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Infinite dimensional equivariant commutative algebra

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I. Introduction

(A) $R = \mathbb{C}[x_1, x_2, \ldots] \circlearrowleft \mathfrak{S}_\infty = \bigcup_{n \geq 1} \mathfrak{S}_n$

Thm (Cohen, 1967) R is $\mathfrak{S}_\infty$-noetherian, i.e., ACC holds for $\mathfrak{S}_\infty$-ideals

Applications:

- Cohen: any variety of metabelian groups is finitely based
- Draisma-Eggermont: $Gr_p(V) \subset \mathbb{P}(\Lambda^p V) \rightsquigarrow Sec_k(Gr_p(V))$
  - Fix k $\exists d$ s.t. $Sec_k(Gr_p(V))$ are cut out by eqns of deg $\leq d$ $\forall p, V$
- Draisma-Kuttler: similar result for Segre embeddings

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(B) $GL_\infty = \bigcup_{n \geq 1} GL_n$

Defn A rep of GL_∞ is polynomial if it appears in a sum of tensor powers of the std rep $\mathbb{C}^\infty = \bigcup \mathbb{C}^n$

Examples:
- $(\mathbb{C}^\infty)^{\otimes k}$
- $\mathrm{Sym}^k(\mathbb{C}^\infty)$
- $\Lambda^k(\mathbb{C}^\infty)$
- $S_\lambda(\mathbb{C}^\infty)$ λ = partition

Defn A GL-algebra is a $\mathbb{C}$-alg R equipped w/action of GL_∞ s.t. it forms a poly rep.

Example
- $R = \mathbb{C}[x_1, x_2, \ldots] = \bigoplus_{k \geq 0} \mathrm{Sym}^k(\mathbb{C}^\infty)$
- $R = \mathbb{C}[x_i, y_i, z_i]_{i \geq 1} = \mathrm{Sym}(\mathbb{C}^\infty \oplus \mathbb{C}^\infty \oplus \mathbb{C}^\infty)$
- $R = \mathrm{Sym}(\mathrm{Sym}^2 \mathbb{C}^\infty) = \mathbb{C}[x_{ij}]_{1 \leq i \leq j < \infty}$
- $R = \mathbb{C}[x_i, y_i] / (x_i y_j - x_j y_i)$

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Thm (Draisma) $R = \mathrm{Sym}(V)$ V fin. len. poly rep

ACC holds for <u>radical</u> GL-ideals.

Rmk ACC for arbitrary GL-ideals unknown, very important problem.

Applications:

- $R = \mathrm{Sym}(\mathrm{Sym}^2 \mathbb{C}^\infty \oplus \mathrm{Sym}^3 \mathbb{C}^\infty)$ $\qquad \underbrace{\mathrm{Spec}(R) = (\mathrm{Sym}^2)^* \times (\mathrm{Sym}^3)^*}$

  space of pairs (f, g) f is homo deg 2 poly in ∞ variables.
  g ——— 3 —

  Parameter space for ideals w/ gens (f, g)

- Erman-Sam-S: proved Stillman's conj using this approach
- Draisma-Lasón-Leykin: finiteness properties of Gröbner bases.

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II. Equivariant comm. alg.

Fix a comm ring R on which a gp G acts.

Goal: "Import" defns from comm alg to equivariant setting

Principal: Phrase things using ideals (no elements) then change "ideal" to "G-idea"

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Classical defn $\mathfrak{p}$ is a prime ideal if $xy \in \mathfrak{p}$ then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$

Equivalent: $\mathfrak{p}$ prime $\iff$ $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$ then $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$ $\forall$ ideals $\mathfrak{a}, \mathfrak{b}$

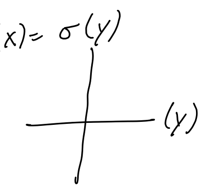
Defn A G-ideal $\mathfrak{p}$ of R is G-prime if $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p} \Rightarrow \mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$ $\forall$ G-ideals $\mathfrak{a}, \mathfrak{b}$

Elemental form: $\mathfrak{p}$ G-prime $\iff$ $x \cdot gy \in \mathfrak{p}$ $\forall g \in G$ either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$

Example: $R = \mathbb{C}[x, y]$ $G = \mathfrak{S}_2$ $\mathfrak{p} = (xy)$ Claim: $\mathfrak{p}$ is G-prime.

