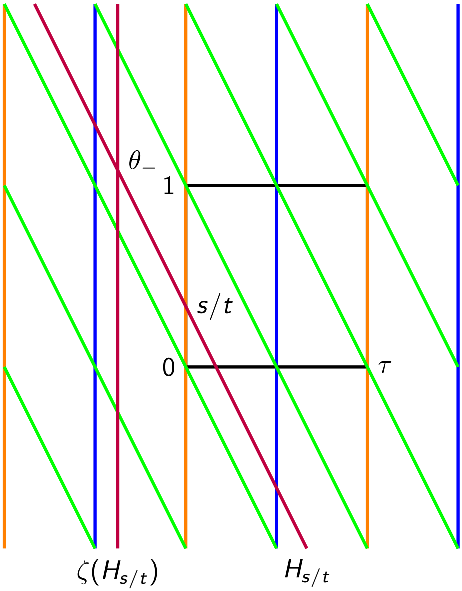

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- Therefore, we also get:

$$\eta = \pm(s z_j + 4s' + 2t) \in \mathbb{Z}[z_j]$$

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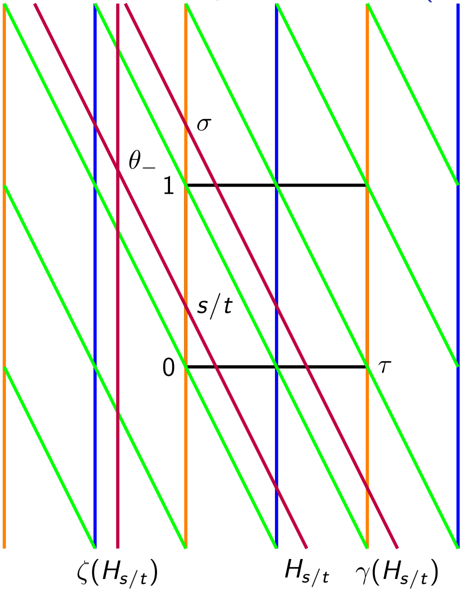

- Move the $H_{s/t}$ by γ

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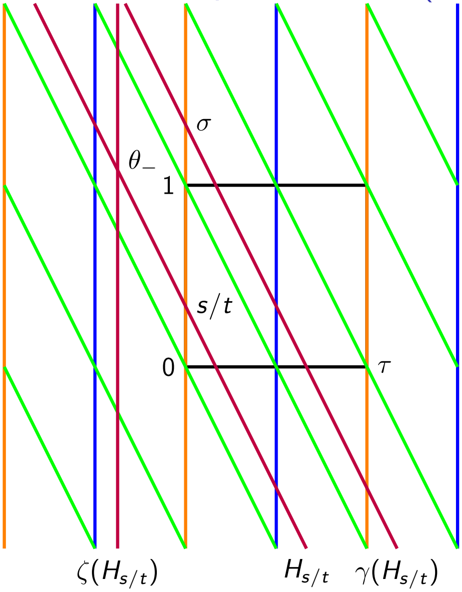

- Move the $H_{s/t}$ by γ
- The point σ is a cusp point and a rational number. Therefore,

$$\frac{\alpha}{\delta} + r(2 - \tau) \in \mathbb{Q}$$

for some $r \in \mathbb{R}$.

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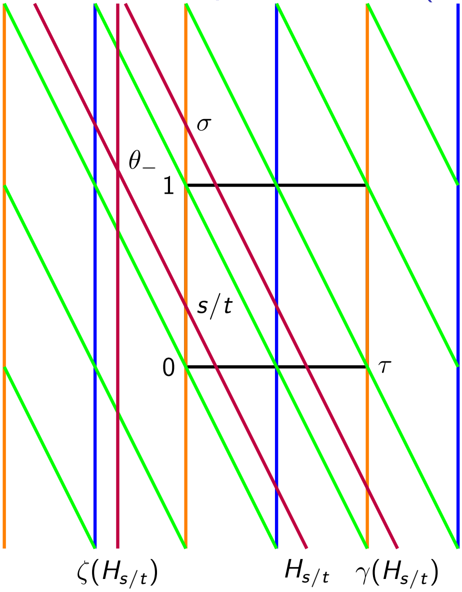

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  for some $r \in \mathbb{R}$.
- Solving for α we get

$$\alpha = \alpha_1 z_j + \alpha_0$$

  for $\alpha_i \in \mathbb{Z}$.