$f \cdot \sigma g \in \mathfrak{p}$ $\forall \sigma \in \mathfrak{S}_2$. Suppose $f \notin \mathfrak{p}$, say $x \nmid f$

$xy \mid f \cdot \sigma g \Rightarrow x \mid \sigma g \; \forall \sigma \Rightarrow x \mid g, y \mid g \Rightarrow xy \mid g \Rightarrow g \in \mathfrak{p}$

$V(\mathfrak{p}) =$ 

Fact: Suppose G is finite, R noetherian. Then:

- if $\mathfrak{p}$ is a prime of R then $\bigcap_{\sigma \in G} \sigma\mathfrak{p}$ is G-prime
- bijection $\{\text{primes}\}/G \xrightarrow{\sim} \{G\text{-primes}\}$.

Example: $R = \mathbb{C}[x_i]_{i \geq 1}$ $G = \mathfrak{S}_\infty$ $\mathfrak{p} = (x_i^2)_{i \geq 1}$ Claim: $\mathfrak{p}$ is G-prime

Say $f \cdot \sigma g \in \mathfrak{p}$ $\forall \sigma \in G$: Choose σ s.t. f and σg have no vars in common.

$\rightsquigarrow$ f or g belongs to $\mathfrak{p}$ $\Rightarrow$ $\mathfrak{p}$ is G-prime.

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Classical defn The radical of an ideal $\mathfrak{a}$ is the set of elts x s.t. $x^n \in \mathfrak{a}$ for some n

$\mathrm{rad}(\mathfrak{a})$ = sum of all ideals $\mathfrak{b}$ s.t. $\mathfrak{b}^n \subset \mathfrak{a}$ for some n

Def The G-radical of a G-ideal $\mathfrak{a}$, denoted $\mathrm{rad}_G(\mathfrak{a})$, is the sum of all G-ideals $\mathfrak{b}$ s.t. $\mathfrak{b}^n \subset \mathfrak{a}$ for some n.

Fact: $$\mathrm{rad}_G(\mathfrak{a}) = \bigcap_{\substack{\mathfrak{a} \subset \mathfrak{p} \\ \mathfrak{p}\ G\text{-prime}}} \mathfrak{p}$$

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Def The *G-spectrum* of R is the set of all G-primes, w/ Zariski top

$$\mathrm{Spec}_G(R)$$

Rmk If $\mathfrak{a}, \mathfrak{b}$ are G-ideals then $V_G(\mathfrak{a}) = V_G(\mathfrak{b})$ iff $\mathrm{rad}_G(\mathfrak{a}) = \mathrm{rad}_G(\mathfrak{b})$

Main problem Understand Spec_G in examples A and B

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III. Example A

$R = \mathbb{C}[x_1, x_2, \ldots]$ $\quad G = \mathfrak{S}_\infty$ $\quad$ (Joint w/ Rohit Nagpal)

To start, let's consider radical G-primes.

$$X = \mathrm{Spec}(R) = \mathbb{C}^\infty = \{ (a_1, a_2, a_3, \ldots) \mid a_i \in \mathbb{C} \}$$

radical G-primes $\Longleftrightarrow$ G-irred. closed sets of X

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Construction 1: bound # of values

- $Z_1 \subset \mathbb{C}^\infty$ consisting of tuples use at most 2 values

$$(a, a, a, b, b, b, b, a, \ldots)$$

- Claim Z_1 is Zariski closed.

Defined by vanishing of 3-variable discriminants

$$(x_i - x_j)(x_i - x_k)(x_j - x_k) = 0 \quad \forall i, j, k$$

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Construction 2: bound multiplicities of values

- $Z_2 \subset Z_1$ where b appears at most once
- This is defined by:

$$(x_i - x_j)(x_k - x_l) = 0 \quad \forall\, i, j, k, l \text{ distinct}$$

Construction 3: alg. relns b/w values

- $Z_3 \subset Z_2$ where $f(a,b) = 0$ $\quad f \in \mathbb{C}[x,y]$
- defined by $\quad (x_i - x_j)(x_i - x_k)\, f(x_j, x_i) = 0$

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$\lambda_1 = \infty$

General construction:

- A partition of ∞ is a tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$, $\lambda_i \in \mathbb{N} \cup \{\infty\}$
- $X_\lambda \subset X$ consisting of pts of type λ $\quad (Z_2 = X_{(\infty,1)})$
- $X_\lambda / \mathfrak{S}_\infty \xleftrightarrow[\sim]{\varphi} \mathbb{A}^{[r]} / \mathrm{Aut}(\lambda)$ $\quad$ $\mathbb{A}^{[r]} \subset \mathbb{A}^r$ coordinates are distinct, $\mathrm{Aut}(\lambda) \subset \mathfrak{S}_r$ fixes λ
- Given a cl. subset Z of $\mathbb{A}^{[r]}/\mathrm{Aut}(\lambda)$ denote $X_\lambda(Z) \subset X$ be the Z. closure of $\varphi^{-1}(Z)$