Proof of Uniqueness of N (Sketch)

- $\det(\gamma) = 1$ implies that:

$$(\alpha_1 z_j + \alpha_0)(s z_j + 4s' + 2t) - t z_j \beta = \pm 1$$

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- There is no such $\gamma \in \Gamma$, and so θ is not a cusp-to-cusp geodesic. This contradicts the geometric observation that totally geodesic surface intersects along cusp-to-cusp geodesics. Therefore, $H_{s/t}$ does not cover a totally geodesic surface in M_j.

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- The proof proceed similarly in the second case.

Thank you!

Lifting N

- Consider the cover $\hat{M}_{j,n}$ corresponding to the kernel of a surjective homomorphism $\phi_n : \Gamma_j \to D_n$ to the dihedral group of order $2n$. Such a homomorphism exists if n divides $2j+1$.
- $|\phi_n(\pi_1(N))| = 2$, so N lifts to n totally geodesic surfaces in $\hat{M}_{j,n}$
- Any element α of order 2 in D_{2n} fixes exactly 1 surface in $\hat{M}_{j,n}$ and exchanges the rest pairwise.
- The cover $\hat{M}_{j,n}/\langle\alpha\rangle$ contains $(n+1)/2$ surfaces.
- We need to show that there are infinitely many prime j such that n divides $2j+1$. We write $n = 2q+1$ and consider the arithmetic progression $\{q+nh\}_{h=0}^{\infty}$. Dirichlet's theorem says that there are infinitely many primes in such an arithmetic progression.

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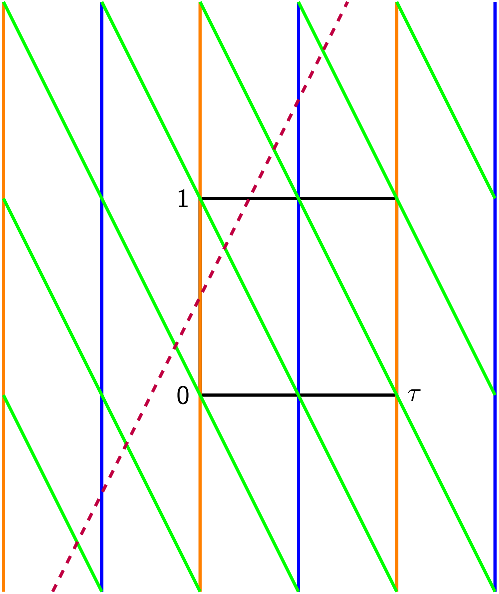

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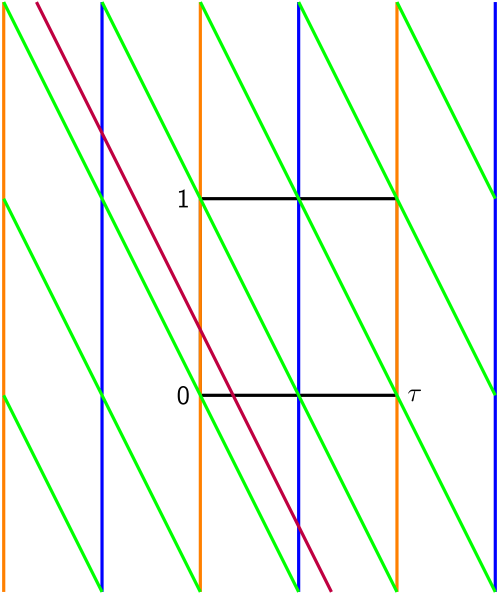

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