Thm (Nagpal - S.) Have bijec

$$\{\text{proper } G\text{-ined subsets of } X\} \longleftrightarrow \left\{(\lambda, Z) \,\middle|\, \begin{array}{l}\lambda \text{ part of } \infty \\ Z \text{ as above} \\ \text{irred}\end{array}\right\}$$

$$X_\lambda(Z) \longleftrightarrow (\lambda, Z)$$

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$$\{G\text{-primes of } R\} \xrightarrow{\text{rad}} \{\text{rad. } G\text{-primes}\}$$

$\mathfrak{q}$

$\cap$

understand

Goal: understand fibers

Let $\mathfrak{q} = \langle x_i - x_j \rangle_{i,j \geq 1} \subset R$ G-prime, radical

Sufficient: understand fiber over $\mathfrak{q}$

Define $\mathfrak{q}_n = \langle (x_i - x_j)^n \rangle_{i,j \geq 1} \subset R$

Thm (Nagpal-S)
- $\mathfrak{q}_n$ is G-prime iff n is odd
- the $\mathfrak{q}_n$ w/ n odd are exactly G-primes w/ radical $\mathfrak{q}$

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Key ingredient:

$A = \mathbb{C}[x_1, \ldots, x_n]$

$I = \langle x_1^k, \ldots, x_n^k \rangle$

$B = \mathbb{C}[x_i - x_j] \subset A$

$J = B \cap I$ contraction.

$= \mathbb{C}[y_2, \ldots, y_n] \quad y_i = x_i - x_1$

$$(y_i - y_j)^{2k-1} = (x_i - x_j)^{2k-1} = \sum_{r=0}^{2k-1} \binom{2k-1}{r} x_i^{r} x_j^{2k-1-r}$$

at least one is $\geq k$

$$\Rightarrow (y_i - y_j)^{2k-1} \in J$$

Thm: $J = \langle (y_i - y_j)^{2k-1} \rangle_{2 \leq i,j \leq n}$

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IV. Example B

$R = \underset{G}{\overset{\|}{GL}}$-algebra. Goal: describe $\mathrm{Spec}_G(R)$

Examples:

- $R = \mathrm{Sym}(\overbrace{\mathbb{C}^\infty \oplus \cdots \oplus \mathbb{C}^\infty}^{d \text{ copies}})$

Prop (Sam-S.) G-primes = G-stable primes.

$$\mathrm{Spec}_G(R) = \coprod_{r=0}^{d} Gr_r(\mathbb{C}^d) \quad (\text{top. is } \underline{\text{not}} \text{ disjoint union})$$

- $R = \bigoplus_{n=0}^{\infty} \Lambda^{2n}(\mathbb{C}^\infty)$ (0) is G-prime. i.e. $R = GL$-domain

$\mathfrak{a}, \mathfrak{b} \subset R$ non-zero G-ideals $\xrightarrow{\text{WTS}}$ $\mathfrak{a}\mathfrak{b} \neq 0$

- $R = \mathrm{Sym}(\mathrm{Sym}^2 \mathbb{C}^\infty)$ $\mathfrak{p}_r$ is rth det ideal. G-stable prime.

$= \bigoplus_{\lambda \text{ even}} S_\lambda(\mathbb{C}^\infty)$ $\mathfrak{p}_{r,s}$ ideal gen'd by $(2s+2) \times (r+1)$ rectangle.

These are G-prime.

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$R = \mathrm{Sym}(\mathrm{Sym}^2)$

$R(\mathbb{C}^\infty) = \mathrm{Sym}(\mathrm{Sym}^2 \mathbb{C}^\infty)$

nilpotent

$R(\mathbb{C}^{\infty|\infty}) = \mathrm{Sym}(\mathrm{Sym}^2 \mathbb{C}^{\infty|\infty}) = \mathrm{Sym}(\mathrm{Sym}^2 \mathbb{C}^{\infty} \oplus \Lambda^2 \mathbb{C}^\infty \oplus \text{cross term})$

$V(\mathfrak{p}_{r,s}(\mathbb{C}^{\infty|\infty}))$ = rank $\leq r$ on even part, rank $\leq s$ on odd part

Thm (S) $R = GL$-alg. $\mathfrak{a}, \mathfrak{b} \subset R$ are GL-ideals

$\mathrm{rad}_G(\mathfrak{a}) = \mathrm{rad}_G(\mathfrak{b}) \iff V(\mathfrak{a}(\mathbb{C}^{\infty|\infty})) = V(\mathfrak{b}(\mathbb{C}^{\infty|\infty}))$